A CLASSIFICATION OF NONEXPANSIVE
BRATTELI-VERSHIK SYSTEMS

Karl Petersen[✉][*][1] and Sandi Shields[✉][2]

[1]Department of Mathematics
CB 3250 Phillips Hall, University of North Carolina, Chapel Hill, NC 27599 USA

[2]Department of Mathematics
College of Charleston, 66 George St., Charleston, SC 29424-0001 USA

(Communicated by Alejandro Maass)

Abstract. We study simple, properly ordered nonexpansive Bratteli-Vershik (BV) systems. Correcting a mistake in an earlier paper, we redefine the classes standard nonexpansive (SNE) and strong standard nonexpansive ($SSNE$). We define also the classes of very well timed and well timed systems, their opposing classes of untimed and very untimed systems (which feature, as subclasses of "Case (2)", in the work of Downarowicz and Maass as well as Hoynes on expansiveness of BV systems of finite topological rank), and several related classes according to the existence of indistinguishable pairs (of some "depth") and their synchronization ("common cuts"). We establish some properties of these types of systems and some relations among them. We provide several relevant examples, including a problematic one that is conjugate to a well timed system while also (vacuously) in the classes "Case (2)". We prove that the class of all simple, properly ordered nonexpansive BV systems is the disjoint union of the ones conjugate to well timed systems and those conjugate to untimed systems, thereby showing that nonexpansiveness in BV systems arises in one of two mutually exclusive ways.

1. **Introduction.** Bratteli-Vershik (BV) systems present visually and combinatorially the hierarchical mechanisms that drive measure-preserving and topological dynamical systems. Vershik [14,15] and Herman-Putnam-Skau [12] showed that every measure-preserving system and every minimal homeomorphism on the Cantor set is isomorphic (in the appropriate sense) to a BV system. As with any system, it is useful, when possible, to code the dynamics as a subshift, so that the methods of symbolic dynamics and formal languages can be brought into action. This is possible exactly when the system is expansive (in the topological setting), or essentially expansive (in the measure-preserving setting; see [1]). There has been considerable progress on the question of when BV systems are expansive or not, or when a sequence of morphisms is recognizable: [1–3, 6, 8, 9], for example. Here we explore the reasons why a BV system might be nonexpansive. We focus on BV systems that are simple (so the topological dynamical system is minimal) and properly ordered (there is a unique minimal path and a unique maximal path), but

not necessarily of finite topological rank (conjugate to one with a bounded number of vertices per level). Our main result, Theorem 5.15, states that for such systems nonexpansiveness arises in one of two mutually exclusive ways, to which we assign the terms *well timed* (Definition 5.3 (1)) and *untimed* (Definition 5.3 (5)).

Many nonexpansive systems are conjugate to odometers, which can be represented by diagrams of bounded width, or seem to be similar to the Gjerde-Johansen example [11, Figure 4], which has unbounded width and is not conjugate to an odometer. For brevity we will refer to this system, which motivated much of this investigation, as the *GJ example*. In an earlier paper [8], abstracting key properties of this example, the authors (including us) proposed a class of systems (standard nonexpansive, SNE) as a model for how nonexpansivesness can arise in BV systems. Proposition 5.4 of [8] asserted that every standard nonexpansive system, according to the definition given there, has unbounded width and cannot be conjugate to an odometer. Example 3.8 below shows that neither of these properties is guaranteed under the old definition of SNE; that definition was flawed, being based on the assumption that two paths with the same sequence of ordinal edge labels had to move simultaneously to new vertices. We cannot even guarantee that SNE systems, as originally defined, are nonexpansive. Here we correct those mistakes by providing a new definition of SNE (Definition 3.5), as well as its invariant version $SSNE$ (Definition 3.6). Nonexpansiveness now follows from Proposition 3.9, and Proposition 3.12 and Theorem 3.13 show that any system satisfying the new definition of SNE is not conjugate to any odometer and has unbounded width.

In Section 4 we study the GJ example in detail, to prove that it is standard nonexpansive and to show how a slight modification ruins this property. For the latter, we use the technique of splitting j-symbols from [6] and prove that the 2-coding of every path in the GJ example is aperiodic (Observation 4.4) and the related fact that a certain factor system gives rise to a recognizable family of morphisms (Observation 4.5).

To understand the possible sources of nonexpansiveness, we define two types of systems, well timed or untimed, as well as their subclasses very well timed and very untimed (Definition 5.3). The GJ example is very well timed. The untimed systems are related to what Downarowicz and Maass called "Case (2)" in [6]. Our Theorem 5.15 states that the family of all nonexpansive systems is the disjoint union of the class of systems which are conjugate to some well timed system and the class of those conjugate to some untimed system. Proving this theorem requires a thorough study of these and related properties, which we carry out in Section 5. Hoynes [13] used a slightly different "Case (2)", and both [6] and [13] used telescoping to arrive at stronger properties. Definition 5.3 states the definitions of these properties and a few of the others that would arise from permuting or negating the quantifiers concerning depth and cuts. Proposition 5.7 clarifies what happens to depth and cuts under telescoping. Working towards the proof of Theorem 5.15, we characterize in Proposition 5.10 the systems that are conjugate to some well timed system as those that are "weakly well timed". We establish relations among these various types of systems. Example 5.6 shows that odometers can be either untimed but not very untimed, or very untimed, depending on how they are presented. Example 5.13 is a system that is both weakly well timed and (vacuously) satisfies the property in each "Case (2)"; this indicates that, as written, the proofs of the main theorem of [6, 13] are slightly incomplete.

