# DENSITY OF GROUP LANGUAGES IN SHIFT SPACES 

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#### Abstract

We study the density of group languages (i.e. rational languages recognized by morphisms onto finite groups) inside shift spaces. The density of a rational language can be understood as the frequency of some "pattern" in the shift space, for example a pattern like "words with an even number of a given letter." In this paper, we handle density of group languages via ergodicity of skew products between the shift space and the recognizing group. We consider both the cases of shifts of finite type (with a suitable notion of irreducibility), and of minimal shifts. In the latter case, our main result is a closed formula for the density which holds whenever the skew product has minimal closed invariant subsets which are ergodic under the product of the original measure and the uniform probability measure on the group. The formula is derived in part from a characterization of minimal closed invariant subsets for skew products relying on notions of cocycles and coboundaries. In the case where the whole skew product itself is ergodic under the product measure, then the density is completely determined by the cardinality of the image of the language inside the recognizing group. We provide sufficient conditions for the skew product to have minimal closed invariant subsets that are ergodic under the product measure. Finally, we investigate the link between minimal closed invariant subsets, return words and bifix codes.


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## 1. Introduction

The study of language density can be traced back to the work of Schützenberger in the 60s 45], Berstel in the 70s [6], and Hansel and Perrin in the 80s [28]. The idea also appears in Eilenberg's monograph [25] and in the monograph by Berstel, Perrin and Reutenauer [7]. These earlier works, motivated mainly by automata theory and the theory of codes, focused on density with respect to Bernoulli measures.

In the present paper, we draw our motivation from symbolic dynamics and turn to ergodic measures on shift spaces. Within this setting, the density of a given rational language can be understood as the frequency of some "pattern" in the shift space - for example a pattern like "words with an even number of a given letter". More precisely, given a shift $X$ on the alphabet $A$ endowed with a shiftinvariant probability measure $\mu$, and $L \subseteq A^{*}$ a rational language, the density $\delta_{\mu}(L)$ of $L$ under the measure $\mu$ is the Cesàro limit of $\mu\left(L \cap A^{n}\right)$ as $n \rightarrow \infty$, whenever it exists.

We restrict our attention to group languages, i.e. languages recognized by morphisms onto finite groups: by fixing a morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$, we consider a language of the form $L=\varphi^{-1}(K), K \subseteq G$. The key idea in this paper is that the density can be expressed in terms of limits of ergodic sums in a skew product (cf. Theorem A), allowing us to show that it exists and calculate it whenever the skew product is ergodic. Indeed, given the shift space $X$ with an ergodic measure $\mu$, we relate the density $\delta_{\mu}(L)$ with the structure of the skew product between $X$ and $G$, denoted $G \rtimes X$, where the skewing function is the cocycle determined by $\varphi$, and where we equip $G \rtimes X$ with the product measure $\nu \times \mu$, where $\nu$ is the uniform probability measure on $G$.

Our first main result, stated in Section 3 as Theorem 3.3, consists of a simple formula for the density when the product measure $\nu \times \mu$ on $G \rtimes X$ is ergodic. The formula resembles an earlier result of Hansel and Perrin within the setting of Bernoulli measures ([28], Theorem 3).

Theorem A. Let $X$ be a shift space on $A$ with an ergodic measure $\mu$ and $\varphi: A^{*} \rightarrow$ $G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\nu \times \mu$ is an ergodic measure for the skew product $G \rtimes X$, then for every $L=\varphi^{-1}(K)$, $K \subseteq G$, the density $\delta_{\mu}(L)$ exists and is given by $\delta_{\mu}(L)=|K| /|G|$.

We first apply this result to the case of shifts of finite type by introducing a suitable notion of irreducibility, namely $\varphi$-irreducibility (Definition 4.1). The next theorem, which is our second main result, is stated later as Corollary 4.12.
Theorem B. Let $X$ be an r-step shift of finite type over the alphabet $A$ and $\mu$ an $r$-step Markov measure fully supported on $X$. Let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. If $X$ is $\varphi$-irreducible, then for every group language $L=\varphi^{-1}(K)$ where $K \subseteq G$, the density $\delta_{\mu}(L)$ exists and is given by $\delta_{\mu}(L)=|K| /|G|$.

There are however many cases where the measure $\nu \times \mu$ is not ergodic, wherein the formula $\delta_{\mu}(L)=|K| /|G|$ might not hold. Three cases where $\nu \times \mu$ is not ergodic are presented in the present paper, in Examples 6.16, 6.17 and 6.18: interestingly in the second case, the formula $\delta_{\mu}(L)=|K| /|G|$ holds nonetheless.

With an eye on such examples, we establish a more general formula (Theorem C below) which holds under the weaker condition that the product measure $\nu \times \mu$ is ergodic on each minimal closed invariant subset of $G \rtimes X$. Let us briefly present the notion, inspired by Proposition 2.1 of 32], which lies at the core of this more general formula. Given a subgroup $H \leq G$, a map $\alpha$ from $X$ to the right coset space $H \backslash G$ is called a cobounding $\operatorname{mar}^{11} \bmod H$ if it satisfies the following coboundary type equation, where $S$ denotes the shift map:

$$
\alpha(S x)=\alpha(x) \varphi\left(x_{0}\right)
$$

As the name suggests, this notion is related with cocycles and coboundaries and more broadly to the long history of cohomology in ergodic theory. Besides 32], we drew inspiration also from [2, 48, 49, 51, 20, 26, 41, 44]. The following is our third main result. It is stated again later in a more precise form, in Section 6, as Theorem 6.13.

Theorem C. Let $X$ be a minimal shift space on $A$ with an ergodic measure $\mu$ and let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\nu \times \mu$ is ergodic on each minimal closed invariant subset of $G \rtimes X$, then there exists a subgroup $H \leq G$ and a cobounding map $\alpha: X \rightarrow H \backslash G$ such that for every $L=\varphi^{-1}(K), K \subseteq G$,

$$
\begin{equation*}
\delta_{\mu}(L)=\frac{1}{|H|} \sum_{k \in K} \sum_{H g \in H \backslash G} \mu\left(\alpha^{-1}(H g)\right) \mu\left(\alpha^{-1}(H g k)\right) . \tag{1.1}
\end{equation*}
$$

Behind this last result is a description of the minimal closed invariant subsets of $G \rtimes X$ in terms of cobounding maps (Proposition 6.4). Among other things, this description entails that $\nu \times \mu$ can be ergodic only when $G \rtimes X$ itself is minimal (Corollary 6.2). The case when $\nu \times \mu$ is ergodic thus lies within the scope of Theorem C] When this happens, (1.1) simplifies and we recover Theorem A.

These results motivated our study of the structure of the skew product, which includes a characterization of minimality in terms of return words (Theorem 5.1). We first give a combinatorial proof of this characterization using the theory of bifix codes, inspired by ideas from earlier works [8, 14]; later on, we show that it can also be recovered using cobounding maps (Proposition 6.9).

Having in mind Theorem C] we also study sufficient conditions for ergodicity of $\nu \times \mu$ on minimal closed invariant subsets of $G \rtimes X$. A first condition is given in

[^0]Corollary 6.12 combined with Proposition 6.11 and a second one, restricted to the case of shift spaces generated by primitive substitutions, is given in Proposition 7.3. We deduce as a corollary of the second condition that substitutive dendric shifts (definitions are recalled in Section 7) have ergodic skew products with all finite groups.

Here is an example of what the existence of density might say for specific systems. Consider the Fibonacci shift (whose definition is recalled in Section 3.3) on the twoletter alphabet $\{a, b\}$, with its unique invariant measure $\mu$. For each word $w$ in the language of the shift space, denote by $|w|_{a}$ the number of occurrences of $a$ in $w$. Then for most long words $w$, for each $r=0, \ldots, m-1$, the probability that $|w|_{a}$ is congruent to $r \bmod m$ is approximately $1 / m$. The convergence holds only in the sense of Cesàro mean. This specific example with $m=2$ is developed in detail in Section 3.3.

There have been previous results of this nature concerning equidistribution modulo $m$. For instance, Veech [48, 49] studied the parity of the number of visits to an interval by the orbit of a point under an irrational rotation, and Jager and Liardet [29] studied the congruence classes of the matrices in $\operatorname{GL}(2, \mathbb{Z})$ associated with continued fraction expansions of real numbers. In each of these cases, it is ergodicity of a relevant skew product that implies equidistribution among cosets. We revisit these results in Section 7.3. Our approach provides still more examples. These examples include the Thue-Morse shift (explored in Examples 2.1 and 6.17) and the case of substitutive dendric shifts (presented in Section 7.2), which includes substitutive Sturmian shifts and substitutive codings of interval exchanges,

Let us give a brief overview of the paper's structure. Section 2 gives some preliminaries on symbolic dynamics. Section 3 recalls the definition of skew products and gives an elementary proof of our first main result, Theorem A. The study of the density for shifts of finite type is handled in Section 4 with a proof of Theorem B and a discussion on various notions of irreducibility. Section 5 contains the material on bifix codes and the characterization of minimal skew products in terms of return words. Our third main result, namely Theorem C] is presented in Section 6, where cobounding maps are studied; the section also contains a simple sufficient condition for ergodicity of minimal closed invariant subsets. In Section 7 we take a closer look at shifts generated by primitive substitutions.

## 2. Symbolic dynamics

Let $A$ be a finite alphabet. Let $\varepsilon$ stand for the empty word of the free monoid $A^{*}$ and $A^{+}=A^{*} \backslash\{\varepsilon\}$. We denote by $A^{\mathbb{Z}}$ the set of two-sided infinite words over $A$. For any word $w$ in the free monoid $A^{*}$ (endowed with concatenation), $|w|$ denotes the length of $w$ and $|w|_{a}$ stands for the number of occurrences of the letter $a$ in the word $w$. We start indexing finite words with 0 , so that a word $w \in A^{*}$ has the form $w=w_{0} w_{1} \cdots w_{n-1}$, where $n=|w|$. Given $0 \leq i \leq j<n$, we let

$$
w_{[i, j)}=w_{i} w_{i+1} \cdots w_{j},
$$

and we extend a similar notation for infinite words.
A factor of a (finite or infinite) word $w$ is defined as a finite concatenation of consecutive letters occurring in $w$, i.e. a word $u$ is a factor of $w$ if there exists indices $i \leq j$ such that $u=w_{[i, j)}$. If $w$ is a finite word, then $u$ is a factor of $w$ precisely
when there are words $p$ and $s$ such that $w=p u s$. When $p=\varepsilon$ (resp., $s=\varepsilon$ ) we say that $u$ is a prefix (resp., suffix) of $w$.

An infinite word $x=\left(x_{n}\right)_{n \in \mathbb{Z}}$ is uniformly recurrent if every word occurring in $x$ occurs in an infinite number of positions with bounded gaps; in other words, for every factor $w$ of $x$, there exists a positive integer $m$ such that for every $n, w$ is a factor of $x_{[n, n+m)}$.

We view sets of two-sided infinite words as dynamical systems under the map $S$, called the shift map, defined by

$$
S\left(\left(x_{n}\right)_{n \in \mathbb{Z}}\right)=\left(x_{n+1}\right)_{n \in \mathbb{Z}} .
$$

A shift space (also shortened to shift) is a pair $(X, S)$ where $X$ is a closed shiftinvariant subset of $A^{\mathbb{Z}}$ for some finite alphabet $A$. We usually shorten $(X, S)$ as $X$ when we refer to the system $(X, S)$. The language of $X$ is defined as the set $\mathcal{L}(X)$ of factors of elements of $X$,

$$
\mathcal{L}(X)=\left\{x_{[i, j)} \mid x \in X, i, j \in \mathbb{Z}, i \leq j\right\} .
$$

When a shift $X$ is said to be defined on the alphabet $A$, we assume that $A \subseteq \mathcal{L}(X)$.
A shift space is said to be minimal if it admits no proper non-empty closed and shift-invariant subset; equivalently the $S$-orbit of every element of $X$ is dense. Note that a shift space $X$ is minimal if and only if every infinite word $x \in X$ is uniformly recurrent. On the other hand, a shift space is called irreducible if there exists an element $x \in X$ with dense $S$-orbit. This is equivalent to the following property of $\mathcal{L}(X)$ : for every $u, v \in \mathcal{L}(X)$, there exists $w \in A^{*}$ such that $u w v \in \mathcal{L}(X)$.

Let $X$ be a shift space on $A$ and fix $w \in \mathcal{L}(X)$. We denote by $\mathcal{R}_{X}(w)$ the set of (right) return words to $w$. It is, by definition, the set of words $r$ such that $r w$ is in $\mathcal{L}(X)$ and has exactly two factors equal to $w$, one as a prefix and the other one as a suffix; that is,

$$
\mathcal{R}_{X}(w)=\left\{r \in A^{*} \mid r w \in \mathcal{L}(X) \cap w A^{*} \backslash A^{+} w A^{+}\right\} .
$$

Let $X \subseteq A^{\mathbb{Z}}$ be a shift space equipped with a Borel probability measure $\mu$. For $w \in \mathcal{L}(X)$, we denote

$$
[w]_{X}=\left\{x \in X \mid x_{[0, n)}=w\right\}
$$

the (right) cylinder defined by $w$. By a slight abuse of notation, we denote by the same symbol $\mu$ the map $\mathcal{L}(X) \rightarrow[0,1]$ which assigns to a word $w \in \mathcal{L}(X)$ the number $\mu\left([w]_{X}\right)$. Thus we have $\mu(\varepsilon)=1$ and

$$
\sum_{a \in A} \mu(w a)=\mu(w) .
$$

The measure $\mu$ is invariant if $\mu\left(S^{-1} U\right)=\mu(U)$ for every Borel set $U \subset X$. Note that $\mu$ is invariant if and only if, for every $w \in \mathcal{L}(X)$,

$$
\sum_{a \in A} \mu(a w)=\mu(w) .
$$

Recall that an invariant measure $\mu$ is ergodic if every Borel set $U$ which is invariant (i.e. $S^{-1} U=U$ ) has measure 0 or 1. A well-known equivalent condition is that $\mu$ is ergodic if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(U \cap S^{-i} V\right)=\mu(U) \mu(V) \tag{2.1}
\end{equation*}
$$

for every pair $U, V$ of Borel sets (40], p. 56, exercise 4(a)).
The shift $X$ is said to be uniquely ergodic if there is a unique invariant probability measure on $X$, in which case this unique measure is necessarily ergodic. By a theorem of Michel, every primitive substitution shift is uniquely ergodic 36]. Other important sufficient conditions for unique ergodicity of shift spaces are due to Boshernitzan [16, 17]. More details on such results may be found in [42, 24].
Example 2.1. The Thue-Morse shift $X=X(\sigma)$ with $\sigma: a \mapsto a b, b \mapsto b a$ is uniquely ergodic by Michel's theorem. Its unique ergodic measure $\mu$ is depicted in Fig. 1


Figure 1. The invariant probability measure on the Thue-Morse shift.

Let $L$ be a rational language on $A$. The aim of this paper is to study the density of $L$, under the measure $\mu$, defined as the following limit whenever it exists:

$$
\delta_{\mu}(L)=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(L \cap A^{i}\right)
$$

In other words, $\delta_{\mu}(L)$ is the Cesàro limit of $\mu\left(L \cap A^{n}\right)$ as $n \rightarrow \infty$. Since $\mu(w)=0$ when $w \notin \mathcal{L}(X)$, we have of course that $\delta_{\mu}(L)=\delta_{\mu}(L \cap \mathcal{L}(X))$ and $\delta_{\mu}(\mathcal{L}(X))=1$. Moreover, it follows from the definition of density that, whenever $L$ and $L^{\prime}$ are disjoint,

$$
\delta_{\mu}\left(L \cup L^{\prime}\right)=\delta_{\mu}(L)+\delta_{\mu}\left(L^{\prime}\right)
$$

## 3. Group languages and skew products

We first recall basic definitions on group languages and skew products in Section 3.1 We then prove Theorem A in Section 3.2. The proof is based on the ergodicity of the skew products under consideration, i.e. uniquely ergodic shifts skewed by finite groups endowed with the uniform probability measure. We also prove that ergodicity implies unique ergodicity (see Proposition 3.5). Lastly, we focus on the case of the Fibonacci shift in Section 3.3, where we prove that the
density considered as a Cesàro mean converges, whereas the sequence of measures $\left(\mu\left(L \cap A^{n}\right)\right)_{n \in \mathbb{N}}$ does not converge in the classical sense.
3.1. First definitions. A group language is a set of the form $L=\varphi^{-1}(K)$ where $\varphi: A^{*} \rightarrow G$ is a morphism onto a finite group $G$ and $K \subset G$. Note that such languages are in particular rational, being recognized by finite groups.

Let $X$ be a minimal shift space equipped with an invariant measure $\mu$. We consider the skew product of $G$ and $X$ with respect to $\varphi$, denoted $G \rtimes_{\varphi} X$ : it is the dynamical system $\left(G \times X, T_{\varphi}\right)$ where $T_{\varphi}$ is the transformation defined by

$$
T_{\varphi}(g, x)=\left(g \varphi\left(x_{0}\right), S x\right)
$$

More generally, $T_{\varphi}$ satisfies, for every $n \in \mathbb{Z}$,

$$
T_{\varphi}^{n}(g, x)=\left(g \varphi^{(n)}(x), S^{n} x\right)
$$

where $\varphi^{(n)}$ is defined by

$$
\varphi^{(n)}(x)= \begin{cases}\varphi\left(x_{[0, n)}\right) & \text { if } n \geq 0  \tag{3.1}\\ \varphi\left(x_{[n, 0)}\right)^{-1} & \text { if } n<0\end{cases}
$$

The map $(n, x) \mapsto \varphi^{(n)}(x)$ is the cocycle defined by $\varphi$. When the morphism $\varphi$ is clear from context, we often simply write $T$ and $G \rtimes X$. Skew products constitute one of the basic extensions of dynamical systems; see e.g. [21, 40].