In Section 6 we describe the possible diagrams for bounded width very untimed systems: they are all "kite shaped", like the example in Figure 3. In the final section we mention several questions suggested by the foregoing, for example whether there are any untimed systems that have unbounded width and whether every well timed system is conjugate to a system that is in SNE.

2. **Some definitions and notation.** We deal with Bratteli-Vershik (BV) systems (X, T). Each system is built on an ordered Bratteli diagram, which is a countably infinite, directed, graded, graph. For each $n = 0, 1, 2, \ldots$ there is a finite nonempty set of vertices V_n. The set V_0 consists of a single vertex, called the "root". The set of edges is the disjoint union of finite nonempty sets $E_n, n \geq 0$, with E_n denoting the set of edges with source in V_n and target in V_{n+1}. We assume that every vertex has at least one outgoing edge, and every vertex other than the root has at least one incoming edge. There can be multiple edges between pairs of vertices. The space X is the set of infinite paths (sequences $x = x_0 x_1 \ldots$, each x_i denoting an edge from level i to level $i + 1$) starting at the root at level $i = 0$. For a path x we denote by $v_i(x)$ the vertex of the path at level i. If $i < j$, we will say that level j is *after* or *later than* level i, which is *earlier* or *before* level j. Since diagrams are often drawn with later levels below earlier ones, we may sometimes say that level j is *below* level i if $i < j$ and use *up* and *down* to refer to relative positions in such diagrams. The space X is a compact metric space when we specify that two paths have distance $1/2^n$ if they agree from levels 0 to n and disagree leaving level n. To avoid degenerate situations we assume that X is homeomorphic to the Cantor set.

The edges entering each vertex are totally ordered by assigning an ordinal in $\{1, 2, \ldots\}$ as a label to each edge. This yields a partial order on the set of infinite paths as follows. Two paths x and y are *comparable* in case they are cofinal: there is a smallest $N > 0$ such that $x_n = y_n$ for all $n \geq N$. In this case $v_N(x) = v_N(y)$, and $x_{N-1} \neq y_{N-1}$; we agree that $x < y$ if $x_{N-1} < y_{N-1}$, and $x > y$ if not. The set of *minimal paths*, meaning those all of whose edges are minimal into all of their vertices, will be denoted by $X_{\min}$, and similarly the set of *maximal paths* will be denoted by $X_{\max}$. The Vershik map T is defined from the set of nonmaximal paths to the set of nonminimal paths by mapping each path x to its *successor*, the smallest $y > x$. We assume that the diagrams are *properly ordered*, which means that there is a unique minimal path and a unique maximal path. This implies that the diagram is *perfectly ordered: the Vershik map* extends to a homeomorphism T on the set X of all infinite paths from the root by mapping the unique maximal path to the unique minimal path. (See [4, 5]).

The system is called *minimal* if every orbit is dense, and it is called *simple* if it has a telescoping for which there are complete connections between adjacent levels. It can be proved that a properly ordered system is minimal if and only if it is simple.

We will say that two systems are *conjugate* if there is an equivariant homeomorphism from one to the other that sends minimal points to minimal points. The relation of *diagram equivalence* is the one generated by graph isomorphism and telescoping. Herman-Putnam-Skau [12, Section 4] (see also [10, p. 70 and p. 72, Theorem 3.6]) characterized conjugacy of minimal pointed topological dynamical systems defined by properly ordered (called there "essentially simple") BV diagrams (with unique minimal points), as follows: two systems are conjugate if and only if their diagrams are equivalent, and this happens if and only if there exists a

diagram Z that telescopes on odd levels to a telescoping of one of the diagrams and on even levels to a telescoping of the other.

We emphasize that in the following, unless stated otherwise, every *system* is a BV system defined by a properly ordered, simple diagram, and every *conjugacy* is an equivariant homeomorphism between two such systems that sends the unique minimal point in one system to the unique minimal point in the other (but we may include reminders about these hypotheses anyway). For convenience we may use the same symbol, such as X or Y, to denote a diagram as well as the system that it defines, and we may also use the same symbol, such as T, for Vershik maps on different systems, or suppress it entirely. By "pair" we mean a pair of distinct elements, unless stated otherwise. The *width* of a level is the number of vertices at that level. Recall that in [6] the *topological rank* of a system (X, T) is defined to be the minimum among all BV systems (Y, T) conjugate to (X, T) of the supremum of the widths of the levels of (Y, T). For further terminology and background, see [7, 8, 10, 12] and the references cited there.

For each $k \geq 1$ denote by A_k the finite alphabet whose elements are the finite paths (segments, strings of edges) from the root to level k. For each $a \in A_k$, the set $[a] = \{x \in X : x_0 \dots x_{k-1} = a\}$ is a clopen *cylinder set*, and $\mathcal{P}_k = \{[a] : a \in A_k\}$ is a partition of X into clopen sets. The map $\pi_k : X \to A_k^{\mathbb{Z}}$ is defined by $(\pi_k x)_n = a$ if and only if $T^n x \in [a]$. The doubly infinite sequence $\pi_k x$ is called the *k-coding of x*. Let $\Sigma_k = \pi_k X$, and denote by σ the shift transformation on $A_k^{\mathbb{Z}}$. Then $\pi_k : (X, T) \to (\Sigma_k, \sigma)$ is a *factor map*: it is continuous, onto, and it commutes with the transformations.

Definition 2.1. For each vertex v at level $k \geq 1$, denote by $P(v)$ the set of paths from the root to v. The elements of $P(v)$ are assigned *ordinal path labels* according to the lexicographic ordering defined by the edge ordering. Define the *dimension* of the vertex v to be $\dim v = |P(v)|$, and concatenate the elements of $P(v)$, in their lexicographic order, as $p_1 \dots p_{\dim v}$ (so that $[p_{j+1}] = T[p_j]$ for $j = 1, \dots, \dim v - 1$). We call the string $p_1 \dots p_{\dim v}$ on symbols from the alphabet A_k the *k-basic block at v* and denote it by $B_k(v)$. Each path from the root to level k determines, by truncation, for each $i < k$ a unique path from the root to level i, so there is a natural factor map $A_k \to A_i$ which converts $B_k(v)$ to a string, which we denote by $B_i(v)$ and call the *i-basic block at v*, of the same length on the alphabet A_i.

Definition 2.2. The *coding by vertices at level $j < n$ of a vertex w at level n*, denoted by $C_j(w)$, is defined as follows. List in their order in the diagram the paths entering w from vertices at level j as $\{p_1, \dots, p_m\}$ and denote the source of p_i by $v_i, i = 1, \dots, m$. Then $C_j(w) = v_1 \dots v_m$. (For $j = n - 1$, this is the "morphism read on $\mathcal{V}_n$" in [7, p. 342]).

Definition 2.3. We say that a BV system is *nonexpansive* (abbreviated NE) if for every $k \geq 1$ there exists a pair of paths with the same k-coding (by finite paths from level 0 to level k).

Thus a system is expansive if and only if there is a $k \geq 1$ such that the map $\pi_k : (X, T) \to (\Sigma_k, \sigma)$ is injective (and hence a conjugacy with its image).

Definition 2.4. We say two paths x and x' are *depth $k \geq 0$* if they have the same k-coding but not the same $(k + 1)$-coding.

If two paths have the same k-coding, then they agree from the root to level k.

For any pair of (distinct) paths x and x' there exists a j such that x and x' do not have the same j-coding. Hence, telescoping can be used to convert a nonexpansive BV system into one with the property that for every $k \geq 1$ there exists a depth k pair of paths.

Definition 2.5. We say two paths x and x' have a *(common) j cut* if there is an integer m such that the initial segments of $T^m x$ and $T^m x'$ are minimal into level j. Then we say that the pair has a cut *at time m*.

We say that two paths *differ at level j* if they follow different edges into level j. Note that if two paths x and x' *differ* at level $k+1$ and have a $k+1$ cut, then m can be chosen so that $T^m x$ and $T^m x'$ are minimal into *distinct* vertices at level $k+1$. (For detailed explanation, see top of page 7 in [13]).

3. Standard nonexpansive systems.

Definition 3.1. Let $k \geq 1$ and $n > k$. We say that two vertices v, w at level n are *k-equivalent*, and write $v \sim_k w$, if they are *strongly uniformly ordered with respect to level k*, meaning that the set of paths from vertices at level k to v is order isomorphic with the set of paths from vertices at level k to w. Equivalently, the k-basic blocks at v and w are identical. (In [8] a *uniformly ordered* level $n+1$ was defined to be a level all of whose vertices had n-basic blocks that were powers of a single block on the alphabet of paths from the root to level n.)

Any pair of vertices that is $(k+1)$-equivalent is also k-equivalent. (If v and w at level $n > k+1$ have equal basic blocks in terms of the paths from the root to level $k+1$, when these blocks are rewritten in terms of paths to level k the results will still be identical.)

Definition 3.2. We say that two paths x and x' are *k-equivalent at level n* if we have $v_n(x) \sim_k v_n(x')$ *and moreover* their paths from the root to level n have *the same ordinal path label*. We say x and x' are *k-equivalent*, and write $x \sim_k x'$, if they agree from the root to level k and, for all $n > k$, x and x' are k-equivalent at level n.

Remark 3.3. Recall that for a path x and $n \geq k \geq 1$, the k-basic block $B_k(v_n(x))$ lists in their assigned order the truncations to levels from 0 to k of paths from the root to $v_n(x)$, say as $q_1 \ldots q_{\dim(v_n(x))}$ (see Definition 2.1). Suppose that i is the index in $[1, \dim(v_n(x))]$ for which q_i is the initial segment of x from the root to level n. This information can be conveyed by writing out the string $B_k(v_n(x))$ with a "dot" immediately preceding q_i. For example, if x follows only minimal edges to level n, then $B_k(v_n(x)) = .q_1 \ldots q_{\dim(v_n(x))}$. Two paths x, x' are k-equivalent at level $n > k$ if $B_k(v_n(x)) = B_k(v_n(x'))$ and, informally, these two basic blocks have the "dot" in the same place.