Lemma 3.1. The skew product $G \rtimes X$ is topologically conjugate to a shift space on $G \times A$ via the map $\Psi: G \rtimes X \rightarrow(G \times A)^{\mathbb{Z}}$ defined by

$$
\Psi(g, x)_{n}=\left(g \varphi^{(n)}(x), x_{n}\right), \quad n \in \mathbb{Z}
$$

Proof. This map is continuous and injective, hence it is an homeomorphism onto its image. Moreover it follows from the definitions that $\Psi \circ T=S \circ \Psi$.

Example 3.2. Let $X$ be the three-element shift defined as the finite orbit of the periodic word $x=(a b c)^{\infty}$, so $X=\{x, y, z\}$ with $y=S x, z=S y$. Let $\varphi: A^{*} \rightarrow$ $\mathbb{Z} / 2 \mathbb{Z}$ be the morphism defined by

$$
\varphi(a)=\varphi(b)=1, \quad \varphi(c)=0
$$

The skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ has six elements. Viewed as a shift over the alphabet $\mathbb{Z} / 2 \mathbb{Z} \times A$, it is the disjoint union of the orbits of two periodic words,

$$
\Psi(0, x)=((0, a)(1, b)(0, c))^{\infty} \quad \text { and } \quad \Psi(1, x)=((1, a)(0, b)(1, c))^{\infty}
$$

The shift $X$ and the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ are depicted in Fig. 2


Figure 2. The finite shift $X$ generated by $x=(a b c)^{\infty}$ and its skew product with $\mathbb{Z} / 2 \mathbb{Z}$ from Example 3.2
3.2. A formula for the density. Throughout this paper, we always equip finite groups with the uniform probability measure (i.e. the normalized Haar measure), denoted $\nu$. Given an ergodic measure $\mu$ on $X$, we equip skew products $G \rtimes X$, with $G$ a finite group, with the product measure $\nu \times \mu$. The next theorem constitutes a first link between skew products and the density function. We later specialize it for shifts finite type (Section 4), and then generalize it for minimal shifts (Section 6).

Theorem 3.3 (Theorem A). Let $X$ be a shift space on $A$ with an ergodic measure $\mu$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\nu \times \mu$ is ergodic, then for every $L=\varphi^{-1}(K), K \subseteq G$, the density $\delta_{\mu}(L)$ exists and is given by $\delta_{\mu}(L)=|K| /|G|$.

Proof. Since the density of disjoint languages is the sum of their density, we may assume that $L=\varphi^{-1}(g)$ for some $g \in G$.

For $h \in G$, let $U_{h}=\{h\} \times X$. Then $(\nu \times \mu)\left(U_{h}\right)=1 /|G|$ for every $h \in G$, by definition of $\nu \times \mu$. Since $L=\varphi^{-1}(g)$, we have

$$
\{h\} \times\left[L \cap A^{i}\right]_{X}=(\{h\} \times X) \cap T^{-i}(\{h g\} \times X)=U_{h} \cap T^{-i} U_{h g}
$$

Next,

$$
\begin{aligned}
\mu\left(L \cap A^{i}\right) & =\sum_{h \in G}(\nu \times \mu)\left(\{h\} \times\left[L \cap A^{i}\right]_{X}\right) \\
& =\sum_{h \in G}(\nu \times \mu)\left(U_{h} \cap T^{-i} U_{h g}\right)
\end{aligned}
$$

Since $\nu \times \mu$ is ergodic, we can use (2.1) to conclude that

$$
\begin{aligned}
\delta_{\mu}(L) & =\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(L \cap A^{i}\right) \\
& =\sum_{h \in G} \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1}(\nu \times \mu)\left(U_{h} \cap T^{-i} U_{h g}\right) \\
& =\sum_{h \in G}(\nu \times \mu)\left(U_{h}\right)(\nu \times \mu)\left(U_{h g}\right)=1 /|G|
\end{aligned}
$$

There are however many examples where the condition from the above theorem (ergodicity of the product measure on the skew product) fails, along with the theorem's conclusion. We give below a simple example of a periodic shift where this happens.

Example 3.4. Let $X$ be the three-element shift and $\varphi$ be as in Example 3.2, and $L=\varphi^{-1}(0)$. The shift $X$ is uniquely ergodic, with ergodic measure $\mu$ given by the uniform probability measure. Moreover,

$$
L \cap \mathcal{L}(X)=(a b c)^{*}\{\varepsilon, a b\} \cup(b c a)^{*}\{\varepsilon\} \cup(c a b)^{*}\{\varepsilon, c\},
$$

and thus,

$$
\mu\left(L \cap A^{i}\right)= \begin{cases}1 & \text { if } i \equiv 0 \bmod 3 \\ \frac{1}{3} & \text { otherwise }\end{cases}
$$

It follows that $\delta_{\mu}(L)$ exists and

$$
\delta_{\mu}(L)=\frac{1}{3}\left(1+\frac{1}{3}+\frac{1}{3}\right)=\frac{5}{9}
$$

instead of $1 / 2$, as Theorem A would have predicted. Observe that the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ is not ergodic, since it has two invariant subsets each of $\nu \times \mu$-measure $1 / 2$.

We can in fact generalize this construction as follows. Take $n \geq 3$ and consider an alphabet $A=\left\{a_{0}, \ldots, a_{n-1}\right\}$ of size $n$. Let $X$ be $n$-element shift space generated by the periodic word $\left(a_{0} \cdots a_{n-1}\right)^{\infty}$ and $\varphi: A^{*} \rightarrow \mathbb{Z} / n \mathbb{Z}$ mapping $a_{0}$ to 0 and $a_{i}$ to 1 for $0<i<n$. Then the density of $\varphi^{-1}(0)$ is $(2 n-1) / n^{2}$, which as $n \rightarrow \infty$ tends to be twice what Theorem A would predict.

In the present setting, ergodicity even implies unique ergodicity. In 48], Veech observed that, if $X$ is a binary coding of an irrational rotation which is uniquely ergodic, then $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ is uniquely ergodic whenever $\nu \times \mu$ is ergodic (see Section 7.3 for more details on binary codings of rotations). Veech's argument carries over to the more general case of $G \rtimes X$ with $G$ a finite group and $X$ a uniquely ergodic shift. We state the result and sketch the argument below.

Proposition 3.5. Let $X$ be a uniquely ergodic shift on $A$ with ergodic measure $\mu$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\nu \times \mu$ is an ergodic measure on $G \rtimes X$, then $G \rtimes X$ is uniquely ergodic.

Proof. Let $\zeta$ be an invariant measure on $G \rtimes X$. Observe that the image measure of $\zeta$ under the component projection $G \rtimes X \rightarrow X$ is an invariant measure on $X$, thus it must be equal to $\mu$ as $X$ is uniquely ergodic. In other words,

$$
\zeta(G \times E)=\mu(E), \quad \text { for every measurable set } E \subseteq X
$$

Let $g \in G$ act on the left of $G \rtimes X$ by $g(h, x)=(g h, x)$. This action is $T$ commuting, as well as measure-preserving as is easily checked on rectangular sets. Thus the measure $g \zeta$ defined by $g \zeta(F)=\zeta(g F)$ is also an invariant measure. We claim that the average measure $\bar{\zeta}=\left(\sum_{g \in G} g \zeta\right) /|G|$ is equal to $\nu \times \mu$. Indeed, for every measurable set $E \subseteq X$ and $h \in G$, we have

$$
\bar{\zeta}(\{h\} \times E)=\frac{1}{|G|} \sum_{g \in G} \zeta(\{g h\} \times E)=\frac{1}{|G|} \zeta(G \times E)=\frac{1}{|G|} \mu(E)
$$

Thus we conclude that $\nu \times \mu=\bar{\zeta}$. If $\nu \times \mu$ is ergodic, then it is an extremal point in the convex set of invariant measures of $G \rtimes X$. Therefore, we conclude that $\bar{\zeta}=\nu \times \mu$ implies $\zeta=\nu \times \mu$ (as in fact $g \zeta=\nu \times \mu$ for every $g \in G)$.
3.3. An example in the Fibonacci shift. We now proceed to illustrate Theorem 3.3 for the language $L=\varphi^{-1}(0)$ where $\varphi:\{a, b\}^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ is the morphism defined by $\varphi(a)=1$ and $\varphi(b)=0$. In other words, we take

$$
L=\left\{\left.w \in\{a, b\}^{*}| | w\right|_{a} \equiv 0 \quad \bmod 2\right\}
$$

The aim of this subsection is to show explicitly that the density of $L$ under the unique ergodic measure of the Fibonacci shift, described below, exists and is equal to $1 / 2$ (Proposition 3.6), while, strikingly, the sequence of measures $\left(\mu\left(L \cap A^{n}\right)\right)_{n \in \mathbb{N}}$ does not converge in the classical sense (Proposition 3.7).

Consider the substitution $\sigma: a \mapsto a b, b \mapsto a$, called the Fibonacci substitution. Observe that the substitution $\sigma$ is primitive; thus the shift space $X=X(\sigma)$ generated by $\sigma$, called the Fibonacci shift, is uniquely ergodic by Michel's theorem. Alternatively this same conclusion follows from Boshernitzan's criterion [16]. Its unique ergodic measure $\mu$ viewed as a map on $\mathcal{L}(X)$ is depicted in Fig. 3.


Figure 3. The invariant probability measure on the Fibonacci shift ( $\lambda=$ the golden ratio). Circled nodes represent elements from the language $L=\left\{\left.w \in\{a, b\}^{*}| | w\right|_{a} \equiv 0 \bmod 2\right\}$.

We next prove that the skew product of the Fibonacci shift with $\mathbb{Z} / 2 \mathbb{Z}$ for the skewing function determined by $\varphi$ is also uniquely ergodic, as an application of Michel's theorem. The argument is a special case of the general method described in Section 7

Proposition 3.6. The skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ is uniquely ergodic.
Proof. Take the substitution $\bar{\sigma}$ defined on the alphabet $\mathbb{Z} / 2 \mathbb{Z} \times A$, whose letters are denoted $a_{0}, a_{1}, b_{0}, b_{1}$ for conciseness,

$$
\bar{\sigma}: a_{0} \mapsto a_{0} b_{1} a_{0} a_{1} b_{1}, a_{1} \mapsto a_{1} b_{0} a_{0} a_{1} b_{0}, b_{0} \mapsto a_{0} b_{1} a_{1}, b_{1} \mapsto a_{1} b_{0} a_{0}
$$

This substitution is primitive and satisfies $\pi \circ \bar{\sigma}=\sigma^{3}$ where $\pi:(\mathbb{Z} / 2 \mathbb{Z} \times A)^{*} \rightarrow A^{*}$ is the natural projection (mapping $a_{i}$ to $a$ and $b_{i}$ to $b$ ). Moreover, $\varphi \circ \sigma^{3}=\varphi$ and $\bar{\sigma}(\mathcal{L}(Y)) \subseteq \mathcal{L}(Y)$, where $Y=\Psi(G \rtimes X)$ is the skew product viewed as a shift space over $\mathbb{Z} / 2 \mathbb{Z} \times A$ via the map $\Psi$ of Lemma 3.1 It follows that $Y$ is the shift space generated by $\bar{\sigma}$. Therefore, by Michel's theorem, $Y$ is uniquely ergodic, hence so is $G \rtimes X$.

As an immediate consequence of Theorem 3.3, we conclude that the density $\delta_{\mu}(L)$ exists and is $1 / 2$. In other words, the sequence $\left(\mu\left(L \cap A^{n}\right)\right)_{n \in \mathbb{N}}$ converges to $1 / 2$ in Cesàro's sense, even though the sequence itself does not converge, as we next show. The rest of the section is devoted to the proof of the following result where $F=(F(n))_{n \in \mathbb{N}}$ is the Fibonacci number sequence (starting with $F(0)=0$, $F(1)=1$ ).

Proposition 3.7. The sequence $\left(\mu\left(L \cap A^{n}\right)\right)_{n \in \mathbb{N}}$ does not have a limit, as

$$
\lim _{n \rightarrow \infty} \mu\left(L \cap A^{F(4 n)}\right)=1, \quad \lim _{n \rightarrow \infty} \mu\left(L \cap A^{F(4 n+2)}\right)=0
$$

The proof relies on a number of key facts about the structure of the language $\mathcal{L}(X)$ of the Fibonacci shift. An important tool is the notion of special word. Recall that a word $w \in \mathcal{L}(X)$ is called left special if $a w, b w \in \mathcal{L}(X)$; it is called right special if instead $w a, w b \in \mathcal{L}(X)$; and it is called bispecial if it is both left and right special. Since it is Sturmian, the Fibonacci shift has the property that for every $n \in \mathbb{N}$, $\mathcal{L}(X) \cap A^{n}$ contains exactly one left and one right special factor (which might or might not coincide). We next establish two lemmas which are central in the proof of Proposition 3.7.

Lemma 3.8. Let $u$ be a left special factor and $v$ be a right special factor in the Fibonacci shift $X$.
(i) The word $u^{\prime}=\sigma^{2}(u b)$ is also left special.
(ii) The word $v^{\prime}=\sigma^{2}(v a)$ is also right special.

Proof. (i). As $u$ is left special, then $a u$ and $b u$ are in $\mathcal{L}(X)$. Let $c$ and $d$ be right extensions of respectively $a u$ and $b u$; let $c^{\prime}$ and $d^{\prime}$ be such that $\sigma^{2}(c)=a b c^{\prime}$ and $\sigma^{2}(d)=a b d^{\prime}\left(c^{\prime}, d^{\prime}\right.$ are either $a$ or $\left.\varepsilon\right)$. Then the following words also belong to $\mathcal{L}(X)$ :

$$
\sigma^{2}(a u c)=a b a \sigma^{2}(u) a b c^{\prime}=a b a u^{\prime} c^{\prime}, \quad \sigma^{2}(b u d)=a b \sigma^{2}(u) a b d^{\prime}=a b u^{\prime} d^{\prime}
$$

In particular $a u^{\prime}$ and $b u^{\prime}$ belong to $\mathcal{L}(X)$.
(ii) The proof of the second part follows similar lines. Let $c$ and $d$ be right extensions of respectively $v a, v b$ (in fact $d=a$ since $b b$ does not occur in $X$ ); let $c^{\prime}$ be such that $\sigma^{2}(c)=a c^{\prime}$. Then we find that the following words belong to $\mathcal{L}(X)$ :

$$
\sigma^{2}(v a c)=v^{\prime} a c^{\prime}, \quad \sigma^{2}(v b a)=\sigma^{2}(v) a b a b a=v^{\prime} b a
$$

Hence both $v^{\prime} a, v^{\prime} b \in \mathcal{L}(X)$.
Lemma 3.9. Let $u_{n}$ and $v_{n}$ denote respectively the left and right special factors of length $n$ in the Fibonacci shift $X$. Then the equality $b u_{n}=v_{n} b$ holds whenever $n=F(2 k+2)-1$ for some $k \geq 0$.

Proof. Consider the following recursively defined sequence of words:

$$
w_{0}=\varepsilon, \quad w_{k+1}=\sigma^{2}\left(w_{k} b\right)
$$

In other words, this is the sequence of words starting with:

$$
\varepsilon, \quad \sigma^{2}(b), \quad \sigma^{4}(b) \sigma^{2}(b), \quad \sigma^{6}(b) \sigma^{4}(b) \sigma^{2}(b), \quad \ldots
$$

It is clear from Lemma 3.8 that this is a sequence of left special factors of $X$. We claim that $\left|w_{k}\right|=F(2 k+2)-1$ for every $k \geq 0$. Indeed, observe that, for every $i \in \mathbb{N},\left|\sigma^{2 i}(b)\right|=F(2 i+1)$ - a fact easily established by induction - and thus,

$$
1+\left|w_{k}\right|=1+\sum_{i=1}^{k}\left|\sigma^{2 i}(b)\right|=\sum_{i=0}^{k} F(2 i+1)=F(2 k+2)
$$

Thus it only remains to show that $b w_{k}=v_{n} b$ where $n=F(2 k+2)-1$; or in other words that removing the last letter from $b w_{k}$ yields a right special factor. We
do so by induction on $k$. The basis $k=0$ is trivial since $w_{0}=\varepsilon$. Assume that the equality $b w_{k}=v_{n} b$ holds for some $k \geq 0$. Then we have:

$$
a b w_{k+1} a=a b \sigma^{2}\left(w_{k} b\right) a=\sigma^{2}\left(b w_{k} b\right) a=\sigma^{2}\left(v_{n} b^{2}\right) a=\sigma^{2}\left(v_{n} a\right) b a
$$

By Lemma 3.8, the word $\sigma^{2}\left(v_{n} a\right)$ is right special, and thus so is the word $a^{-1} \sigma^{2}\left(v_{n} a\right)$ obtained by removing its leading letter $a$. Then it remains only to observe that $a^{-1} \sigma^{2}\left(v_{n} a\right) b=w_{k+1} b$.

From the proof of the above lemma we also deduce that, for $n=F(2 k+2)-1$,

$$
\varphi\left(u_{n}\right)=\varphi\left(v_{n}\right)= \begin{cases}0 & \text { if } k \text { is odd } \\ 1 & \text { if } k \text { is even }\end{cases}
$$

Proof of Proposition 3.7. Let $\leq_{\text {lex }}$ denote the lexicographic order on words. It follows immediately from the description of $\leq_{\text {lex }}$ on $\mathcal{L}(X) \cap A^{n}$ by Perrin and Restivo 39] (Theorem 2 therein), that for every $w \in \mathcal{L}(X)$ with $|w|=n$,

$$
\varphi(w)= \begin{cases}\varphi\left(v_{n} a\right) & \text { if } w \leq_{\operatorname{lex}} v_{n} a \\ \varphi\left(v_{n} b\right) & \text { if } w \geq_{\operatorname{lex}} v_{n} b\end{cases}
$$

Moreover Proposition 2 from [39] states that the lexicographically maximal element of $\mathcal{L}(X) \cap A^{n+1}$ is $b u_{n}$. In particular, if $b u_{n}=v_{n} b$, then $v_{n} b$ is maximal in $\mathcal{L}(X) \cap A^{n+1}$, and for such values of $n$,

$$
\mu\left(L \cap A^{n+1}\right)=\mu\left(v_{n} b\right)=\mu\left(b u_{n}\right) \quad \text { or } \quad \mu\left(L \cap A^{n+1}\right)=1-\mu\left(v_{n} b\right)=1-\mu\left(b u_{n}\right)
$$

Letting $w_{k}$ be the left special factor of length $n=F(2 k+2)-1$, we deduce from Lemma 3.9 and the observation thereafter that:

$$
\mu\left(L \cap A^{F(4 k)}\right)=1-\mu\left(b w_{2 k+1}\right), \quad \mu\left(L \cap A^{F(4 k+2)}\right)=\mu\left(b w_{2 k}\right)
$$

To conclude the proof it remains only to observe that $\lim _{n \rightarrow \infty} \mu\left(b w_{k}\right)=0$, which is a straightforward consequence of [23], Proposition 13.

## 4. Densities in shifts of finite type

The aim of this section is to apply our first density formula within the setting of shifts of finite type. We provide a simple condition which guarantees topological transitivity of the skew product with a finite group; this in turn implies ergodicity for the product of the uniform probability measure on the group and a Markov measure on the shift. This includes the previously mentioned case of Bernoulli measures studied by Schützenberger, Berstel, and Hansel and Perrin [45, 6, 28].

The question of ergodicity for skew products over Bernoulli measures was studied by Kakutani 30], and the more general Markov case was studied by Bufetov 18]. The latter introduced a condition that he called "strongly connected" to characterize ergodicity of certain skew products involving Markov measures (Theorem 4 in [18]). Restating Bufetov's criterion in our setting, for consistency of terminology, we rename this condition strong irreducibility, abbreviated SI; his result was also recently extended in a preprint by Lummerzheim et al. (34], Theorem 4.3).

In Section4.1 we define $\varphi$-irreducibility of a subshift with respect to a morphism onto a finite group, which characterizes topological transitivity in skew products. Section4.1 treats skew products of shift spaces over Markov measures, using the fact that irreducibility implies ergodicity. In Section 4.3 we discuss strong irreducibility, which provides topological transitivity simultaneously for all skew products.
4.1. Shifts of finite type. Recall that a shift $X$ is an $r$-step shift of finite type (SFT, $r \geq 1$ ) if there is a list $F \subseteq A^{r+1}$ of forbidden factors of length $r+1$, with the property that an infinite word $x \in A^{\mathbb{Z}}$ belongs to $X$ precisely when none of its factors of length $r+1$ are in $F$. Recall also that a shift $X$ is topologically transitive if for every pair $(U, V)$ of nonempty open sets in $X$, there is $n>0$ for which $S^{n} U \cap V \neq \varnothing$. This is equivalent with the irreducibility of $X$, i.e. for every $u, v \in \mathcal{L}(X)$, there is $w \in \mathcal{L}(X)$ such that $u w v \in \mathcal{L}(X)$. For more on shifts of finite type, see e.g. [33, 40]. Topological transitivity will be used to prove ergodicity for $r$-step Markov measures fully supported on $r$-step shifts of finite type in Section 4.2,

Definition 4.1. Let $X$ be a shift on $A, G$ a finite group, and $\varphi: A^{*} \rightarrow G$ be a morphism onto $G$. We say that $X$ is $\varphi$-irreducible if, for all $u, v \in \mathcal{L}(X)$, there exists $w \in A^{*}$ such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=1_{G}$.
Remark 4.2. For a morphism $\varphi: A^{*} \rightarrow G$ as above, let

$$
\operatorname{ker}(\varphi)=\left\{(u, v) \in A^{*} \times A^{*} \mid \varphi(u)=\varphi(v)\right\} .
$$

Observe that if $\psi: A^{*} \rightarrow H$ is another morphism onto a finite group $H$ such that $\operatorname{ker}(\varphi) \subseteq \operatorname{ker}(\psi)$, then $\varphi$-irreducibility implies $\psi$-irreducibility. Moreover for the morphism $\varphi: A^{*} \rightarrow 1$ onto the trivial group, the above definition agrees with "ordinary" irreducibility, i.e. topological transitivity for $X$. In particular, $\varphi$-irreducibility always implies irreducibility.

We shall prove the following result, which shows that $\varphi$-irreducibility is precisely the notion needed for topological transitivity of the original shift to propagate to skew products with finite groups.

Theorem 4.3. Let $X$ be an r-step shift of finite type on $A$ and let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group. Then $X$ is $\varphi$-irreducible if and only if the skew product $G \rtimes X$ is topologically transitive.

Before doing so, we establish an intermediate result involving the following notion.

Definition 4.4. We say that $X$ is fiber ergodic with respect to $\varphi$ if, for every $g, h \in G$ there exists $w \in \mathcal{L}(X)$ such that $g \varphi(w)=h$; or equivalently the restriction of $\varphi$ to $\mathcal{L}(X)$ is onto.

It is not hard to see that topological transitivity of $G \rtimes X$ implies fiber ergodicity, though the converse is false, as shown by the following example.
Example 4.5. Take again the skew product from Example 3.2 based on the threeelement shift generated by the periodic infinite word $(a b c)^{\infty}$ taken with respect to the morphism $\varphi:\{a, b, c\}^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=\varphi(b)=1, \varphi(c)=0$. Then $X$ is fiber ergodic but $G \rtimes X$ is not topologically transitive.

On the other hand, fiber ergodicity follows from $\varphi$-irreducibility for shifts of finite type, as shown next.

Lemma 4.6. Let $X$ be a shift of finite type on $A$ and let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group. If $X$ is $\varphi$-irreducible, then it is fiber ergodic with respect to $\varphi$.

Proof. Let $r \geq 1$ such that $X$ is an $r$-step SFT. Take $g \in G$. Since $\varphi$ is onto, we may find letters $a_{1}, \ldots, a_{l} \in A$ such that $\varphi\left(a_{1} \cdots a_{l}\right)=g$. For $i=1, \ldots, l$ let $u_{i}$ be a word of length $r+1$ in $\mathcal{L}(X)$ starting with $a_{i}$. Let $t_{i}$ and $v_{i}$ be the suffix and prefix
of length $r$ of $u_{i}$. By assumption, there exists for each $i=1, \ldots l-1$ a word $w_{i}$ such that $t_{i} w_{i} v_{i+1} \in \mathcal{L}(X)$ and $\varphi\left(t_{i} w_{i}\right)=1_{G}$. Since $X$ is an $r$-step shift, it follows that the word $z=u_{1} w_{1} \cdots w_{l-1} u_{l} w_{l}$ belongs to $\mathcal{L}(X)$, and

$$
\varphi(z)=\varphi\left(a_{1}\right) \varphi\left(t_{1} w_{1}\right) \cdots \varphi\left(a_{l}\right) \varphi\left(t_{l} w_{l}\right)=\varphi\left(a_{1} \cdots a_{l}\right)=g
$$

Proof of Theorem 4.3. For $g \in G$, define a relation $\prec_{g}$ on $\mathcal{L}(X)$ by $u \prec_{g} v$ if there exists $w \in \mathcal{L}(X)$ such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=g$. Observe that $u \prec_{g} v$ precisely when $T^{m}\left(\left\{1_{G}\right\} \times[u]_{X}\right)$ intersects $\{g\} \times[v]_{X}$ for some $m \geq|u|$. Therefore $G \rtimes X$ is topologically transitive precisely when all relations $\prec_{g}, g \in G$, are total. In particular, whenever this is the case, $\prec_{1_{G}}$ must contain all pairs $u, v \in \mathcal{L}(X)$, which is precisely the definition of $\varphi$-irreducibility. Thus topological transitivity of $G \rtimes X$ implies $\varphi$-irreducibility of $X$. It remains to prove the converse.

Assume that $X$ is $\varphi$-irreducible, so that $\prec=\prec_{1_{G}}$ contains all pairs of words in $\mathcal{L}(X)$. Take $u, v \in \mathcal{L}(X)$ and $g \in G$; we need to show that $u \prec_{g} v$.

By fiber ergodicity (which holds thanks to Lemma 4.6), there is $u^{\prime} \in \mathcal{L}(X)$ such that $\varphi\left(u^{\prime}\right)=g$. Since $u \prec u^{\prime}$, there is $z \in \mathcal{L}(X)$ such that $u z u^{\prime} \in \mathcal{L}(X)$ and $\varphi(u z)=1_{G}$. Then $w_{0}=z u^{\prime}$ satisfies $u w_{0} \in \mathcal{L}(X)$ and $\varphi\left(u w_{0}\right)=g$.

Extend $u w_{0}$ to a word $w u_{0} v^{\prime} \in \mathcal{L}(X)$ with $\left|v^{\prime}\right| \geq r+1$. Since $v^{\prime} \prec v$, there is a word $v^{\prime \prime}$ such that $v^{\prime} v^{\prime \prime} v \in \mathcal{L}(X)$ and $\varphi\left(v^{\prime} v^{\prime \prime}\right)=1_{G}$. Then all subwords of length $r+1$ of $u w_{0} v^{\prime} v^{\prime \prime} v$ are in $\mathcal{L}(X)$, and hence $u w_{0} v^{\prime} v^{\prime \prime} v \in \mathcal{L}(X)$. Finally, letting $w=w_{0} v^{\prime} v^{\prime \prime}$, we have $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=g$. This shows that $u \prec_{g} v$, concluding the proof.

Finally we note in passing the following useful property of $\varphi$-irreducibility in shifts of finite type. It shows that for shifts of finite type, the task of verifying the condition in Definition 4.1 can be reduced to a finite set of words.
Proposition 4.7. Let $X$ be an r-step shift of finite type on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group. Then for $X$ to be $\varphi$-irreducible, it suffices that, for all $u, v \in \mathcal{L}(X)$ with $|u|=|v|=r$, there exists $w \in A^{*}$ such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=1_{G}$.
Proof. Let $u, v \in \mathcal{L}(X)$. If $|v|<r$ then we may simply replace $v$ by one of its extensions in $\mathcal{L}(X)$ of length $r$. If $|v|>r$, then we may likewise replace it by its prefix of length $r$. Thus we may assume moving forward that $|v|=r$.

First we assume that $|u|>r$. Let $u=p q=p^{\prime} q^{\prime}$ where $|q|=\left|p^{\prime}\right|=r$. By assumption, there are words $w, w^{\prime}$ such that $q w^{\prime} p^{\prime}, q w v \in \mathcal{L}(X)$ and $\varphi(q w)=$ $\varphi\left(q w^{\prime}\right)=1_{G}$. Observe that, for every $n \in \mathbb{N}, u\left(w^{\prime} u\right)^{n} w v$ has all of its factors of length $r$ in $\mathcal{L}(X)$. Thus the word $z_{n}=\left(w^{\prime} u\right)^{n} w$ is such that $u z_{n} v \in \mathcal{L}(X)$, while $\varphi\left(u z_{n}\right)=\varphi(p)^{n}$. Taking $n=|G|$, we get $\varphi\left(u z_{n}\right)=1_{G}$, as needed.

It remains to handle the case where $|u|<r$. Take a word $p$ such that $|p|=r-|u|$ and $p u \in \mathcal{L}(X)$. Then we may find words $w, w^{\prime}$ such that $p u w^{\prime} u w v \in \mathcal{L}(X)$ with $\varphi\left(p u w^{\prime}\right)=1_{G}$ and $\varphi\left(p u w^{\prime} u w\right)=1_{G}$, thus $\varphi(u w)=1_{G}$.

Example 4.8. Consider once again the three-element shift $X$ from Example 3.2 generated by the periodic word $(a b c)^{\infty}$, which is an irreducible 1-step shift of finite type. Let $\varphi:\{a, b, c\}^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=\varphi(b)=1$ and $\varphi(c)=0$.

Observe that every word $w$ such that $a w b \in \mathcal{L}(X)$ is of the form $(b c a)^{n}$ for some $n \geq 0$. In particular it follows that for every such word $w, \varphi(a w)=1$, thus $X$ is not $\varphi$-irreducible (though notice that it is fiber-ergodic). It is however $\varphi$-irreducible if $\varphi$ is similarly defined but takes values instead in $\mathbb{Z} / 3 \mathbb{Z}$.


Figure 4. The SFT and skew product from Example 4.9

Example 4.9. Let $X$ be the golden mean shift, i.e. the 1 -step SFT formed by sequences in $\{a, b\}^{\mathbb{Z}}$ avoiding the factor $b b$. Take the morphism $\varphi:\{a, b\}^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$, $\varphi(a)=1, \varphi(b)=0$. Then $X$ is $\varphi$-irreducible, as evidenced by the fact that $a a b, a a a, b a, b a a b \in \mathcal{L}(X)$. It is also not hard to verify directly that the skew product viewed as an SFT under the topological conjugacy $\Psi$ from Lemma 3.1 is indeed irreducible. The shift $X$ and the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ are depicted in Fig. (4)
4.2. Markov measures. Using the topological transitivity condition from Theorem 4.3, we establish ergodicity for skew products involving invariant Markov measures. Recall that a measure $\mu$ on $A^{\mathbb{Z}}$ is an $r$-step Markov measure, $r \geq 1$, when for every word word $w \in A^{*}$ of length $m \geq r$ such that $\mu\left(x_{[0, m)}=w\right) \neq 0$ and every letter $a \in A$,

$$
\mu\left(x_{m}=a \mid x_{[0, m)}=w\right)=\mu\left(x_{m}=a \mid x_{[m-r, m)}=w_{[m-r, m)}\right)
$$

Note that invariant Markov measures are also called stationary. The support of an $r$-step invariant Markov measure $\mu$ is an $r$-step shift of finite type. We say that an $r$-step Markov measure $\mu$ is irreducible when for all $u, v \in A^{r}$, there exists $m>0$ such that

$$
\mu\left(x_{[m, m+r)}=v \mid x_{[0, r)}=u\right)>0
$$

Notice that this is equivalent to irreducibility of the shift of finite type which supports $\mu$. It is well known that a Markov measure $\mu$ is ergodic if and only if it is irreducible. The 1-step case may be found in [40], pp. 51-53, while the general $r$-step case can be reduced to $r=1$ by passing to the higher block shift (whose definition is recalled after Corollary 4.12). We shall prove the following:
Theorem 4.10. Let $X$ be an r-step shift of finite type, $r \geq 1$, and let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group with uniform probability measure $\nu$. If $X$ is $\varphi$ irreducible, then for every r-step Markov measure $\mu$ fully supported on $X, \nu \times \mu$ is an ergodic measure on the skew product $G \rtimes X$.

Recall from Lemma 3.1 that for every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$ and every shift $X$, the skew product $X \rtimes G$ is topologically conjugate to a subshift of $(G \times A)^{\mathbb{Z}}$ via the map

$$
\Psi(g, x)_{n}=\left(g \varphi^{(n)}(x), x_{n}\right), \quad n \in \mathbb{Z}
$$

Lemma 4.11. Let $X$ be an $r$-step shift of finite type, $r \geq 1$, and $\varphi: A^{*} \rightarrow G$ be $a$ morphism onto a finite group $G$ with uniform probability measure $\nu$.
(i) $\Psi(G \rtimes X)$ is an $r$-step shift of finite type.
(ii) For every r-step Markov measure $\mu$ fully supported on $X$, the image measure $\Psi_{*}(\nu \times \mu)=(\nu \times \mu) \circ \Psi^{-1}$ is a Markov measure fully supported on $\Psi(G \rtimes X)$.
Proof. (i). Take a word $\omega \in(G \times A)^{*}$ with $\omega_{i}=\left(g_{i}, w_{i}\right)$ and $m=|\omega| \geq r+1$. Observe that $\omega$ belongs to the language of $\Psi(G \rtimes X)$ as long as $w=w_{0} \cdots w_{m-1}$
belongs to $\mathcal{L}(X)$ and $g_{i} \varphi\left(w_{i}\right)=g_{i+1}$. Both of those conditions only need to be verified on factors of length $r+1$ of $\omega$, hence $\Psi(G \rtimes X)$ is indeed an $r$-step shift of finite type.
(ii) Fix an $r$-step Markov measure on $X$ and let $\pi=\Psi_{*}(\nu \times \mu)$. Fix a word $\omega \in(G \times A)^{*}$, with $\omega_{i}=\left(g_{i}, w_{i}\right), m=|\omega| \geq r$, and $w=w_{0} \cdots w_{m-1}$. Assume that $\omega$ belongs to the language of $\Psi(G \rtimes X)$, which means that $w \in \mathcal{L}(X)$ and $g_{i+1}=g_{i} \varphi\left(w_{i}\right), i=0, \ldots, m-2$. Take a letter $\alpha=(g, a) \in G \times A$. In case $g \neq g_{m-1} \varphi(a)$ then it is clear that

$$
\pi\left(\xi_{m}=\alpha \mid \xi_{[0, m)}=\omega\right)=\pi\left(\xi_{m}=\alpha \mid \xi_{[m-r, m)}=\omega_{[m-r, m)}\right)=0
$$

Thus we may suppose from now on that $g=g_{m-1} \varphi(a)$. Then we have

$$
\begin{aligned}
\pi\left(\xi_{m}=\alpha \mid \xi_{[0, m)}=\omega\right) & =\frac{\pi\left(\xi_{m}=\alpha, \xi_{[0, m)}=\omega\right)}{\pi\left(\xi_{[0, m)}=\omega\right)} \\
& =\frac{(\nu \times \mu)\left(\left\{g_{0}\right\} \times[w a]\right)}{(\nu \times \mu)\left(\left\{g_{0}\right\} \times[w]\right)} \\
& =\frac{\mu\left(x_{m}=a, x_{[0, m)}=w_{[0, m)}\right)}{\mu\left(x_{[0, m)}=w\right)} \\
& =\mu\left(x_{m}=a \mid x_{[0, m)}=w\right)
\end{aligned}
$$

Using a similar argument together with the invariance of $\pi$, we find

$$
\pi\left(\xi_{m}=\alpha \mid \xi_{[m-r, m)}=\omega_{[m-r, m)}\right)=\mu\left(x_{m}=a \mid x_{[m-r, m)}=w_{[m-r, m)}\right)
$$

The fact that $\pi$ is $r$-step Markov now follows directly from the fact that $\mu$ is.
Proof of Theorem 4.10. In light of Lemma4.11, the image measure $\Psi_{*}(\nu \times \mu)$ is an $r$ step Markov measure fully supported on a shift of finite type, which is topologically transitive by Theorem4.3. Therefore, $\Psi_{*}(\nu \times \mu)$ must be ergodic, and since $\Psi$ is a topological conjugacy we deduce that the measure $\nu \times \mu$ is ergodic as well.