Example 3.4. We illustrate some of these definitions in Figure 1. The diagram D in this figure has two vertices, a and b, at level 1 and three vertices at every level $j \geq 2$. We identify each vertex at level 1 with the path of length 1 (edge) from the root to that vertex. The diagram is stationary after level 2. Each vertex is connected to every vertex at the previous level by exactly one edge. We shall consider two different orderings on D.

First, for all $j \geq 2$ and every vertex v at level j, order the edges with target v in increasing order from left to right. Then for any $k < j$, all pairs of vertices at level
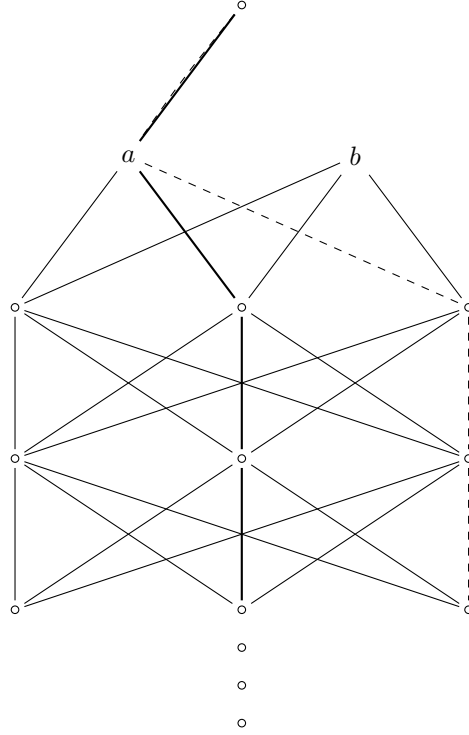
FIGURE 1. An illustration of some of the definitions

j are strongly uniformly ordered with respect to level k.

The 1-basic blocks for the two paths x and x' (indicated by the bold and dashed paths, respectively) at the first four levels after the root are as follows.

$$
\begin{aligned}
B_1(v_1(x)) &= .a \\
B_1(v_1(x')) &= .a \\
B_1(v_2(x)) &= .ab \\
B_1(v_2(x')) &= .ab \\
B_1(v_3(x)) &= ab.abab \\
B_1(v_3(x')) &= abab.ab \\
B_1(v_4(x)) &= (ab)^3 ab.abab(ab)^3 \\
B_1(v_4(x')) &= (ab)^3 (ab)^3 abab.ab
\end{aligned}
\tag{1}
$$

In fact, at all levels the 1-basic blocks are the same for both paths. However, after level 2, the dots for the respective paths are not in the same place of their common basic block. Therefore, x and x' are not 1-equivalent. (Hence, they are not k-equivalent for any $k \geq 1$.)

Note: Suppose that at all levels $j \geq 2$, we add a fourth vertex $v(j,4)$ that is connected to every vertex at level $j-1$ by a single edge and then order the edges with target $v(j,4)$ from left to right. In the new diagram, x' would no longer be maximal after level 2. In other words, x' would no longer be in the orbit of a maximal path. In this case, the 1-coding of both paths would be the same bi-infinite periodic

sequence $(ab)^\infty.ab(ab)^\infty$. (This shows that two paths with the same 1-coding need not be 1-equivalent.) In fact, x and x' would be depth 2 with a common 1-cut. While instructional, drawing such a diagram with all its intersecting edges is messy. So we chose to illustrate k-equivalence, or lack thereof, with a simpler diagram.

We now consider a different ordering on D that we obtain from the ordering considered above by swapping labels 2 and 3 on edges with target $v(j,3)$, for all $j > 2$. This gives

$$
\begin{aligned}
B_1(v_1(x)) &= B_1(v_1(x')) = .a \\
B_1(v_2(x)) &= B_1(v_2(x')) = .ab \\
B_1(v_3(x)) &= B_1(v_3(x')) = ab.abab \\
B_1(v_4(x)) &= B_1(v_4(x')) = (ab)^3ab.abab(ab)^3
\end{aligned}
\tag{2}
$$

In fact, with this new ordering on D the 1-basic blocks at every level are the same with the dot in the same place for both paths. Therefore, x and x' are 1-equivalent with the same 1-coding $(ab)^\infty.ab(ab)^\infty$.

Definition 3.5. We say that a nonexpansive BV system is *standard nonexpansive (SNE)* if for every $k \geq 1$ there is a pair of k-equivalent paths.

Definition 3.6. We say that two paths x and x' are *k-same*, and write $x \approx_k x'$, if $T^j x \sim_k T^j x'$ for all $j \in \mathbb{Z}$. We say that a system is *strong standard nonexpansive (SSNE)* if for every $k \geq 1$ there is a pair of distinct k-same paths.

Remark 3.7. The paper [8] introduced a different definition of standard nonexpansive, which we here call the "old definition": For every k there should exist a pair of paths that agree from the root to a level $n > k$, have the same sequence of edge labels, and have the same k-basic blocks at all levels after k. In the new definition (3.5) we replace the requirement that the sequences of edge labels be the same with the stronger requirement that the two paths always have the "dot" in the same place of their equal k-basic blocks.