We now derive the following corollary of Theorem 3.3, it corresponds to our second main result.

Corollary 4.12 (Theorem B). Let $X$ be an $r$-step shift of finite type, $r \geq 1$, and let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group with uniform probability measure $\nu$. Assume that $X$ is $\varphi$-irreducible. Then for every r-step Markov measure $\mu$ fully supported on $X$ and every language $L=\varphi^{-1}(K), K \subseteq G$, the density $\delta_{\mu}(L)$ exists and is given by $\delta_{\mu}(L)=|K| /|G|$.

Let us briefly describe how ergodicity for $r$-step Markov measures can also be reduced to the 1 -step case by passing to the higher block shift. Let $\mu$ be an invariant $r$-step Markov measure, $r \geq 1$, with support a shift of finite type $X$. Let $A_{X}^{[r]}=$ $\mathcal{L}(X) \cap A^{r}$, which we view as an alphabet. We define the map $\beta_{r}: A^{\mathbb{Z}} \rightarrow\left(A_{X}^{[r]}\right)^{\mathbb{Z}}$ by

$$
\beta_{r}(x)_{i}=x_{[i, i+r)} .
$$

The image $\beta_{r}(X)$ forms a shift space on $A_{X}^{[r]}$ denoted $X^{[r]}$, which is called the higher block shift. The image measure of $\mu$ under $\beta_{r}$, denoted $\mu^{[r]}$, is an invariant 1-step Markov measure on this shift space. Moreover, $\mu$ is ergodic exactly when $\mu^{[r]}$ is.

The density of a group language also carries over to the higher block shift, as follows. Given a morphism $\varphi: A^{*} \rightarrow G$ onto a finite group, let $\varphi^{[r]}: A_{X}^{[r]} \rightarrow G$ be the morphism defined by $\varphi(w)=\varphi\left(w_{0}\right)$ for $w=w_{0} \cdots w_{r-1} \in A_{X}^{[r]}$. To the group
language $L=\varphi^{-1}(K), K \subseteq G$, corresponds the group language $L^{[r]}=\left(\varphi^{[r]}\right)^{-1}(K)$. It is straightforward to check that $\mu^{[r]}\left(L^{[r]} \cap\left(A_{X}^{[r]}\right)^{i}\right)=\mu\left(L \cap A^{i}\right)$, and as a result:

$$
\delta_{\mu}(L)=\delta_{\mu[r]}\left(L^{[r]}\right)
$$

4.3. Strong irreducibility. We now relate $\varphi$-irreducibility to a condition introduced by Bufetov [18], originally under the name strongly connected. The term strictly irreducible was used in Lummerzheim et al. [34]. We make a compromise between the two and use the term strongly irreducible. Bufetov's condition is an all-or-nothing type of condition. We shall see below that if it holds, then all skew products by morphisms onto finite groups are topologically transitive, while its failure gives in general little information about the topological transitivity of such skew products.
Definition 4.13. Let $X$ be a shift on an alphabet $A$. For $r \geq 1$, we define the relation $\sim_{r}$ on $\mathcal{L}(X)$ by $u \sim_{r} v$ if there exists $w \in A^{r}$ such that $w u, w v \in \mathcal{L}(X)$. Since $\sim_{r}$ is symmetric, its transitive closure, which we denote by $\simeq_{r}$, is an equivalence relation.

An $r$-step shift of finite type is called strongly irreducible if it is irreducible and the relation $\simeq_{r}$ is total, meaning that it has a single equivalence class.

Remark 4.14. It may happen that $\simeq_{r}$ has a single equivalence class even in the absence of irreducibility. This is the case in the 1-step shift of finite type $X$ on the alphabet $A=\{a, b\}$ consisting of sequences avoiding the factor $b a$.

Recall the notation $\prec=\prec_{1_{G}}$, used in the proof of Theorem 4.3 for the relation defined by $u \prec v$ if and only if there exists $w \in \mathcal{L}(X)$ such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=1_{G}$.

Proposition 4.15. In an irreducible $r$-step $S F T$, $r \geq 1$, the equivalence relation $\simeq_{r}$ is contained in the relation $\prec$.

Proof. Let $X$ be an irreducible $r$-step SFT. We start by showing that given $u, v \in$ $\mathcal{L}(X)$ with $u \sim_{r} v$, there is a word $w$ such that $|w| \geq r, w u, w v \in \mathcal{L}(X)$, and $\varphi(w)=1_{G}$. Find a word $z,|z|=r$, such that $z u, z v \in \mathcal{L}(X)$, and then find a word $w_{0}$ such that $u w_{0} z \in \mathcal{L}(X)$. Then $\left(u w_{0} z\right)^{n} v \in \mathcal{L}(X)$ for all $n \geq 1$, and taking $n=|G|$ we find $\varphi\left(\left(u w_{0} z\right)^{n}\right)=1_{G}$. Thus we may take $w=\left(w_{0} z u\right)^{n-1} w_{0} z$.

Suppose next $u=u_{0} \sim_{r} u_{1} \sim_{r} \cdots \sim_{r} u_{n-1} \sim_{r} u_{n}=v$. We want to find a word $w$ such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=1_{G}$. Using the above claim, there are words $w_{i}$ such that $u_{i} w_{i} u_{i+1} \in \mathcal{L}(X), \varphi\left(u_{i} w_{i}\right)=1_{G}$, and $\left|w_{i}\right| \geq r$. Then defining $w=w_{0} u_{1} w_{1} \cdots u_{n-1} w_{n-1}$ produces the requisite word such that $u w v \in \mathcal{L}(X)$ and $\varphi(u w)=1_{G}$.

We then deduce the following. It is our version of Bufetov's theorem ( 18$]$, Theorem 5) and its generalization by Lummerzheim et al. (34] Theorem 4.3), which we here specialize to the case of skew products with finite groups and morphisms, but generalize to the case of higher step shifts.

Theorem 4.16. Let $X$ be an irreducible $r$-step $S F T$ on $A, r \geq 1$. If $X$ is strongly irreducible, then it is $\varphi$-irreducible for every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$. When $r=1$, the converse holds.
Proof. For the first part of the statement, notice that when $X$ is strongly irreducible then by Proposition 4.15 the relation $\prec=\prec_{1_{G}}$ must be total for every morphism
$\varphi: A^{*} \rightarrow G$ onto a finite group $G$, which is precisely the definition of $X$ being $\varphi$-irreducible.

We now suppose that $r=1$. We will prove the contrapositive: assuming that $X$ is not strongly irreducible, then it is not $\varphi$-irreducible for some $\varphi$. The construction presented is essentially the same as the one used in the proof Theorem 4.3 in 34 ].

Fix an equivalence class $C$ of $\simeq_{1}$ restricted to $A \times A$. For each $a \in A$, observe that the set of right extensions of $a$ in $X$,

$$
\mathrm{R}(a)=\{b \in A \mid a b \in \mathcal{L}(X)\}
$$

is contained in a class of $\simeq_{1}$; thus either $\mathrm{R}(a) \subseteq C$ or $\mathrm{R}(a) \subseteq A \backslash C$. Let

$$
B=\{a \in C \mid \mathrm{R}(a) \subseteq A \backslash C\} \cup\{a \in A \backslash C \mid \mathrm{R}(a) \subseteq C\}
$$

Define a morphism $\varphi: A^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ by $\varphi(a)=1$ if $a \in B$ and $\varphi(a)=0$ otherwise. Let $a \in C$ and $b \in A$; take a word $w$ such that $a w b \in \mathcal{L}(X)$. We claim that $\varphi(a w)=0$ if and only if $b \in C$. Observe that this claim finishes the proof, as it implies that $a \nprec b$ whenever $b \in A \backslash C$.

To establish the claim, we argue by induction on $|w|$. We first consider the case $|w|=0$. We have to check that $\varphi(a)=0$ if and only if $b \in C$. But $\varphi(a)=0$ if and only if $a \notin B$, which is in turn equivalent to $\mathrm{R}(a) \subseteq C$, i.e. $b \in C$. For the induction step, suppose that $w=w^{\prime} c, c \in A$, with $a w^{\prime} b \in \mathcal{L}(X)$, and that the claim holds for $w^{\prime}$. First assume that $\varphi(a w)=0$. If $c \in C$, then by induction $\varphi\left(a w^{\prime}\right)=0$ and $\varphi(a w)=\varphi(c)=0$, and as $b \in \mathrm{R}(c)$, we get $b \in C$. If on the other hand $c \in A \backslash C$, then $\varphi\left(a w^{\prime}\right)=1$, since the claim hods for $w^{\prime}$ and $c \in A \backslash C$, hence $\varphi(a w)=1+\varphi(c)=0$, and so $\varphi(c)=1$, and $c \notin B$. As $b \in \mathrm{R}(c)$, it again follows that $b \in C$. Conversely, assume that $b \in C$. If $c \in C$, then as $b \in \mathrm{R}(c)$, we get $c \notin B$ and $\varphi(c)=0$, while by induction $\varphi\left(a w^{\prime}\right)=0$; hence $\varphi(w)=0+0=0$. Likewise, if $c \in A \backslash C$, then $c \in B$ and $\varphi\left(a w^{\prime}\right)=1=\varphi(c)$ and $\varphi(a w)=1+1=0$.

At this time we are unsure whether the converse holds when $r>1$. Nonetheless, the first part of the above combined with Theorem 4.3 yields the following immediate corollary.
Corollary 4.17. Let $X$ be a strongly irreducible (hence irreducible) r-step SFT on $A, r \geq 1$. For every morphism $\varphi: A^{*} \rightarrow G$, assumed to be onto the finite group $G$, the skew product $G \rtimes X$ is topologically transitive.

To end this section we give a few examples of shifts of finite type which are or are not strongly irreducible. But first we make the following simple observation, similar to Proposition 4.7 .
Proposition 4.18. Let $X$ be an irreducible $r$-step SFT on $A, r \geq 1$. Then for $X$ to be strongly irreducible, it suffices that $u \simeq_{r} v$ for all pairs $u, v \in \mathcal{L}(X)$ with $|u|=|v|=r$.
Proof. Let $u^{\prime}$ and $v^{\prime}$ be words in $\mathcal{L}(X)$ of arbitrary lengths. Choose some words $u, v$ such that $|u|=|v|=r, u$ is a prefix of $u^{\prime}$ or vice-versa, and $v$ is a prefix of $v^{\prime}$ or vice-versa. By assumption, we may find words $w_{0}, \ldots, w_{n-1}$ and $t_{1}, \ldots, t_{n-1}$, all of length $r$, such that with $t_{0}=u$ and $t_{n}=v$,

$$
w_{i} t_{i}, w_{i} t_{i+1} \in \mathcal{L}(X), \quad 0 \leq i \leq n-1
$$

Then notice that $w_{0} u^{\prime}$ is also in $\mathcal{L}(X)$; indeed this is obvious when $u^{\prime}$ is a prefix of $u$, and otherwise it follows from the fact that $X$ is an $r$-step SFT. Likewise $w_{n-1} v^{\prime} \in \mathcal{L}(X)$, which shows that $u^{\prime} \simeq_{r} v^{\prime}$.


Figure 5. The SFT and skew product from Example 4.20

Example 4.19. Continuing from Example 4.9, it is not hard to see that the golden mean shift is in fact strongly irreducible. As a result, all skew products with skewing functions given by morphisms onto finite groups are topologically transitive, including the one in Example 4.9 .

Example 4.20. Let $X$ be the 1 -step SFT on $A=\{a, b, c\}$ formed by the sequences avoiding the words $c a, a b, b b, c c$. Then $X$ is irreducible but not strongly irreducible. Indeed, the relation $\simeq_{1}$ has two classes in $A \times A$, namely $\{b\}$ and $\{a, c\}$. Taking for instance $C=\{b\}$, the set $B$ in the second part of the proof of Theorem 4.16 equals $\{b, c\}$. Consider now the morphism $\varphi: A^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=0, \varphi(b)=\varphi(c)=$ 1. One has that $X$ is irreducible, whereas $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ is not. If $w \in A^{*}$ and $b w a \in$ $\mathcal{L}(X)$, we must have $\varphi(b w)=1$ and therefore $X$ is not $\varphi$-irreducible. The shift $X$ and the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ are depicted in Fig. 5.

## 5. Characterization of minimality and return words

The aim of this section is to provide a first characterization of minimality for skew products (Theorem5.1), stated in terms of return words (Section 2). Its proof, given in Section 5.2, strongly relies on the deep links between the skew products under consideration and bifix codes, recalled in Section 5.1

Theorem 5.1. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. The following conditions are equivalent.
(i) For every $n>0$ and $x \in X,\left\{\varphi\left(x_{[0, m)}\right) \mid m \in \mathbb{N}, x_{[m, m+n)}=x_{[0, n)}\right\}=G$.
(ii) The skew product $G \rtimes X$ is minimal.
(iii) For every $u \in \mathcal{L}(X)$, the restriction of $\varphi$ to $\mathcal{R}_{X}(u)^{*}$ is surjective.

We will later provide a further characterization of minimality in Theorem 6.5 stated in terms of cobounding maps, in the flavour of Anzai's theorem on ergodicity of skew products [2]. We will also provide an alternate proof of the equivalence between (ii) and (iii) (Remark 6.10).

Remark 5.2. The condition (i) is reminiscent of the WELLDOC property (which stands for well distributed occurrences) studied by Balková et al. [5] in the context of pseudorandom number generators. For convenience, we recall that an infinite word $X$ is said to have the WELLDOC property when it satisfies the following condition where Ab denotes the Parikh abelianization map $A^{*} \rightarrow \mathbb{Z}^{|A|}$ :

$$
\forall k \geq 2, \forall n \geq 0,\left\{\operatorname{Ab}\left(x_{[0, m)}\right) \bmod k \mid m \geq 0, x_{[m, m+n)}=x_{[0, n)}\right\}=(\mathbb{Z} / k \mathbb{Z})^{|A|}
$$

It follows from the above theorem that the WELLDOC property is equivalent to all skew products of the form $(\mathbb{Z} / k \mathbb{Z})^{d} \rtimes X$ being ergodic.
5.1. More on the theory of codes. We now recall some basic notions from the theory of codes. For more details we refer the reader to the monograph (7].

Recall that a prefix code is a subset $U \subseteq A^{*}$ where no word is a strict prefix of another; likewise a suffix code is a subset of $A^{*}$ where no words is a strict suffix of another. Let $X$ be a shift space. A prefix code $U \subseteq \mathcal{L}(X)$ is called $X$-complete if every word $w \in \mathcal{L}(X)$ is either a prefix of a word in $U$ or has a prefix which is an element of $U$; replacing prefix by suffix in this definition, we obtain the corresponding notion of $X$-complete suffix code.
Example 5.3. Let $X$ be the Fibonacci shift considered in Section 3.3. Then the set $U=\{a, b a\}$ is an $X$-complete prefix code, though clearly not a suffix code.

A set $U$ is a bifix code if it is both a prefix and a suffix code. It is $X$-complete if it is both an $X$-complete prefix code and an $X$-complete suffix code. When $X=A^{\mathbb{Z}}$, we simply say complete instead of $A^{\mathbb{Z}}$-complete, be it for prefix, suffix, or bifix codes.

Let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. If $H \leq G$ is a subgroup, then the submonoid $M=\varphi^{-1}(H)$ of $A^{*}$ is generated by a complete bifix code $Z$ called a group code. More precisely, $Z$ is the set of nonempty words in $M$ with no non-trivial prefix in $M$.

Let $Z$ be a group code. Let $P$ (resp. $S$ ) be the set of proper prefixes (resp. suffixes) of the words in $Z$. Note that $P($ resp. $S)$ is also the set of words which have no prefix (resp. suffix) in $Z$. The $Z$-degree $d(u)$ of a word $u \in A^{*}$ is defined as any of the following numbers, which all coincide:
(i) the number of suffixes of $u$ which are in $P$;
(ii) the number of prefixes of $u$ which are in $S$;
(iii) the number of $Z$-parses of $w$, that is, the number of triples $(s, z, p)$ such that $w=s z p$ with $s \in S, z \in Z^{*}$ and $p \in P$.
It follows from the third definition that for every $u, v, w \in A^{*}$, we have

$$
\begin{equation*}
d(v) \leq d(u v w) \tag{5.1}
\end{equation*}
$$

Indeed, if $(s, z, p)$ is a $Z$-parse of $v$, let $u s=s^{\prime} z^{\prime}$ with $s^{\prime} \in S$ and $z^{\prime} \in Z^{*}$, and $p w=z^{\prime \prime} p^{\prime}$ with $z^{\prime \prime} \in Z^{*}$ and $p^{\prime} \in P$. Then $\left(s^{\prime} z^{\prime} z z^{\prime \prime}, p^{\prime}\right)$ is a $Z$-parse of $u v w$ which extends $(s, z, p)$. This shows that every parse of $v$ extends to a parse of $u v w$.

Proposition 5.4. Let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$, and $H$ be a subgroup of index $d$ in $G$. Let $Z$ be the group code such that $\varphi^{-1}(H)=Z^{*}$, and let $S$ be the set of proper suffixes of elements of $Z$. For every $p, q \in S$ such that $q$ is a proper prefix of $p, \varphi(p) H \neq \varphi(q) H$. In particular, every word has $Z$-degree at most d.

Proof. Let $p=q x$ and assume by contradiction that $\varphi(p) H=\varphi(q) H$. Then it follows that $x$ is in $\varphi^{-1}(H)$, hence it is a nonempty word of $Z^{*}$, and thus it has a suffix in $Z$. Set $x=s r$ with $r \in Z$. Let $z \in Z$ be such that $z=t p$. Then,

$$
z=t p=t q x=t q s r
$$

so $r \in Z$ is a proper suffix of $z \in Z$, which contradicts the fact that $Z$ is bifix.
Let $X$ be a minimal shift space and let $U=Z \cap \mathcal{L}(X)$. The $X$-degree of $U$, denoted $d_{X}(U)$, is the maximal value of the $Z$-degrees of all words in $\mathcal{L}(X)$. The
following is essentially a reformulation of [8], Theorem 4.2.11; we include a proof for the convenience of the reader.
Theorem 5.5. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. Let $H$ be a subgroup of index $d$ in $G$ and $Z$ be the group code such that $\varphi^{-1}(H)=Z^{*}$. The set $U=Z \cap \mathcal{L}(X)$ is a finite $X$-complete bifix code of $X$-degree $d_{X}(U) \leq d$.