Proposition 5.4 of [8] asserted that every standard nonexpansive system, according to the definition given there, has unbounded width. The following example shows that this is not correct. In Theorem 3.13 below we show that the new definition of SNE given here does suffice to guarantee unbounded width.

Example 3.8. Figure 2 shows part of a diagram with ordinal labels on edges of two paths (x and x') indicated by small numbers. The figure displays the codings of vertices by vertices at the previous level, which indicate the connections and edge orderings. This example satisfies the old definition of SNE, but not the new one. It has bounded width and is conjugate to an odometer (since after telescoping to the levels with exactly two vertices, all levels are uniformly ordered, see [8]); so it is a counterexample to Prop 5.4 of [8] and the assertion in its proof that any system conjugate to an odometer cannot have pairs of paths with the same edge labels.

In this example (see Figure 2), for which levels $0-2$ are not shown, the dark path (x) and the dotted path (x') are identical from the root to the vertex labeled b at level 3, where they diverge. They continue below with the same sequence of ordinal edge labels. At level 1 (not shown) there are two vertices, u and v, and at level 2 there are four vertices, with codings by vertices of the previous level u^2, u^3, uv^2, v^2u. The vertices u and v connect to a and b via edges in the order that leads to identical codings by vertices u^2uvvu^3 at a and u^3vvuu^2 at b. At all levels 2 and later, the paths have identical codings by vertices on the symbols u and v, but the "dots" for

the two paths are in different places after level 4. After level 0, the diagram repeats levels and edges with period two, so it has bounded width.

Thus these two paths have the same sequence of edge labels and the same basic blocks at corresponding vertices, so they conform to the old definition of SNE. But the dots in their basic blocks are not in the same place, so they do not satisfy the new definition.

To find such pairs of paths for values of k larger than 1, use the periodicity of the diagram. For example, to deal with 1 replaced by 3 (and hence also by 2), we may form a pair of paths that is identical from the root to the vertex labeled B at level 5, where it splits into two paths following the same sequence of edge labels as before $(22121212\ldots)$. These paths always enter vertices with the same codings by level 3 vertices a, b, but not with the dot in the same place (after level 6). Then for 3 replaced by 5 (and hence also 4), take two paths that follow identical edges from the root to the vertex D at level 7, where they split and follow the same sequence of edges $22121\ldots$. These paths always enter vertices with the same codings by level 5 vertices A, B, and hence also by the vertices at level 4, but not with the dots in the same place. Etc.

The strict synchronizing structure of k-equivalent pairs of paths guarantees that they cannot be separated by their k-codings.

Proposition 3.9. *In any system, if two paths are k-equivalent at infinitely many levels, then they have the same k-codings.*

Proof. Suppose that x, x' is a pair of distinct paths for which there is an infinite increasing sequence (n_j) such that for each j the paths x, x' are k-equivalent at level n_j. Their (identical) k-basic blocks at their n_j-level vertices $v_{n_j}(x), v_{n_j}(x')$ have lengths increasing to infinity, $j \geq 1$. If the lengths of the segments of the blocks both to the left and right of their dots increase unboundedly, x and x' will have identical k-codings. Suppose that the segments to one side of the dot, for example the left, stay bounded. Then there are $m \in [0, \infty)$ and j_0 such that the segment to the left has length m at all levels n_j for $j \geq j_0$. At these levels $T^{-m}x, T^{-m}x'$ are minimal paths from the root to the vertices $v_{n_j}(x), v_{n_j}(x')$. Since n_j is arbitrarily large, $T^{-m}x$ and $T^{-m}x'$ consist entirely of minimal edges, so both equal the minimal path $x_{\min}$, contradicting our assumption that the paths are distinct. Analogously, using uniqueness of the maximal path, the segments of the basic blocks to the right cannot stay bounded. $\square$

Corollary 3.10. *For systems under our standing assumptions (BV systems, each defined by its own simple, properly ordered diagram), standard nonexpansive implies nonexpansive.*

Proposition 3.11. *If two paths x and x' are k-equivalent, then for every $n > k$ they have an n cut, and there is a $j \geq k$ such that they are depth j. Hence, in any system, if two k-equivalent paths are depth $n \geq k$, then they have an $n + 1$ cut.*

Proof. For any path and any $n > k$, the position of the dot in the k-basic block at level n represents the smallest number of applications of T^{-1} applied to that path necessary to produce a path that is minimal into level n. Since x and x' are k-equivalent, the dot is in the same position for their k-basic blocks at level n, so the paths x and x' have an n cut.
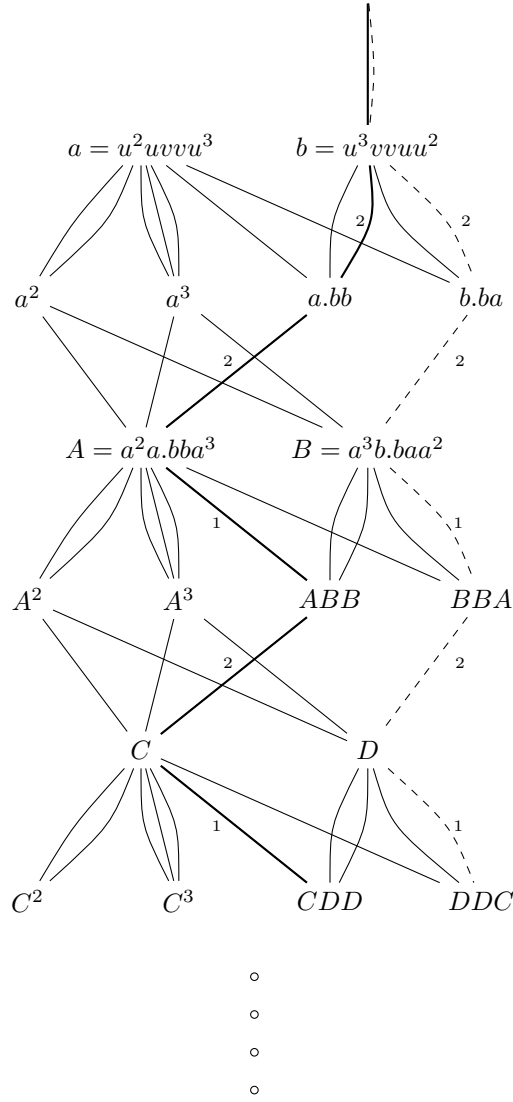
FIGURE 2. Part of a diagram whose system satisfies the old defi-
nition of SNE but not the new one

Given paths x and x' in a system that are k-equivalent, by Proposition 3.9 they
have the same k-codings. There exists a largest $j \geq k$ such that for all $m \in \mathbb{Z}$, $T^m x$
and $T^m x'$ agree to level j. It follows that x and x' are depth j. $\qquad \square$

Note that the converse of each statement in Proposition 3.11 is not true: it can
happen for a depth k pair and some $n > k$ that the k-basic block at $v_n(x)$ is a proper
prefix of the k-basic block at $v_n(x')$, and then the paths would not be k-equivalent
but could still have an n cut.

The following Proposition will be extended in Theorem 5.14.

Proposition 3.12. *No SNE system can be conjugate to any odometer.*

Proof. Suppose that (X, T) is a system that is conjugate to an odometer. By [8, Theorem 5.3] it has a telescoping that has infinitely many uniformly ordered levels. We claim that for $k = 1$ no pair in the telescoped system can be k-equivalent, so the telescoped system in fact cannot be SNE.

Suppose that x, x' is a 1-equivalent pair in the telescoped system. Since $x \neq x'$ but their first edges are equal, $x_0 = x'_0$, the pair first disagrees at some level $j > 1$, that is to say, they follow different edges from level $j - 1$ to level j. If they enter the same vertex at level j along these different edges, they cannot have their dots in the same place. So the paths are at different vertices at level j, and then, by induction, they must be at different vertices at each level n for all $n \geq j$.

Look at the first uniformly ordered level $n > j$. At the vertices $v_n(x)$ and $v_n(x')$, because of 1-equivalence we have $(n - 1)$-basic blocks of the same length, and then because of uniform order these $(n - 1)$-basic blocks are equal. Since x, x' are at different vertices at level $n - 1$, their dots are at different places in the $(n-1)$-basic blocks at $v_n(x), v_n(x')$. When these $(n - 1)$-basic blocks are expanded to 1-basic blocks, the dots for x, x' will be at different places, contradicting 1-equivalence of x and x'. $\qquad\square$

The following Theorem follows directly from the preceding Proposition combined with the main result of [6], but we give a direct proof here in order to re-establish Proposition 5.4 of [8] in our context, with the new definition of SNE.

Theorem 3.13. *SNE implies unbounded width.*

Proof. Starting with $j_0 = 1$, using Proposition 3.11, we can find two paths $x^{(0)}, y^{(0)}$ that are j_0-equivalent and follow different edges into some level $j_1 > j_0$. As above, because they are j_0-equivalent, they must pass through different vertices at level j_1, and hence they pass through different j_0-equivalent vertices for all $j \geq j_1$.

If they pass through j_1-equivalent vertices at infinitely many levels, they must have the dot in the same place in their basic blocks at those levels, because when the j_1-basic blocks are expanded to j_0-basic blocks, they are the same and have the dot in the same place, so the dot had to be in the same place for the j_1-basic blocks. Then $x^{(0)}$ and $y^{(0)}$ would be j_1-equivalent at these infinitely many levels, and so by Proposition 3.9 would have the same j_1-coding, but they do not. Thus for all large enough n, $x^{(0)}$ and $y^{(0)}$ pass through vertices that are j_0-equivalent and not j_1-equivalent.

Continuing, given $K \geq 1$, for each $k = 0, \ldots, K - 1$ we can find N, integers $j_0 < j_1 < \cdots < j_K$, and pairs $x^{(k)}, y^{(k)}$ that for each $n \geq N$ and $k = 0, \ldots, K - 1$ pass through vertices that are j_k-equivalent and not j_{k+1}-equivalent:

$$v_n(x^{(k)}) \sim_{j_k} v_n(y^{(k)}) \quad \text{while} \quad v_n(x^{(k)}) \nsim_{j_{k+1}} v_n(y^{(k)}). \tag{3}$$

But these pairs of vertices at any level $n \geq N$ are all different, because not j_k-equivalent implies not j_{k+1}-equivalent. If there are at most R vertices at every level and $K > R^2$, this is a contradiction. $\qquad\square$

4. **Modifying the GJ example to ruin the property SNE.** We will show that the example of Gjerde and Johansen [11, Figure 4], which has unbounded width and is not conjugate to any odometer, is SNE (even $SSNE$) by specifying for any k exactly which pairs of paths are k-equivalent. We will also determine for any k which pairs are depth k.