The proof makes use of the following simple observation.
Lemma 5.6. Let $v$ be a word with maximal $Z$-degree. Then no word of $U$ can be of the form uvw with $u$ or $w$ nonempty.
Proof. Assume the contrary. Then we would have $d(u v w)>d(v)$ because every $Z$ parse of $v$ extends to a $Z$-parse of $u v w$, while the latter has the additional $Z$-parse $(\varepsilon, u v w, \varepsilon)$.

Proof of Theorem 5.5. Since $Z$ is a bifix code, the same is true for $U$. Consider a word $v \in \mathcal{L}(X)$ of maximal $Z$-degree. By Lemma 5.6, there is no word in $U$ containing $v$ as a strict factor, thus $U$ must be finite. Indeed, since $X$ is minimal, every long enough element of $\mathcal{L}(X)$ is of the form $u v w$ with $u, w$ nonempty and thus cannot be a factor of a word in $U$; hence the length of the words in $U$ is bounded.

Next, we show that $U$ is an $X$-complete prefix code. Consider a nonempty word $u \in \mathcal{L}(X)$. Since $X$ is minimal, there is a word $w$ such that $u w v \in \mathcal{L}(X)$. Set $u=a u^{\prime}$ where $a$ is a letter. Since $d\left(a u^{\prime} w v\right) \geq d\left(u^{\prime} w v\right) \geq d(v)$ and since $d(v)$ is maximal; we have $d\left(a u^{\prime} w v\right)=d\left(u^{\prime} w v\right)$. This forces the word $a u^{\prime} w v$ to have a prefix in $Z$, hence in $U$. Thus either $u$ has a prefix which is an element of $U$ or it is a prefix of an element of $U$. Hence $U$ is an $X$-complete prefix code. The proof that $U$ is an $X$-complete suffix code is similar.

Example 5.7. Let $X$ be the Fibonacci shift on $A=\{a, b\}$ (as in Section 3.3) and let $\varphi: A^{*} \rightarrow S_{3}$ be the morphism onto the symmetric group $S_{3}$ defined by $\varphi(a)=$ $(12), \varphi(b)=(13)$, where permutations are written in usual cycle notation. Let $H$ be the subgroup of $G$ formed of the permutations fixing 1 . Let $Z$ be the group code such that $\varphi^{-1}(H)=Z^{*}$; it is given by $Z=a b^{*} a \cup b a^{*} b$. The elements of the infinite set $Z$ are represented in the tree found in Fig. 6 as the labels of paths from the root to the leaves. The $X$-complete bifix code $U=Z \cap \mathcal{L}(X)$, equal to $\{a a, a b a, b a a b, b a b\}$, is depicted in Fig. 6.


Figure 6. Representation of the $X$-complete bifix code $U$ of Example 5.7. Nodes are labeled by the image of 1 under the permutation given by the label of the path. Elements of $U$ correspond to paths ending in double-circled nodes.

The next result uses ideas found in the proof of the main result of [14].
Proposition 5.8. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be $a$ morphism onto a finite group $G$. Let $Z$ be the group code such that $\varphi^{-1}\left(1_{G}\right)=Z^{*}$, and $U=Z \cap \mathcal{L}(X)$. Assume that for every $u \in \mathcal{L}(X)$, the restriction of $\varphi$ to $\mathcal{R}_{X}(u)^{*}$ is surjective. Then the $X$-degree of $U$ is $|G|$.

Proof. Let $F_{A}$ be the free group $A$ and let $K$ be the subgroup of $F_{A}$ generated by $U$. Let $u \in \mathcal{L}(X)$ be of maximal $X$-degree. Let $Q$ be the set of prefixes of $u$ which are suffixes of some element of $U$ (so $|Q|$ is the $X$-degree of $U$ ). Recall that, for distinct elements $p$ and $q \in Q$, the cosets $p K$ and $q K$ are distinct by Proposition 5.4.

Let us define

$$
V=\left\{v \in F_{A} \mid v Q \subset Q K\right\}
$$

For every $v \in V$, the map $\pi(v): p \mapsto q$ defined by $v p \in q K$ is a permutation. Indeed, let $v p, v p^{\prime} \in q K$ for some $q \in Q$. Then $v^{-1} q$ is in $p K \cap p^{\prime} K$, and thus $p=p^{\prime}$.

We claim that set $V$ is a subgroup of $F_{A}$. First, let $v \in V$. Then for any $q \in Q$, since $\pi(v)$ is a permutation of $Q$, there is a $p \in Q$ such that $v p \in q K$. Then $v^{-1} q \in$ $p K$. This shows that $v^{-1} \in V$. Next, if $v$ and $w \in V$, then $v w Q \subset v Q K \subset Q K$ and thus $v w \in V$. Since it is clear that $V$ contains the identity element, this proves the claim.

Next, $V$ contains $\mathcal{R}_{X}(u)$. Indeed, let $q \in Q$ and $y \in \mathcal{R}_{X}(u)$. Since $q$ is a prefix of $u, y q$ is a prefix of $y u$, and since $y u$ is in $\mathcal{L}(X)$ (by definition of $\mathcal{R}_{X}(u)$ ), $y q$ is also in $\mathcal{L}(X)$. Since, by Theorem [5.5] $U$ is an $X$-complete bifix code, it is an $X$-complete suffix code. This implies that $y q$ is a suffix of a word in $U^{*}$, and thus there is a suffix $r$ of $U$ such that $y q \in r U^{*}$. We verify that the word $r$ is a suffix of $u$. Since $y \in \mathcal{R}_{X}(u)$, there is a word $y^{\prime}$ such that $y u=u y^{\prime}$. Consequently, $r$ is a prefix of $u y^{\prime}$, and in fact the word $r$ is a prefix of $u$. Indeed, one has $|r| \leq|u|$, since otherwise $u$ would be in the set of internal factors of $U$, and this cannot be the case by Lemma 5.6 (recall that $u$ has maximal $Z$-degree). Thus we have $r \in Q$. Since $U^{*} \subset K$ and $r \in Q$, we have $y q \in Q K$, hence $y \in V$.

Since $V$ contains $\mathcal{R}_{X}(u)$, the restriction of $\varphi$ to $V$ is surjective. Since $\pi(v)$ is the identity on $Q$ if and only if $v$ is in $K$, the index of $K \cap V$ in $V$ is $|Q|$. On the other hand, since the restriction of $\varphi$ to $V$ is surjective, the index of $K \cap V$ in $V$ is equal to $|G|$. Thus $|Q|=|G|$.

The next example shows that the converse is false.
Example 5.9. Take $A=\{a, b\}, \varphi: A \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ defined by $\varphi(a)=1, \varphi(b)=0$, and $X$ equal to the Thue-Morse shift, considered in Example 2.1. The bifix code $U=\varphi^{-1}(0) \cap \mathcal{L}(X)$ is equal to $\{b, a a, a b a, a b b a\}$. It has $X$-degree 2 , but $\mathcal{R}_{X}(u)$ is contained in $\varphi^{-1}(0)$ for every $u$ sufficiently long. This last claim can be checked by finding one such word $u$ and then using the return preservation property (9].

On the other hand, the assumption that the restriction of $\varphi\left(\mathcal{R}_{X}(u)^{*}\right)=G$ for all $u \in \mathcal{L}(X)$ cannot be dropped from Proposition 5.8. Indeed, consider again the Thue-Morse shift $X$ and let $\varphi:\{a, b\}^{*} \rightarrow \mathbb{Z} / n \mathbb{Z}$ be a morphism with $\varphi(a)=1=$ $-\varphi(b)$. Set $Z^{*}=\varphi^{-1}(0)$ and $U=Z \cap \mathcal{L}(X)$. For every $n \geq 3$, we have $U=$ $\{a b, b a, a a b b, b b a a, a a b a b b, b b a b a a\}$, while $d_{X}(U)=3$ and thus $d_{X}(U)<d(Z)=n$ as soon as $n>3$.
5.2. Proof of the characterization of minimality. We are now ready to give the proof of the main theorem of this section.

Proof of Theorem 5.1. (i) implies(ii) Let $Y \subset G \rtimes X$ be a minimal closed invariant subset. Fix $g \in G$ and $x \in X$; let us show that $(g, x) \in Y$.

First, note that the projection $P_{X}(Y)=X$, since $X$ is minimal. Thus we may find $h$ such that $(h, x) \in Y$. By (i) we can find a sequence of positive integers $\left(m_{n}\right)_{n \in \mathbb{N}}$ such that $\varphi\left(x_{\left[0, m_{n}\right)}\right)=h^{-1} g$ and $x_{[0, n)}=x_{\left[m_{n}, m_{n}+n\right)}$. It follows that

$$
T^{m_{n}}(h, x) \rightarrow(g, x), \quad \text { as } n \rightarrow \infty
$$

showing that $(g, x) \in Y$.
(ii) implies (iii) Fix a word $u \in \mathcal{L}(X)$, an element $g \in G$, and consider the two clopen subsets $\left\{1_{G}\right\} \times[u]_{X}$ and $\{g\} \times[u]_{X}$. By minimality of $G \rtimes X$, there exists $n \in \mathbb{Z}$ such that:

$$
\left(\left\{1_{G}\right\} \times[u]_{X}\right) \cap T^{-n}\left(\{g\} \times[u]_{X}\right) \neq \varnothing
$$

Choose a point $\left(1_{G}, x\right)$ in that intersection, let $w=x_{[0, n)}$ if $n \geq 0$, and $w=$ $x_{[n, 0)}$ otherwise. Then we have $T^{n}\left(1_{G}, x\right)=\left(\varphi(w)^{ \pm 1}, S^{n} x\right) \in\{g\} \times[u]_{X}$, and thus $w \in \mathcal{R}_{X}(u)^{*}$ and $g^{ \pm 1} \in \varphi\left(\mathcal{R}_{X}(u)^{*}\right)$.
(iii) implies (i). Fix $x \in X, n \geq 0$ and $g \in G$. We want to prove that we can find $m \geq 0$ with $x_{[0, m)} \in \varphi^{-1}(g)$ and $x_{[0, n)}=x_{[m, m+n)}$.

Let $u=x_{[0, n)}$ and $Y=\mathcal{D}_{u}(X)$ be the derivative shift of $X$ with respect to $u$ (whose definition may be found in [24], p.291). Thus $Y$ is a shift space on an alphabet $B=B_{u}$ with a bijection $\theta_{u}: B \rightarrow \mathcal{R}_{X}(u)$ such that $\theta_{u}(Y)=X$. Let $y \in Y$ be such that $\theta_{u}(y)=x$. Consider moreover the morphism $\psi=\varphi \circ \theta_{u}$, guaranteed to be onto by condition (iii). Let $Z$ be the group code on $B$ such that $Z^{*}=\psi^{-1}\left(1_{G}\right)$, and $U=Z \cap \mathcal{L}(Y)$. By Theorem 5.5, the set $U$ is a $Y$-complete bifix code and thus it is, in particular, nonempty.

Next, observe that the morphism $\psi$ also satisfies the condition of Proposition 5.8 the restriction of $\psi$ to every $\mathcal{R}_{Y}(v)^{*}$ is onto. Indeed, for every $v \in \mathcal{L}(Y)$, the morphism $\theta_{v}$ coding the return words to $v$ satisfies $\theta_{u} \circ \theta_{v}=\theta_{w}$ with $w=\phi_{u}(v) u$. Thus, we may apply Proposition 5.8 to conclude that the $Y$-degree of $U$ is equal to $|G|$. Since $U$ is a $Y$-complete prefix code, $y$ has arbitrary long prefixes in $U^{*}$. If such a prefix $v$ is long enough, it has $Z$-degree equal to the $Y$-degree of $U$, that is $|G|$.

We claim that $v$ has a prefix $p$ such that $\psi(p)=g$. Indeed, since the $Z$-degree of $v$ is $|G|$, it has $|G|$ prefixes which belong to the set of proper suffixes of the elements of $Z$. By Lemma [5.6, all these prefixes have distinct images by $\psi$, and thus one such prefix $p$ is mapped to $g$ by $\psi$. Letting $m=\left|\theta_{u}(p)\right|$, we find that $x_{[0, n)}=u=x_{[m, m+n)}$ and moreover $\varphi\left(x_{[0, m)}\right)=\psi(p)=g$.

Remark 5.10. We remark that there is also a direct proof that (ii) implies (i), which sheds some light on these equivalences. Assume minimality of the skew product $G \rtimes X$ and fix $x \in X, n>0$. Let $w_{n}=x_{[0, n)}$ and $g \in G$ be given. We need to find a factor $v_{n}$ such that $x \in\left[w_{n} v_{n} w_{n}\right]_{X}$ and $\varphi\left(w_{n} v_{n}\right)=g$. By minimality of $G \rtimes X$, there is a sequence $\left(n_{j}\right)_{j \in \mathbb{N}}$ such that $T^{n_{j}}\left(1_{G}, x\right)=\left(\varphi^{\left(n_{j}\right)}(x), S^{n_{j}} x\right) \rightarrow(g, x)$. Thus, for large enough $j$, the sequences $S^{n_{j}} x$ and $x$ agree on arbitrarily long prefixes $p_{j}$, and $x \in\left[p_{j} q_{j} p_{j}\right]_{X}$ for some factor $q_{j}$ such that $\left|p_{j} q_{j}\right|=n_{j}$. If $j$ is large enough, then the given prefix $w_{n}$ of $x$ is a prefix of $p_{j}$, and (since $G$ is finite) $\varphi\left(p_{j} q_{j}\right)=\varphi^{\left(n_{j}\right)}(x)=g$.

Example 5.11. Consider once more the Fibonacci substitution $\sigma: a \mapsto a b, b \mapsto a$ and the Fibonacci shift $X$. Let $\varphi: A^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ be the morphism $\varphi: a \mapsto 1, b \mapsto 0$. We saw in Section 3.3 that the skew product $G \rtimes X$ is minimal and uniquely ergodic. We provide here another argument for this which uses bifix codes.

First let $Z$ be the bifix code such that $Z^{*}=\varphi^{-1}(0)$ and $U=Z \cap \mathcal{L}(X)$. We find $U=\{a a, a b a, b\}$. Consider an alphabet $B=\{u, v, w\}$ and define a morphism $\phi: B^{*} \rightarrow A^{*}$ by

$$
\phi: u \mapsto a a, v \mapsto a b a, w \mapsto b
$$

By construction, the image of $\phi$ has the same intersection with $\mathcal{L}(X)$ as the submonoid $\varphi^{-1}(0)=Z^{*}$. Moreover, we have $\sigma^{3} \circ \phi=\phi \circ \tau$ where $\tau$ is the morphism

$$
\tau: u \mapsto v v w u w, v \mapsto v v w u w u w, w \mapsto v
$$

Letting $Y$ be the shift space generated by $\tau$, the skew product $G \rtimes X$ can be identified with the tower $\widehat{Y}$ relative to the function $f(x)=\left|\phi\left(x_{0}\right)\right|$ (see Section 1.1.3 in 24] for details). Since $\tau$ is primitive, $Y$ is uniquely ergodic by Michel's theorem, and so is $\widehat{Y}$.
5.3. Average length. We finish the section with a brief discussion on the notion of average length. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. Let $Z$ be the group code such that $\varphi^{-1}\left(1_{G}\right)=Z^{*}$, and $U=Z \cap \mathcal{L}(X)$, considered in Proposition 5.8. The average length of $U$ is the quantity given by

$$
\ell(U)=\sum_{u \in U}|u| \mu(u)
$$

Example 5.12. The bifix code $U$ in Example 5.11 has average length $\ell(U)=\mu(\varepsilon)+$ $\mu(a)+\mu(a b)=1+\frac{1}{\lambda}+\frac{1}{\lambda^{2}}=2$ where $\lambda$ is the golden ratio (cf. Fig. (9).

In general, one has also

$$
\ell(U)=\sum_{w \in P} \mu(w)
$$

where $P$ is the set of proper prefixes of some element of $U$. Moreover, under the hypotheses of Theorem 3.3, i.e. ergodicity of the skew product $G \rtimes X$ where $L=$ $Z^{*}=\varphi^{-1}\left(1_{G}\right)$, the following equalities hold:

$$
\delta_{\mu}(L)=|G|=\frac{1}{\ell(U)}
$$

We will see that ergodicity of the skew product entails minimality (Corollary 6.2), hence the restriction of $\varphi$ on the sets $\mathcal{R}_{X}(u)^{*}$ is onto. We thus are in the scope of Corollary 4.3.8 from [8], which states that $\ell(U)=|G|$.

## 6. Minimal subsets and modular cobounding maps

We now generalize to the non-minimal case some of the results from the previous sections, in particular Theorem 3.3 which allows a simple expression of the density under the assumption of ergodicity, and Theorem 5.1 which characterizes minimality. We rely on the key notion of modular cobounding map (Definition 6.3), closely related with the notion of coboundary. It allows in particular to recover a further characterization of minimality (Theorem 6.5) inspired by Anzai's theorem on ergodicity of skew products [2].

We prove first that the minimal skew products under consideration are finite disjoint unions of their minimal closed invariant subsets, all of which have the same measure (Proposition 6.1). We conclude that ergodicity implies minimality (Corollary 6.2). Moreover, Proposition 6.11 together with Corollary 6.12 provide sufficient conditions for ergodicity on every minimal closed invariant subset of $G \rtimes X$. We prove Theorem 6.13 (our third main result, i.e. TheoremC) in Section 6.4 and conclude with examples in Section 6.5.
6.1. Modular coboundaries. We start by examining the general structure of skew products in terms of minimal closed invariant subsets.