Denote the system in the GJ example by X. Given $j > 0$ there are $2j$ vertices in the diagram at level j. For any $i \leq 2j$, let $v(j, i)$ denote the i'th vertex from the left (beginning with $i = 1$) in level j. To specify the edge ordering, begin with edges into each vertex ordered left to right. Then for all $j > 1$ and all odd $i > 2$, on the edges entering $v(j, i)$ swap the labels i and $i - 1$. For $i = 1, 2, \ldots$ and $j > i$, paths that at level j pass only through vertices $v(j, 2i)$ or $v(j, 2i + 1)$ will be said to constitute the i'th Morse component $MC(i)$ of the diagram. In particular, $MC(i)$ begins at level $i + 1$.

Observation 4.1. *(1) For any $i \geq 1$, at any level of $MC(i)$ the i-basic blocks are the same (at both vertices), as can be verified by writing them out. It can also be verified that the $(i + 1)$-basic blocks at the vertices of $MC(i)$ differ after level $i + 1$.*
(2) In the diagram for X, each vertex is connected to every vertex at the previous level by exactly one edge, so all vertices at any level have the same dimension. From this it can be argued inductively that for every $j \geq 1$ two paths have the same ordinal path label from the root into level j precisely when they have the same sequence of edge labels into level j.

Proposition 4.2. *The following statements hold for the GJ example, X.*
(1) If two edges at level $n \geq 2$ have the same label, then they have different targets. If they also have different sources, then they are contained in $MC(j)$ for some $j < n$.
(2) A pair of paths x, x' with the same sequence of edge labels first differ at level $j \geq 1$ if and only if they enter a Morse component at level j and remain in that Morse component at all levels j and higher, never again meeting the same vertex.
(3) If two paths have the same sequence of edge labels and first differ at level j, then for some $k < j$ they enter $MC(k)$ at level j and are k-equivalent and not $(k + 1)$-equivalent. Conversely, if two paths are k-equivalent and not $(k + 1)$-equivalent, then they have the same sequence of edge labels and enter $MC(k)$ at the level where they first differ.
(4) For any $k \geq 2$, a pair of paths is depth k if and only if those paths are k-equivalent but not $(k + 1)$-equivalent. (Equivalently, in view of (3), two paths are depth k if and only they have the same sequence of edge labels and enter $MC(k)$ at the level where they first differ.)

Proof. Proof of (1): Fix $n \geq 2$. All edges with source $v(n, 1)$ are labeled 1 and all edges with source $v(n, 2n)$ are labeled $2n$. For all $j = 1, ..., n - 1$, all edges with source $v(n, 2j)$ are labeled $2j$, with the exception of the single edge between $v(n, 2j)$ and $v(n + 1, 2j + 1)$, which is labeled $2j + 1$. Likewise, all edges with source $v(n, 2j + 1)$ are labeled $2j + 1$, with the exception of the single edge between $v(n, 2j + 1)$ and $v(n + 1, 2j + 1)$, which is labeled $2j$. If two edges between levels n and $n+1$ have the same label 1 or $2n$, they must have the same source (and different targets). If two edges between levels n and $n + 1$ have the same label and different sources, then for some $j = 1, ..., n - 1$ one of these edges has source $v(n, 2j)$, the other has source $v(n, 2j + 1)$, and the two edges have different targets and lie in $MC(j)$.

Proof of (2): Two paths with the same sequence of edge labels cannot first differ into level 1, because the edges out of $v(1, 1)$ all have label 1, while the edges out of $v(1, 2)$ all have label 2; so the paths first differ into level $j \geq 2$, arriving at different vertices at level j. Leaving level j, the two paths traverse edges with the same label and different sources. By part (1), both of these edges are in the same Morse component and have different targets. Hence, both paths also traverse

different edges with the same label and different sources into level $j+2$. By repeated applications of part (1) we get that the two paths enter a Morse component at level j and remain in that Morse component at all levels j and higher.

Conversely, if a pair of paths x, x' with the same sequence of edge labels enters a Morse component at level j and remains in that Morse component at all levels j and higher, then x and x' first differ at level j. This is because, from above, the pair cannot differ before entering a Morse component, so they meet different vertices at a first level $n > j$. But since two edges in a Morse component with the same label have different sources, $n = j$. In fact, x and x' do not ever meet the same vertex in that Morse component, since the edges entering any vertex have distinct labels.

Proof of (3): If paths x and x' have the same sequence of edge labels, then by (2) x and x' enter a Morse component $MC(k)$ at the first level $j > k$ where they differ, and they will meet distinct vertices in $MC(k)$ at all levels j and higher. Hence, by Observation 4.1 (1), they will have the same k-basic blocks at levels j and higher, whereas their $(k + 1)$-basic blocks at level $j + 1$ will differ. By Observation 4.1 (2), they will have the same ordinal path label from the root into all levels. Thus the dot will be in the same place in their identical k-basic blocks at all levels $k + 1$ and higher: the pair will be k-equivalent but not $(k + 1)$-equivalent.