Proposition 6.1. Let $X$ be a minimal shift space on $A$ with an invariant measure $\mu$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. The skew product $G \rtimes X$ is a disjoint union of its minimal closed invariant subsets, which are finite in number and equal in $\nu \times \mu$-measure.
Proof. Let $G$ act on the left of $G \rtimes X$ by $g(h, x)=(g h, x)$. This action is $T$ commuting and continuous, so $G$ acts by automorphisms. In particular, it permutes the set of minimal closed invariant subsets.

Let us fix a pair $(g, x) \in G \times X$ and show that it is contained in some minimal closed invariant subset of $G \rtimes X$. By Zorn's lemma, there exists at least one minimal closed invariant subset, say $Y \subseteq G \rtimes X$. Its projection on $X$ is also a minimal closed invariant subset, thus by minimality of $X$, it must be $X$ itself. It follows that $(h, x) \in Y$ for some $h \in G$, and then $g h^{-1} Y$ is a minimal closed invariant subset containing $(g, x)$.

This also shows that $G$ acts transitively on the set of all minimal closed invariant subsets, thus it must be finite and with cardinality dividing $|G|$. Moreover, this action by $G$ is measure-preserving, as is easily checked on rectangular sets: indeed, for every measurable sets $E \subseteq G$ and $F \subseteq X$, we have

$$
(\nu \times \mu)(g(E \times F))=(\nu \times \mu)(g E \times F)=(\nu \times \mu)(E \times F) .
$$

Thus all minimal closed invariant subsets must have the same measure.
The above result has the following straightforward consequence.
Corollary 6.2. Let $X$ be a minimal shift space on $A$ with an invariant measure $\mu$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\nu \times \mu$ restricts to an ergodic measure on a closed invariant subset $Y$, then $Y$ must be minimal. In particular, ergodicity of $G \rtimes X$ with respect to $\nu \times \mu$ implies minimality of the skew product $G \rtimes X$.

We now proceed to describe the minimal closed invariant subsets of $G \rtimes X$, using the following key notion inspired by 32].
Definition 6.3. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. A cobounding map for $\varphi$ on $X$ is a continuous map $\alpha: X \rightarrow$ $H \backslash G$ to the set of right cosets of a subgroup $H \leq G$ such that, for all $x \in X$,

$$
\alpha(S x)=\alpha(x) \varphi\left(x_{0}\right)
$$

We also say that $\alpha$ is a cobounding map $\bmod H$. Observe that if $\alpha$ is a cobounding map then $\alpha\left(S^{n} x\right)=\alpha(x) \varphi^{(n)}(x)$ for all $n \in \mathbb{Z}$, with $\varphi^{(n)}$ as in (3.1).

As the name suggests, this definition is related to cohomological ideas in ergodic theory. These ideas have a rich history, as evidenced for instance by 2, 48, 49, 51,

20, 41, 44, 32]. In particular we may view a cobounding map as a "certificate" that (the cocycle defined by) $\varphi$ is a coboundary mod $H$. The next proposition clarifies the link between cobounding maps and closed invariant subsets. It is a special case of a result of Lemańczyk and Mentzen (32], Proposition 2.1).

Proposition 6.4. Let $X$ be a shift space on $A$ with an invariant measure $\mu$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$ with uniform probability measure $\nu$. Let $\alpha$ be a cobounding map mod $H$ for $\varphi$, and $Y_{\alpha}=\{(g, x) \mid \alpha(x)=H g\}$.
(i) The set $Y_{\alpha}$ is a closed invariant subset of $G \rtimes X$ of $\nu \times \mu$-measure $1 /[G: H]$.
(ii) If $Y$ is a minimal closed invariant subset of $G \rtimes X$, then $Y=Y_{\alpha}$ for some cobounding map $\alpha$.

Proof. (i). Note that $Y_{\alpha}=\bigcup_{H g \in H \backslash G} H g \times \alpha^{-1}(H g)$. Since $\alpha$ is continuous, each $\alpha^{-1}(H g)$ is closed, so is $Y_{\alpha}$. Fix a pair $(g, x) \in Y_{\alpha}$, which means that $g \in H g=\alpha(x)$. Hence $g \varphi^{(n)}(x) \in \alpha(x) \varphi^{(n)}(x)=\alpha\left(S^{n} x\right)$, and $T^{n}(g, x)=\left(g \varphi^{(n)}(x), S^{n} x\right) \in Y_{\alpha}$. Thus $Y_{\alpha}$ is invariant. It has measure

$$
\begin{aligned}
(\nu \times \mu)\left(Y_{\alpha}\right) & =\sum_{H g \in H \backslash G}(\nu \times \mu)\left(H g \times \alpha^{-1}(H g)\right) \\
& =\frac{|H|}{|G|} \sum_{H g \in H \backslash G} \mu\left(\alpha^{-1}(H g)\right)=\frac{1}{[G: H]}
\end{aligned}
$$

(ii) Let $Y$ be a minimal closed invariant subset of $G \rtimes X$. Consider the subgroup $H=\{h \in G \mid h Y=Y\}$, and for each $x \in X$ let

$$
\alpha(x)=\{h \in H \mid(h, x) \in Y\} .
$$

Fix an element $g \in \alpha(x)$. We claim that $\alpha(x)=H g$. On the one hand, it is clear that for $h \in H,(h g, x) \in h Y=Y$, thus $h g \in \alpha(x)$. This shows that $H g \subseteq \alpha(x)$. On the other hand, for $k \in \alpha(x)$ we have that $(g, x)=\left(g k^{-1} k, x\right) \in Y \cap \bar{g} k^{-1} Y$. But note that $g k^{-1} Y$ is also a closed invariant subset of $G \rtimes X$, since $Y$ is and $T$ commutes with the left action of $G$ on $G \rtimes X$. This shows that $Y \cap g k^{-1} Y$ is a non-empty closed invariant subset of $Y$, and thus $Y \cap g k^{-1} Y=Y$ by minimality of $Y$.

For every $(h, y) \in Y$, we conclude that $h=k g^{-1} h^{\prime}$ for some $h^{\prime} \in G$ such that $\left(h^{\prime}, y\right) \in Y$, which implies that $k g^{-1}(h, y)=\left(h^{\prime}, y\right) \in Y$. This means that $k g^{-1} \in H$ and thus $k=k g^{-1} g \in H g$, as claimed. In particular, this shows that $\alpha$ is a map $X \rightarrow H \backslash G$ and that $Y=Y_{\alpha}$. It remains to show that $\alpha$ is a cobounding map.

To establish continuity, it suffices to show that $\alpha^{-1}(H g)$ is closed for each $g \in G$. But observe that $\alpha^{-1}(\mathrm{Hg})$ can be written in terms of the two component projections $P_{G}: G \rtimes X \rightarrow G$ and $P_{X}: G \rtimes X \rightarrow X:$

$$
\alpha^{-1}(H g)=\{x \in X \mid g \in \alpha(x)\}=\{x \in X \mid(g, x) \in Y\}=P_{X}\left(P_{G}^{-1}(g) \cap Y\right)
$$

Since $Y$ is a closed subspace, $P_{G}$ is continuous, and $P_{X}$ is a closed map, we conclude that $\alpha^{-1}(H g)$ is indeed closed.

We end the proof by showing that $\alpha(x) \varphi^{(n)}(x)=\alpha\left(S^{n} x\right)$ for every $x \in X$ and $n \in \mathbb{Z}$. Take first $h \in \alpha(x)$, so $(h, x) \in Y$. Since $Y$ is invariant,

$$
\left(h \varphi^{(n)}(x), S^{n} x\right)=T^{n}(h, x) \in Y
$$

and therefore $h \varphi^{(n)}(x) \in \alpha\left(S^{n} x\right)$. This shows that $\alpha(x) \varphi^{(n)}(x) \subseteq \alpha\left(S^{n} x\right)$. As this inclusion holds for all $x \in X$ and $n \in \mathbb{Z}$, it also holds for $S^{n} x$ and $-n$, which yields:

$$
\alpha\left(S^{n} x\right) \varphi^{(n)}\left(S^{n} x\right) \subseteq \alpha\left(S^{-n} S^{n} x\right)=\alpha(x)
$$

Since $\varphi^{(n)}\left(S^{n} x\right)=\varphi^{(n)}(x)^{-1}$, this proves the other inclusion.
6.2. A characterization of minimality via cobounding maps. There is always at least one cobounding map, namely the constant map $X \rightarrow G \backslash G$, which we call the trivial cobounding map. The corresponding closed invariant subset is then the whole skew product. It is immediately apparent that the existence of a nontrivial cobounding map thus forbids minimality of the skew product. In fact, we have the following consequence of Proposition 6.4. which is reminiscent of Anzai's theorem on ergodicity of skew products [2] (see also [40], Chapter 2, Theorem 4.8).
Theorem 6.5. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. The skew product $X \rtimes G$ is minimal if and only if there exists no non-trivial cobounding maps for $\varphi$.

Proof. We prove the contrapositive implications. If there is a non-trivial cobounding map $\alpha: X \rightarrow H \backslash G$, then the subset $Y_{\alpha}$ from Proposition 6.4 is a proper, non-empty, closed invariant subset of $G \rtimes X$. Conversely if $G \rtimes X$ is not minimal then it has (by Zorn's lemma) a proper minimal closed invariant subset $Y$, which must then be of the form $Y=Y_{\alpha}$ for some cobounding map $\alpha: X \rightarrow H \backslash G$, by (ii) of Proposition 6.4. Since $Y_{\alpha}$ is proper, $\alpha$ must be non-trivial.

Given two cobounding maps $\alpha$ and $\beta$, respectively $\bmod H$ and $K$, write $\alpha \leq \beta$ if $H \leq K$ and $\alpha^{-1}(H g) \subseteq \beta^{-1}(K g)$ for all $g \in G$. This gives a partial order on cobounding maps, which corresponds directly to the ordering of the corresponding closed invariant subsets under inclusion.

Proposition 6.6. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. For any two cobounding maps $\alpha$ and $\beta, Y_{\alpha} \subseteq Y_{\beta}$ if and only if $\alpha \leq \beta$.

Therefore, the minimal closed invariant subsets of $G \rtimes X$ correspond to the minimal cobounding maps under $\leq$.
Proof. Assume $\alpha: X \rightarrow H \backslash G$ and $\beta: X \rightarrow K \backslash G$. If $\alpha \leq \beta$, then $H g \subseteq K g$ and $\alpha^{-1}(H g) \subseteq \beta^{-1}(K g)$ for every $g \in G$, thus

$$
Y_{\alpha}=\bigcup_{H g \in H \backslash G} H g \times \alpha^{-1}(H g) \subseteq \bigcup_{K g \in K \backslash G} K g \times \beta^{-1}(K g)=Y_{\beta}
$$

Conversely, assume that $Y_{\alpha} \subseteq Y_{\beta}$ and fix $H g \in H \backslash G$. Then $H g \times \alpha^{-1}(H g) \subseteq$ $K g^{\prime} \times \beta^{-1}\left(K g^{\prime}\right)$ for some $g^{\prime} \in G$. In particular, $g \in K g^{\prime}$ so we may assume $g=\overline{g^{\prime}}$. We then deduce that $H \subseteq K$ and $\alpha^{-1}(H g) \subseteq \beta^{-1}(K g)$.

Remark 6.7. The left action of $G$ on $G \rtimes X$ corresponds to the left action on cobounding maps given by $(g \alpha)(x)=g(\alpha(x))$, where $g \alpha$ is viewed as a cobounding map $\bmod H^{g}=g H^{-1}$. This cobounding map is such that $g \alpha(x)=H^{g} h \Longleftrightarrow$ $\alpha(x)=H g^{-1} h$, hence $X_{g \alpha}=g X_{\alpha}$. This shows that the passage from minimal closed invariant subsets to minimal cobounding maps preserves the left action of $G$.

Corollary 6.8. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. Let $H$ be a subgroup of $G$ such that there exists a minimal cobounding map $\alpha: X \rightarrow H \backslash G$. Then, any other minimal cobounding map is modulo a conjugate of $H$ and the number of minimal closed invariant subsets of the skew product $G \rtimes X$ equals $[G: H]$.
6.3. Cobouding maps and return words. We have established in Theorem 5.1] a characterization of minimality in terms of return words. Without surprise, we also find links between cobounding maps and return words. This makes the relationship between (ii) and (iii) in Theorem 5.1 more transparent (see Remark 6.10 below).

Proposition 6.9. Let $X$ be a minimal shift space on $A$ and $\varphi: A^{*} \rightarrow G$ be $a$ morphism onto a finite group $G$. Let $H$ be a subgroup of $G$.
(i) If a cobounding map $\alpha: X \rightarrow H \backslash G$ takes constant value $H g$ on a cylinder $[u]_{X}$, then $\varphi\left(\mathcal{R}_{X}(u)\right) \subseteq g^{-1} H g$.
(ii) If a word $u \in \mathcal{L}(X)$ satisfies $\varphi\left(\mathcal{R}_{X}(u)\right) \subseteq H$, then there exists a cobounding map $\alpha: X \rightarrow H \backslash G$ which takes constant value $H$ on $[u]_{X}$.
(iii) $A$ cobounding map $\alpha: X \rightarrow H \backslash G$ is minimal if and only if $\varphi\left(\mathcal{R}_{X}(u)\right)$ generates $g^{-1} H g$ whenever $u$ is such that $\alpha$ takes constant value $H g$ on $[u]_{X}$.

Proof. (i). Consider $w \in \mathcal{R}_{X}(u)^{*}$. Taking $x \in[w u]_{X} \subseteq[u]_{X}$, we find:

$$
\alpha(x)=\alpha\left(S^{|w|} x\right)=\alpha(x) \varphi(w)=H g \varphi(w)
$$

while the stabilizer of $H g$ under the right action of $G$ on $H \backslash G$ is exactly $g^{-1} H g$.
(ii) Let $u \in \mathcal{L}(X)$ be such that $\varphi\left(\mathcal{R}_{X}(u)\right) \leq H$. For $x \in X$, let

$$
C_{x}=\left\{j \geq 0 \mid x_{[j, j+|u|)}=u\right\}
$$

Observe that under the assumption that $\varphi\left(\mathcal{R}_{X}(u)\right) \subseteq H$, the value of $H \varphi\left(x_{[0, j)}\right)^{-1}$ is identical for every $j \in C_{x}$. Hence, we may define a map $\alpha: X \rightarrow H \backslash G$ by

$$
\alpha(x)=H \varphi\left(x_{[0, j)}\right)^{-1}, \quad \text { where } j \in C_{x}
$$

Since $X$ is a minimal shift space, there exists a constant $m>0$ independent of $x$ such that $\min \left(C_{x}\right)<m$; hence $\alpha$ is continuous, since its value is determined by the first $m$ letters. Moreover, fixing $x \in X$ and $j \in C_{x}$ with $j \geq 1$, we find that

$$
\alpha(S x)=H \varphi\left(x_{[1, j)}\right)^{-1}=H \varphi\left(x_{[0, j)}\right)^{-1} \varphi\left(x_{0}\right)=\alpha(x) \varphi\left(x_{0}\right) .
$$

(iii). Assume first that $\alpha$ is minimal. Fix $h \in H$ and suppose that $\alpha$ takes constant value $H g$ on some cylinder $[u]_{X}$. We need to show that $g^{-1} h g$ belongs to $\left\langle\varphi\left(\mathcal{R}_{X}(u)\right)\right\rangle$. Since $\{g\} \times[u]$ and $\{h g\} \times[u]$ are two non-empty clopen subsets of the minimal closed invariant subset $Y_{\alpha}$, we may find $k \in \mathbb{Z}$ with $(\{g\} \times[u]) \cap T^{-k}(\{h g\} \times[u]) \neq \varnothing$. Choose $x$ in that intersection and let $w=x_{[0, k)}$ if $k \geq 0$ and $w=x_{[k, 0)}$ otherwise. It follows that:

$$
(h g, x)=T^{k}(g, x)=\left(g \varphi(w)^{ \pm 1}, S^{k} x\right)
$$

hence $\varphi(w)=g^{-1} h^{ \pm 1} g$. But $w$ is a concatenation of elements of $\mathcal{R}_{X}(u)$, thus $g^{-1} h g$ belongs to the subgroup of $G$ generated by $\varphi\left(\mathcal{R}_{X}(u)\right)$.

To prove the converse, we consider a cobounding map $\beta: X \rightarrow K \backslash G$ such that $\beta \leq \alpha$. Let $u$ be a word such that both $\alpha$ and $\beta$ are constant on $[u]_{X}$, say with $\alpha\left([u]_{X}\right)=H g$ and $\beta\left([u]_{X}\right)=K h$. Note that $h \in K h \subseteq H g$, so we may assume that $h=g$. By part (i) of the statement, $\varphi\left(\mathcal{R}_{X}(u)\right)$ generates a subgroup of $g^{-1} K g$, while it also generates $g^{-1} H g$ by our assumption. Thus $H=K$ and $\beta=\alpha$ on $[u]_{X}$.

As we may partition $X$ into a union of such cylinders, we get $\alpha=\beta$, thus showing that $\alpha$ is minimal.

Remark 6.10. From this result combined with Theorem 6.5, we deduce an alternate proof for the equivalence between (ii) and (iii) in Theorem 5.1. Indeed, on the one hand, if the skew product is not minimal then Theorem 6.5 states that there exists a non-trivial cobounding map; hence it follows from (i) of Proposition 6.9 that for a sufficiently long word $u$, all return words in $\mathcal{R}_{X}(u)$ are mapped inside some proper subgroup of $G$. On the other hand, if the image of some return set $\mathcal{R}_{X}(u)$ fails to generate $G$, then by (ii) of Proposition 6.9, there exists a non-trivial cobounding map, hence the skew product cannot be minimal by Theorem 6.5,

Roughly speaking, the smaller the subgroup, the more restrictive the coboundary condition, hence the most stringent cobounding maps are the cobounding maps mod 1. One important fact is that having such cobounding maps turns out to be sufficient for the measure $\nu \times \mu$ to be ergodic on the minimal closed invariant subsets of $G \rtimes X$.

Proposition 6.11. Let $X$ be a minimal shift space on $A$ with an ergodic measure $\mu$ and $\varphi: A^{*} \rightarrow G$ a morphism onto a finite group $G$ with uniform probability measure $\nu$. If $\alpha: X \rightarrow G$ is a cobounding map mod 1, then the product measure $\mu \times \nu$ is ergodic on $Y_{\alpha}$.

Proof. The map $\gamma: X \rightarrow Y_{\alpha}, \gamma(x)=(\alpha(x), x)$, is a homeomorphism which intertwines $S$ and $T$ and satisfies $\mu(E) /|G|=(\nu \times \mu)(\gamma(E))$ for every measurable set $E \subseteq X$. Thus $\left(Y_{\alpha}, T, \nu \times \mu\right)$ is measure-theoretically isomorphic to $(X, S, \mu)$, and since the latter is ergodic, so is the former.

We moreover observe that cobounding maps mod 1 are minimal by Proposition 6.9 (iii) In this special case, Proposition 6.9 also yields the following.

Corollary 6.12. Let $X$ be a minimal shift space on $A$ with and $\varphi: A^{*} \rightarrow G a$ morphism onto a finite group $G$. The following conditions are equivalent:
(i) $\varphi$ has a cobounding map mod 1 on $X$.
(ii) $\varphi\left(\mathcal{R}_{X}(u)\right)=1$ for every long enough $u \in \mathcal{L}(X)$.
(iii) $\varphi\left(\mathcal{R}_{X}(u)\right)=1$ for some word $u \in \mathcal{L}(X)$.

Proof. That (i) implies (ii) follows from Proposition 6.9 (i) (ii) implies (iii) is trivial; (iii) implies (i) follows from Proposition 6.9 (ii).
6.4. Generalized density formula. The next theorem is our third main result. It generalizes the formula for density given in Theorem 3.3

Theorem 6.13 (TheoremC). Let $X$ be a minimal shift space on $A$ with an ergodic measure $\mu$ and $\varphi: A^{*} \rightarrow G$ a morphism onto a finite group $G$ with uniform probability measure $\nu$. Suppose that $\nu \times \mu$ is ergodic on every minimal closed invariant subset of $G \rtimes_{\varphi} X$. Then for every language $L=\varphi^{-1}(K), K \subseteq G$, the density $\delta_{\mu}(L)$ exists and for every minimal cobounding map $\alpha: X \rightarrow H \backslash G$,

$$
\begin{equation*}
\delta_{\mu}(L)=\frac{1}{|H|} \sum_{k \in K} \sum_{H g \in H \backslash G} \mu\left(\alpha^{-1}(H g)\right) \mu\left(\alpha^{-1}(H g k)\right) . \tag{6.1}
\end{equation*}
$$

Remark 6.14. In the special case where $G \rtimes X$ is ergodic, then the trivial cobounding map is minimal and (6.1) simplifies:

$$
\delta_{\mu}(L)=\frac{1}{|G|} \sum_{k \in K} \sum_{G g \in G \backslash G} \mu\left(\alpha^{-1}(G g)\right) \mu\left(\alpha^{-1}(G g k)\right)=\frac{|K|}{|G|}
$$

Thus we recover Theorem 3.3.
Most of the proof of Theorem 6.13 boils down to the following technical lemma.
Lemma 6.15. Let $\alpha: X \rightarrow H \backslash G$ be a minimal cobounding map. Let $U_{g}=\{g\} \times X$ and $U_{g}^{\alpha}=U_{g} \cap Y_{\alpha}$. For every $i \in \mathbb{N}, h \in G$,

$$
\mu\left(\varphi^{-1}(h) \cap A^{i}\right)=[H: G] \sum_{g \in G}(\nu \times \mu)\left(U_{g}^{\alpha} \cap T^{-i} U_{g h}^{\alpha}\right)
$$

Proof. Let $L=\varphi^{-1}(h)$. We start by observing that

$$
\mu\left(L \cap A^{i}\right)=\sum_{g \in G}(\nu \times \mu)\left(U_{g} \cap T^{-i} U_{g h}\right)
$$

Consider for every $g \in G$ the partition $U_{g}=\bigcup_{\beta} U_{g}^{\beta}$, where $U_{g}^{\beta}=U_{g} \cap Y_{\beta}$ and $\beta$ ranges over all minimal cobounding maps. By invariance of the subsets $Y_{\beta}, U_{g}^{\beta} \cap$ $T^{-i} U_{g h}=U_{g}^{\beta} \cap T^{-i} U_{g h}^{\beta}$ and

$$
\mu\left(L \cap A^{i}\right)=\sum_{\beta} \sum_{g \in G}(\nu \times \mu)\left(U_{g}^{\beta} \cap T^{-i} U_{g h}^{\beta}\right)
$$

where the first sum is over all minimal cobounding maps $\beta$.
List all minimal cobounding maps $\left\{\beta_{1}, \ldots, \beta_{n}\right\}$; in particular $n=[H: G]$. Since $G$ acts transitively on the set of minimal cobounding maps, we may consider for each $i=1, \ldots, n$ an element $g_{j}$ such that $\beta_{j}=g_{j} \alpha$. Then we find

$$
\begin{aligned}
\mu\left(L \cap A^{i}\right) & =\sum_{j=1}^{n} \sum_{g \in G}(\nu \times \mu)\left(U_{g}^{g_{j} \alpha} \cap T^{-i} U_{g h}^{g_{j} \alpha}\right) \\
& =\sum_{j=1}^{n} \sum_{g \in G}(\nu \times \mu)\left(U_{g_{j} g}^{\alpha} \cap T^{-i} U_{g_{j} g h}^{\alpha}\right) \\
& =[H: G] \sum_{g \in G}(\nu \times \mu)\left(U_{g}^{\alpha} \cap T^{-i} U_{g h}^{\alpha}\right) .
\end{aligned}
$$

Proof of Theorem 6.13. Let $\alpha: X \rightarrow H \backslash G$ be a cobounding map mod $H$ which is minimal. Using the above lemma together with ergodicity of $Y_{\alpha}$,

$$
\begin{aligned}
\delta_{\mu}\left(\varphi^{-1}(h)\right) & =[H: G] \sum_{g \in G} \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1}(\nu \times \mu)\left(U_{g}^{\alpha} \cap T^{-i} U_{g h}^{\alpha}\right) \\
& =[H: G]^{2} \sum_{g \in G}(\nu \times \mu)\left(U_{g}^{\alpha}\right)(\nu \times \mu)\left(U_{g h}^{\alpha}\right) \\
& =\frac{[H: G]^{2}}{|G|^{2}} \sum_{g \in G} \mu\left(\alpha^{-1}(H g)\right) \mu\left(\alpha^{-1}(H g h)\right) \\
& =\frac{1}{|H|} \sum_{H g \in H \backslash G} \mu\left(\alpha^{-1}(H g)\right) \mu\left(\alpha^{-1}(H g h)\right)
\end{aligned}
$$

6.5. Examples. We finish the section with three examples that illustrate various aspects of Theorem 6.13
Example 6.16. We continue with the tree-element shift from Example 3.2 Let $x=$ $(a b c)^{\infty}, y=(b c a)^{\infty}, z=(c a b)^{\infty}$ and $X=\{x, y, z\}$ with its unique invariant measure $\mu$. Let $\varphi: X \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ defined by $\varphi: a, b \mapsto 1, c \mapsto 0$. For $|u| \geq 3, \mathcal{R}_{X}(u)$ is a singleton consisting of either one of $a b c, b c a$ or $c a b$, which are all mapped to the identity element. Accordingly the skew product has two closed invariant subsets which correspond to the following cobounding maps mod 1 , defined on cylinders of length 3 ,

$$
\alpha:[a b c] \mapsto 0, \quad[b c a] \mapsto 1, \quad[c a b] \mapsto 0, \quad \beta:[a b c] \mapsto 1, \quad[b c a] \mapsto 0,[c a b] \mapsto 1
$$

By Corollary 6.12, the measure $\nu \times \nu$ restricts to ergodic measures on the minimal closed invariant subsets $Y_{\alpha}$ and $Y_{\beta}$. Using Theorem 6.13 we recover the following values for the density of $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$, confirming the conclusion reached in Example 3.4

$$
\begin{gathered}
\delta_{\mu}\left(\varphi^{-1}(0)\right)=\mu\left(\alpha^{-1}(0)\right)^{2}+\mu\left(\alpha^{-1}(1)\right)^{2}=\frac{4}{9}+\frac{1}{9}=\frac{5}{9} \\
\delta_{\mu}\left(\varphi^{-1}(0)\right)=2 \mu\left(\alpha^{-1}(0)\right) \mu\left(\alpha^{-1}(1)\right)=\frac{4}{9} .
\end{gathered}
$$

Example 6.17. Let $X$ be the Thue-Morse shift with its unique invariant measure $\mu$ (see Example 2.1) and let $\varphi: A^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=1, \varphi(b)=0$.

The morphism $\varphi$ has two cobounding maps mod 1 on $X$, hence by Corollary 6.12 $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ has two minimal closed invariant subsets on which the measure $\nu \times \mu$ is ergodic. The cobounding maps take constant values on the cylinders of length 7 ; one of them is depicted in Fig. 7


Figure 7. One of the two cobounding maps $X \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ on the Thue-Morse shift for the morphism $\varphi:\{a, b\}^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=1$, $\varphi(b)=0$. The map is constant on cylinders of length 7 .

Observe that the cobounding maps are fair, in the sense that the preimages $\alpha^{-1}(g)$ have the same $\mu$-measure for all elements of the group. Therefore, even though the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ is not ergodic, the shift space $X$ is still evenly distributed with respect to $\varphi$, in the sense that that that that

$$
\delta_{\mu}\left(\varphi^{-1}(1)\right)=\delta_{\mu}\left(\varphi^{-1}(0)\right)=1 / 2
$$



Figure 8. Cobounding map $\bmod H=\langle(12)\rangle$ on the shift of Example 6.18 for the morphism $\varphi:\{a, b, c\}^{*} \rightarrow S_{3}, \varphi(a)=\varphi(c)=$ $(123), \varphi(b)=(12)$. The map is constant on cylinders of length 4.

Example 6.18. Consider the following substitution and morphism onto $S_{3}$ defined on the alphabet $A=\{a, b, c\}$ :

$$
\sigma: a \mapsto a a b, b \mapsto a c b, c \mapsto b a, \quad \varphi(a)=\varphi(c)=\left(\begin{array}{ll}
1 & 2
\end{array}\right), \varphi(b)=(12) .
$$

Let $X$ be the shift space generated by the primitive substitution $\sigma$. We claim that the skew product $S_{3} \rtimes X$ has three minimal closed invariant subsets. These minimal subsets correspond to three cobounding maps

$$
\alpha_{12}: X \rightarrow\langle(12)\rangle \backslash S_{3}, \quad \alpha_{13}: X \rightarrow\langle(13)\rangle \backslash S_{3}, \quad \alpha_{23}: X \rightarrow\langle(23)\rangle \backslash S_{3}
$$

The cobounding map $\alpha_{12}$ is depicted in Fig. 8 , and the other two can be deduced from $\alpha_{12}$ using the natural left action of $S_{3}$ on cobounding maps. Note that in all cases the subgroup involved has index 3 in $S_{3}$, in accordance with Corollary 6.8 ,

We now briefly sketch a proof of the fact that the above cobounding maps are indeed minimal. In the present case, this is equivalent to showing that there are no cobounding maps mod 1. Recall that by Corollary 6.12, there is a cobounding map $\bmod 1$ if and only if $\mathcal{R}_{X}(u)$ has trivial image under $\varphi$ for all sufficiently long word $u$.

By the first main result from [9], since $\sigma$ is a bifix encoding, there exists a constant $K>0$ such that for all $u \in \mathcal{L}(\sigma)$ with $|u| \geq K$,

$$
\mathcal{R}_{X}(\sigma(u))=\sigma\left(\mathcal{R}_{X}(u)\right)
$$

Using the formula provided in [9] we find the upper bound $K \leq 6$, but direct computations show that we can take $K=2$; in fact, the one-letter word $u=c$ is the only word which fails the above equality. Now take the sequence of words $u_{n}=\sigma^{n}(a)$; we claim that $(23) \in \mathcal{R}_{X}\left(u_{n}\right)$, for infinitely many $n$. Indeed, observe that the following equalities holds:

$$
\varphi \circ \sigma=\varphi \circ \sigma^{8}, \quad \varphi \circ \sigma(a)=\varphi \circ \sigma(b)=\varphi \circ \sigma(c)=(23)
$$

As $\mathcal{R}_{X}\left(u_{0}\right)=\{a, b a, b b a, c b a\}$, it follows that $(23)=\varphi\left(\sigma^{7 k+1}(a)\right)$ belongs to $\mathcal{R}_{X}\left(u_{7 k+1}\right)$ for all $k \geq 0$. This shows that $\mathcal{R}_{X}\left(u_{n}\right)$ have non-trivial images for
infinitely many $n$, thus $\varphi$ has no cobounding map mod 1 on $X$. This confirms that the above cobounding maps are minimal.

## 7. ERGODICITY FOR PRIMITIVE SUBSTITUTIONS

In this section we focus on the special case of shift spaces defined by primitive substitutions. Our main result is a sufficient condition for the minimal closed invariant subsets of skew products to be uniquely ergodic (Proposition 7.3). As a corollary, we deduce that substitutive dendric shifts have ergodic skew products with all finite groups (Theorem 7.11). Note that the family of dendric shifts, studied in [8, 13, 14, 11, 12] encompasses several classical families of shifts, such as Sturmian shifts, codings of interval exchanges, and Arnoux-Rauzy shifts.
7.1. Skew products based on primitive shifts. Let us fix a primitive substitution $\sigma$ on a finite alphabet $A$ and let $X=X(\sigma)$ be the shift space defined by $\sigma$. The shift $X$ is a minimal shift space and we recall that it is uniquely ergodic by Michel's theorem.

Definition 7.1. Let $\varphi: A^{*} \rightarrow G$ be a morphism onto a finite group $G$. We say that that the primitive substitution $\sigma$ is invertible under $\varphi$ if:

$$
\begin{equation*}
\exists n \geq 1, \varphi \circ \sigma^{n}=\varphi \tag{7.1}
\end{equation*}
$$

Example 7.2. Let $\sigma: a \mapsto a b, b \mapsto a$ be the Fibonacci substitution and consider the $\operatorname{morphism} \varphi: A^{*} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \varphi(a)=1, \varphi(b)=0$. One checks that

$$
\sigma^{3}(a)=a b a a b, \quad \sigma^{3}(b)=a b a
$$

hence $\varphi \circ \sigma^{3}=\varphi$, so $\sigma$ is invertible under $\varphi$. This property has already been used in the proof of Proposition 3.6. In fact $\sigma$ has the much stronger of being invertible under every homomorphism onto a finite group, as we shall later see (Lemma 7.5).

Proposition 7.3. Let $\sigma$ be a primitive substitution, $X$ be its shift space, and $\varphi: X \rightarrow G$ be a morphism onto a finite group $G$. If $\sigma$ is invertible under $\varphi$, then the minimal closed invariant subsets of $G \rtimes X$ are uniquely ergodic.

Proof. Up to replacing $\sigma$ by some power, we may assume without loss of generality (since this does not change the shift space) that $\varphi \circ \sigma=\sigma$ and that $\sigma$ has a fixed point $y \in X$.

Let $\Psi$ be the topological conjugacy from Lemma 3.1. The fact that $\varphi=\varphi \circ$ $\sigma$ entails the existence of a substitution $\bar{\sigma}$ on $(G \times A)^{*}$ such that $\Psi(g, \sigma(x))=$ $\bar{\sigma}(\Psi(g, x))$, namely, when $\sigma(a)=b_{0} \cdots b_{n-1}$,

$$
\bar{\sigma}(g, a)=\left(g_{0}, b_{0}\right)\left(g_{1}, b_{1}\right) \cdots\left(g_{n-1}, b_{n-1}\right), \quad \text { where } g_{1}=g, g_{i+1}=g_{i} \varphi\left(b_{i}\right)
$$

Observe that, for every $g \in G$, the infinite word $z=\Psi(g, y)$ is a uniformly recurrent fixed point of $\bar{\sigma}$, since

$$
\bar{\sigma}(z)=\bar{\sigma}(\Psi(g, y))=\Psi(g, \sigma(y))=\Psi(g, y)=z
$$

Let $B$ be the subset of letters in $G \times A$ appearing in $z$; it follows that $\bar{\sigma}$ restricts to a substitution on $B$. Moreover, $z$ belongs to a minimal subset of $(G \times A)^{\mathbb{Z}}$, hence it must be uniformly recurrent. Since $\bar{\sigma}$ is a growing substitution fixing a uniformly recurrent word, it must be primitive and have for shift space the closed orbit of $z$, which is $\Psi(Y)$. In particular $\Psi(Y)$ is uniquely ergodic by Michel's theorem, and so is $Y$.

Observe that Example 6.18 fails both the invertibility property (7.1) and the property of Corollary 6.12. At this time we do not know whether or not the product measure is ergodic on the minimal closed invariant subsets of the skew product. In contrast, here is an example which satisfies both (7.3) and Corollary 6.12,
Example 7.4. Consider the following substitution and morphism onto $\mathbb{Z} / 2 \mathbb{Z}$ defined on the alphabet $A=\{a, b, c, d\}$ :

$$
\sigma: a \mapsto b a a, b \mapsto a d c, c \mapsto c d c, d \mapsto a d, \quad \varphi: a, b \mapsto 0, c, d \mapsto 1
$$

Then $\varphi \circ \sigma=\varphi$, hence the minimal closed invariant subsets of $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$ are uniquely ergodic. The substitution $\bar{\sigma}$ satisfying $\Psi(g, \sigma(x))=\bar{\sigma}(\Psi(g, x))$ is defined as follows on the alphabet $\mathbb{Z} / 2 \mathbb{Z} \times A$, written $a_{0}, a_{1}, b_{0}, b_{1}$, etc. for convenience,

$$
\begin{array}{llll}
a_{0} \mapsto b_{0} a_{0} a_{0}, & b_{0} \mapsto a_{0} d_{0} c_{1}, & c_{0} \mapsto c_{0} d_{1} c_{0}, & d_{0} \mapsto a_{0} d_{0}, \\
a_{1} \mapsto b_{1} a_{1} a_{1}, & b_{1} \mapsto a_{1} d_{1} c_{0}, & c_{1} \mapsto c_{1} d_{0} c_{1}, & d_{1} \mapsto a_{1} d_{1}
\end{array}
$$

This substitution splits in two primitive substitutions respectively, defined on the alphabets $B=\left\{a_{0}, b_{0}, c_{1}, d_{0}\right\}$ and $C=\left\{a_{1}, b_{1}, c_{0}, d_{1}\right\}$. Each corresponds to one of the two minimal closed invariant subsets of the skew product $\mathbb{Z} / 2 \mathbb{Z} \rtimes X$.

In what follows, we say that a substitution $\sigma: A^{*} \rightarrow A^{*}$ is invertible if its extension to an endomorphism on the free group $F_{A}$ is an automorphism. We now state two further properties when the substitution is assumed to be invertible.
Lemma 7.5. Let $\sigma$ be a primitive substitution. If $\sigma$ is invertible, then it is invertible under every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$.

Proof. Let $\operatorname{Aut}\left(F_{A}\right)$ be the automorphism group of $F_{A}$ and $\operatorname{Hom}\left(F_{A}, G\right)$ be the set of morphisms $F_{A} \rightarrow G$. For $\tau \in \operatorname{Aut}\left(F_{A}\right)$, denote by $\tau_{*}$ the self-map of $\operatorname{Hom}\left(F_{A}, G\right)$ defined by

$$
\tau_{*}(\varphi)=\varphi \circ \tau
$$

Since $\rho_{*} \circ \tau_{*}=(\tau \circ \rho)_{*}$, the map $\tau \mapsto \tau_{*}$ is a morphism from $\operatorname{Aut}\left(F_{A}\right)$ to the symmetric group on $\operatorname{Hom}\left(F_{A}, G\right)$. Moreover observe that $\operatorname{Hom}\left(F_{A}, G\right)$ is a finite set, being in bijection with the set of maps $A \rightarrow G$. As a result, for every automorphism $\tau$ of $F_{A}, \tau_{*}$ is a permutation of $\operatorname{Hom}\left(F_{A}, G\right)$ with finite order; in other words, there exists $n \geq 0$ such that $\tau_{*}^{n}=$ id, i.e. $\varphi \circ \tau^{n}=\varphi$ for every morphism $F_{A} \rightarrow G$. Applying this to the extension of $\sigma$ to an automorphism of $F_{A}$ yields the result.

In what follows, we say that a substitution is aperiodic if it generates a shift space that contains no finite orbit. We next establish the following lemma that will be used in the next section. Observe that the generation property expressed below implies the one stated in Theorem 5.1.
Lemma 7.6. Let $\sigma$ be a primitive aperiodic substitution and $X$ the shift generated by $\sigma$. If $\mathcal{R}_{X}(u)$ generates $F_{A}$ for every $u \in \mathcal{L}(X)$, then $\sigma$ is invertible.

Proof. First, fix a point $x \in X$ which is periodic under $\sigma$, meaning $\sigma^{p}(x)=x$ for some $k>0$ ([24], Proposition 1.4.8). Let $w$ be a word of the form $w=x_{[-n, n)}$ and let $u=x_{[0, n)}$ be its suffix of length $n$, for some $n \geq 0$. As Almeida and Costa observed in the proof of [1], Proposition 5.5, it follows from Mossé's recognizability theorem (38], Theorem 3.1bis) that when $n$ is large enough, $u \mathcal{R}_{X}(w) u^{-1} \subseteq \operatorname{im}\left(\sigma^{k}\right)$. Since by assumption $\mathcal{R}_{X}(w)$ generates $F_{A}$, we conclude that $\sigma^{k}$ is a surjective endomorphism of $F_{A}$. Since finitely generated free groups have the Hopfian property ([35], Proposition 3.5), it follows that $\sigma^{k}$ is invertible, hence so is $\sigma$.

For the sake of completeness, we give an example showing that the converse of the above lemma fails, i.e. invertibility does not guarantee that all $\mathcal{R}_{X}(u)$ generate $F_{A}$.
Example 7.7. Let $\sigma$ be the primitive substitution from Example 7.4 defined on the four-letter alphabet $A=\{a, b, c, d\}$ by:

$$
\sigma: a \mapsto b a a, b \mapsto a d c, c \mapsto c d c, d \mapsto a d .
$$

This is an invertible substitution (as may be checked by straightforward computations), but nonetheless the following is not a generating set of $F_{A}$ :

$$
\mathcal{R}_{X}(a)=\{a, b a, d c b a, d c d c a, d c d c b a\}
$$

In fact this set of return words generate the rank 3 subgroup of $F_{A}$ with basis $\{a, b, d c\}$.
7.2. Skew products based on dendric shifts. We turn now to the dendric case. First, we recall the definition. Let $X$ be a shift space. For $w \in \mathcal{L}(X)$, let $\mathrm{L}(w)=\{a \in A \mid a w \in \mathcal{L}(X)\}$ and $\mathrm{R}(w)=\{a \in A \mid w a \in \mathcal{L}(X)\}$. We denote by $\mathrm{E}(w)$ the graph with vertices the disjoint union of $\mathrm{R}(w)$ and $\mathrm{L}(w)$ and edges the pairs $(a, b) \in A \times A$ such that $a w b \in \mathcal{L}(X)$; it is called the extension graph of $w$. A shift space $X$ is dendric if for every $w \in \mathcal{L}(X)$, the extension graph $\mathrm{E}(w)$ is a tree. For instance, every Sturmian shift is dendric [13].

An important result concerning dendric shifts is the so-called Return Theorem by Berthé et al. which we quote next.
Theorem 7.8 (13], Theorem 4.5). Let $X$ be dendric shift on an alphabet A. For every $w \in \mathcal{L}(X)$, the set $\mathcal{R}_{X}(w)$ is a basis of the free group on $A$.

Therefore it follows from Theorem 5.1 that every skew product of a dendric shift and a finite group is minimal.

Example 7.9. The Fibonacci shift (see Section 3.3) is Sturmian and therefore dendric. There, we have $\mathrm{R}(a)=L(a)=\{a, b\}$ and the graph $\mathrm{E}(a)$ is shown in Fig. 9. Moreover $\mathcal{R}_{X}(a)=\{a, a b\}$, which is obviously a basis of the free group on $\{a, b\}$.


Figure 9. The extension graph $\mathrm{E}(a)$ in the Fibonacci shift.

We also give an example of a shift space which is not dendric.
Example 7.10. Let $X$ be the Thue-Morse shift from Example 2.1 generated by the two-letter substitution $\sigma: a \mapsto a b, b \mapsto b a$. For the word $w=a b a$, we find $\mathrm{L}(w)=\mathrm{R}(w)=\{a, b\}$ and the graph $\mathrm{E}(w)$, depicted in Fig. 10, is not connected.

The following result generalizes the proof of unique ergodicity for the skew product of Section 3.3.
Theorem 7.11. Let $X$ be a dendric shift generated by a primitive substitution. Then the skew product $G \rtimes X$ is uniquely ergodic for every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$.


Figure 10. The extension graph $\mathrm{E}(w), w=a b a$, in the ThueMorse shift.

The proof uses Lemmas 7.5 and 7.6. Observe that dendric shift spaces in particular fall under the scope of Lemma [7.6, thanks to the Return Theorem of Berthé et al., stated above as Theorem 7.8 .

Proof. Assume that $X$ is generated by the primitive morphism $\sigma$. Observe that $\sigma$ must be aperiodic, as dendric spaces cannot contain finite orbits. Moreover by the Return Theorem, $\left\langle\mathcal{R}_{X}(w)\right\rangle=F_{A}$ for all $w \in \mathcal{L}(X)$. Therefore we may apply Lemmas 7.5 and 7.6 to conclude that $\sigma$ is invertible, and as a result invertible under every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$. Applying Proposition 7.3 , it follows that the skew product $G \rtimes X$ has uniquely ergodic minimal closed invariant subsets. As the skew product is also minimal by the Return Theorem (Theorem 7.8) and Theorem 5.1, this completes the proof.

We thus obtain, as a direct application of Theorem 3.3, the following result about the density of group languages in substitutive dendric shifts.

Corollary 7.12. Let $X$ be a dendric shift generated by a primitive substitution and let $\mu$ be its unique ergodic measure. For every morphism $\varphi: A^{*} \rightarrow G$ onto a finite group $G$ and every language $L=\varphi^{-1}(K), K \subseteq G$, the density $\delta_{\mu}(L)$ exists and is equal to $|K| /|G|$.
7.3. Skew products based on Sturmian shifts. We end with a discussion about the links between our results applied in the case of Sturmian shifts, and earlier works on skew products based on irrational rotations, mostly by Veech 48, 49] and Jager and Liardet 29]. Due to the nature of the examples involved, and to stay consistent with the relevant literature, it is convenient here to use alphabets consisting of natural numbers, such as $\{0,1\}$ and $\{1,2\}$.

Among the first classical examples of skew products, skew translations (i.e. skew products with base an irrational rotation on the unit circle) and their ergodic properties have been widely investigated; see e.g. [48, 49, 46, 27] and the classical references [21, 40]. In particular, they have been used to produce examples of interval exchanges that are not uniquely ergodic [31]. Such examples are based on skew products of irrational rotations associated with the group $\mathbb{Z} / 2 \mathbb{Z}$, that are minimal and not uniquely ergodic, with the skewing function being the characteristic function of an interval 48, 49].

More precisely, let $\alpha$ be an irrational number in $[0,1]$. We consider the rotation $R_{\alpha}: x \mapsto x+\alpha$ modulo 1 defined on $\mathbb{T}=\mathbb{R} / \mathbb{Z}$. Let $I=[0, \beta)$ be a semi-open interval of $\mathbb{T}$. Let $\chi_{I}: \mathbb{T} \rightarrow\{0,1\}$ be the characteristic function of $I$, i.e. $\chi_{I}(x)=1$ if and only if $x \in I$. Let $m \geq 2$ and let $G=\mathbb{Z} / m \mathbb{Z}$. Let $\varphi:\{0,1\}^{*} \rightarrow G$ be the morphism defined by $0 \mapsto 0,1 \mapsto 1$. We then consider the skew product $G \rtimes_{I} \mathbb{T}$ of $R_{\alpha}$ defined as

$$
(k, x) \mapsto\left(k+\varphi \circ \chi_{I}(x), x+\alpha\right)=\left(k+\varphi \circ \chi_{I}(x), R_{\alpha}(x)\right) .
$$

Such skew products over rotations are closely related to symbolic skew products such as those considered in the present paper, and more precisely, to binary codings of rotations. Let $I^{\mathrm{c}}$ stand for the complement of $I$ in $\mathbb{T}$. Let $x$ be the infinite word in $\{0,1\}^{\mathbb{Z}}$ obtained by coding the orbit of 0 under $R_{\alpha}$ with respect to the partition $\mathcal{I}=\left\{I, I^{\mathrm{c}}\right\}$, i.e. for any $n \in \mathbb{Z}, x_{n}=1$ if and only if $R_{\alpha}^{n}(0) \in I$, or else, $x_{n}=\chi_{I}\left(R_{\alpha}^{n}(0)\right)$ for all $n \in \mathbb{Z}$. Let $(X, S)$ be the shift space generated by $x$. Then this shift is minimal and uniquely ergodic since $\alpha$ is irrational and $I$ is semi-open. When the length of $I$ equals $\alpha$ or $1-\alpha$, such binary codings of rotations are Sturmian.

Not all intervals $I$ lead to ergodic skew products. In fact, there exists an interval $I$ such that the skew product $G \rtimes_{I} \mathbb{T}$ of $R_{\alpha}$ is minimal, and but it is uniquely ergodic if and only if $\alpha$ is badly approximable, by [48]. Then, in [49], it is proved that skewing a badly approximable rotation by a finite group action over any finite number of intervals with rational endpoints still provides a uniquely ergodic skew product. However, by [19], there exist examples of $\mathbb{Z} / 2 \mathbb{Z}$ skew products of badly approximable rotations that are minimal and not uniquely ergodic, with the skewing function being defined over a finite number of intervals.

In most examples considered in [48, 49], intervals have lengths that do not belong to $\mathbb{Z} \alpha+\mathbb{Z}$. We consider here the complementary case of Sturmian shifts. We can then apply Corollary 7.12 when they are furthermore assumed to be generated by a substitution, such as exemplified below. Note also that substitutive Sturmian shifts have been characterized in [50, 22] (in particular the parameter $\alpha$ of the underlying rotation is quadratic).

Example 7.13. Let $X$ be the Fibonacci shift over $\{0,1\}$. We consider the skew product $\mathbb{Z} / m \mathbb{Z} \rtimes X$, where $\varphi$ is the morphism $\{0,1\}^{*} \rightarrow \mathbb{Z} / m \mathbb{Z}$ given by $0 \mapsto 0$, $1 \mapsto 1$. In particular, one has $\varphi^{(n)}(x)=\left|x_{0} \cdots x_{n-1}\right|_{1}$ modulo $m$ for $n \geq 0$. By Corollary $7.12, \mathbb{Z} / m \mathbb{Z} \rtimes X$ is uniquely ergodic, which yields equidistribution results on the congruence of the number of visits of $R_{\alpha}$ to the interval $[0, \alpha)$, where $\alpha=$ $\frac{\sqrt{5}-1}{2}$. In other words, for every $x \in X, k \in \mathbb{Z} / m \mathbb{Z}$ and $a \in\{0,1\}$, one has:

$$
\frac{1}{N} \operatorname{Card}\left\{0 \leq n \leq N-\left.1| | x_{0} \cdots x_{n-1}\right|_{a} \equiv k \quad \bmod m\right\} \rightarrow \frac{1}{m}
$$

or in other words,
$\frac{1}{N} \operatorname{Card}\{0 \leq n \leq N-1 \mid \operatorname{Card}\{i \mid 0 \leq i<n, i \alpha \in[0, \alpha)\} \equiv k \quad \bmod m\} \rightarrow \frac{1}{m}$.
Finally, the next example concerns the work of Jager and Liardet [29].
Example 7.14. Let $X$ be the Fibonacci shift $X$ over the alphabet $\{1,2\}$. We consider an example of a skew product with a non-Abelian skewing group, namely $G(2)=$ $\mathrm{GL}(2, \mathbb{Z} / 2 \mathbb{Z})$, i.e. the group of $2 \times 2$ matrices with entries in $\mathbb{Z} / 2 \mathbb{Z}$ and determinant 1 (the example could also be carried out with $G(m)$ for arbitrary $m \geq 2$, but we treat only the case $m=2$ for simplicity).

The group $G(2)$ is isomorphic to the group of permutations $S_{3}$. Let

$$
\varphi: A^{*} \rightarrow G(2), \quad k \mapsto\left(\begin{array}{cc}
0 & \frac{1}{k} \\
1 & k
\end{array}\right)
$$

where $\bar{k}$ stands for the congruence class of the integer $k$ modulo 2. The map $\varphi:\{0,1\}^{*} \rightarrow G(2)$ is onto. We consider the skew product $G \rtimes X$ in relation to
equidistribution properties modulo 2 for convergents of continued fraction expansions, by comparing with [29], which handles the case of a random real number; see also [47, 37, 15] for related works.

Let $x=\left(x_{n}\right)_{n} \in X$. Consider the real number in the unit interval $[0,1]$ that admits $\left(x_{n}\right)_{n \geq 1}$ as its sequence of partial quotients and let $\left(p_{n}(x) / q_{n}(x)\right)_{n}$ stand for the associated sequence of rational approximations. One has $q_{-1}(x)=0, p_{-1}(x)=1$, $q_{0}(x)=1, p_{0}(x)=0$, and for all positive $n, q_{n+1}(x)=x_{n+1} q_{n}(x)+q_{n-1}(x)$ and $p_{n+1}(x)=x_{n+1} p_{n}(x)+p_{n-1}(x)$. For $n \geq 0$, one has

$$
\varphi^{(n)}(x)=\left(\begin{array}{cc}
p_{n-1}(x) & p_{n}(x) \\
q_{n-1}(x) & q_{n}(x)
\end{array}\right)
$$

By Corollary 7.12, the skew product $G(2) \rtimes X$ is uniquely ergodic. We deduce the following equidistribution results for the sequence

$$
\left(\begin{array}{ll}
p_{n-1}(x) & p_{n}(x) \\
q_{n-1}(x) & q_{n}(x)
\end{array}\right)
$$

in the group $G(2)$. This is the counterpart of (3.11) from [29]. For every $k=1,2$ and for every $x \in X$

$$
\left.\begin{array}{l}
\lim _{N \rightarrow \infty} \frac{1}{N} \operatorname{Card}\left\{1 \leq n \leq N \mid q_{n}(x) \equiv 0\right. \\
\bmod m\}=\frac{1}{3} \\
\lim _{N \rightarrow \infty} \frac{1}{N} \operatorname{Card}\left\{1 \leq n \leq N \mid q_{n}(x) \equiv 1\right.
\end{array} \quad \bmod m\right\}=\frac{2}{3} .
$$

In fact, the distribution in the group $G(2)$ of the sequence of continued fraction convergents $\left(p_{n} / q_{n}\right)_{n}$ whose sequence of partial quotients is given by elements of the Fibonacci shift $X$ behaves like that of a random irrational number. We recover the well-known fact that certain residue classes are attained more frequently than others. This statement can be considered a modulo $m$ counterpart of Lévy's theorem stating that $\lim _{n} \frac{\log q_{n}}{n}=\frac{\pi^{2}}{12 \log 2}$ a.e. Higher-dimensional continued fractions can be considered via skew products for primitive dendric shifts on larger alphabets, such as codings of interval exchanges; consider for instance the Jacobi-Perron algorithm whose equidistribution properties modulo $m$ are studied in [10] for random numbers.

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[^0]:    ${ }^{1}$ The term cobounding appears for instance in a paper by Baggett et al. [4]; the terms transfer function and intertwining have also been used with similar meanings [4, 3, 43].