Conversely, we show that any pair of k-equivalent paths x, x' that are not $(k+1)$-equivalent enter $MC(k)$ at the first level where they differ and have the same sequence of edge labels. By the definition of k-equivalence, such paths agree to level k and their k-basic blocks at levels $k + 1$ and higher are the same with the dot in the same place. In particular, x and x' have the same ordinal path label from level k into any higher level and they agree to level k, so they have the same ordinal path label from the root into any level. Hence, by Observation 4.1 (2), they have the same sequence of edge labels. Denote by j the first level at which x and x' differ (so that $j > k$). Then, by (2), for some $i < j$ both paths enter $MC(i)$ at level j and are contained in that component at all higher levels. By the preceding paragraph, x and x' are i-equivalent but not $(i + 1)$-equivalent. Therefore, $i = k$.

Proof of (4): Suppose that $k \geq 2$ and x, x' is a depth k pair. Then x and x' have the same k-coding and therefore the same 2-coding. As shown later in Example 4.6, this means that x, x' are 2-equivalent, in other words, at all levels after level 2, they have the same 2-basic block with the dot in the same place. At any level $n \geq k$, the k-basic block $B_k(v)$ at any vertex v factors via the 1-block code mentioned in Definition 2.1 onto $B_2(v)$, and the result has the same length with the dot in the same position. It follows that the k-basic blocks at $v_n(x)$ and $v_n(x')$ have equal length and the dot in the same position. Since x and x' have the same k-coding, then their k-basic blocks at all levels must be the same. Therefore, x and x' are k-equivalent. Furthermore, x and x' are not $(k+1)$-equivalent, since at some time in their orbits they follow different paths into level $(k + 1)$ (by definition of depth k), whereas by Proposition 3.9 $(k + 1)$-equivalent paths must have the same $(k + 1)$-coding. Hence, for all $k \geq 2$, any depth k pair is k-equivalent but not $(k + 1)$-equivalent.

Conversely, for all $k \geq 2$, any pair of paths that is k-equivalent has the same k-coding by Proposition 3.9. If that pair is not $(k + 1)$-equivalent, then, it is not i-equivalent for any $i > k$. As just shown, this means it cannot be depth i for any $i > k$. Therefore that pair is depth k. $\qquad\square$

Corollary 4.3. *The GJ example is standard nonexpansive*

Proof. By Part (4) of Proposition 4.2, given $k \geq 2$ two paths with the same sequence of edge labels that enter $MC(k)$ at the level where they first differ are depth k. For example, the path x that passes through the vertices $v(1, j)$ for all $j, 1 \leq j \leq k$, then $v(j, 2k)$ for all $j > k$ and the path y that passes through the vertices $v(1, j)$ for $1 \leq j \leq k$, then $v(j, 2k + 1)$ for all $j > k$, are depth k. $\qquad\square$

The following observations are related to Example 4.6.

Observation 4.4. *In the GJ example, the 2-coding of every path is aperiodic (also sometimes called "nonperiodic").*

Proof. We will show that the forward coding of the minimal path $x_{\min}$ by vertices at level 2, $(v_2(T^j x), j \geq 0)$, is aperiodic. This implies that the 2-coding of $x_{\min}$ is aperiodic, and hence, since the orbit of $x_{\min}$ is dense, the 2-coding of every path in X is aperiodic. Fix a large $n \geq 3$. The idea is to reduce $C_2(v(n, 1))$ to a long initial block of the famous Prouhet-Thue-Morse sequence, which is known to be aperiodic.

For $n \geq 2$ we have

$$C_{n-1}(v(n, 1)) = v(n - 1, 1)v(n - 1, 2)v(n - 1, 3) \ldots v(n - 1, 2n - 2), \qquad (4)$$

but note that for $n \geq 4$

$$C_{n-2}(v(n - 1, 3)) = v(n - 2, 1)v(n - 2, 3)v(n - 2, 2)v(n - 2,) \ldots v(n - 2, 2n - 4). \quad (5)$$

This switch in order of adjacent symbols occurs at every level $n - 1 \geq 3$. Working towards the expansion of $C_2(v(n, 1))$ as a string on symbols $v(2, m), m = 1, 2, 3, 4$, in Equation 4 replace each $v(n - 1, i)$ by $C_{n-2}(v(n - 1, i))$, then in the result (which is $C_{n-2}(v(n, 1))$) replace each $v(n - 2, i)$ by $C_{n-3}(v(n - 2, i))$, etc., until we finally arrive at $C_2(v(n, 1))$. Note that, reading from left to right in any $C_j(v(n, i)), 2 \leq j < n$, from time to time the symbols $v(j, 2)$ and $v(j, 3)$ switch order.

We will now repeat this process of repeatedly expanding $C_{n-1}(v(n, 1))$, deliberately losing some information at each step, to produce for each $j = n - 1, \ldots, 2$ a string $\tilde{C}_j$ on the alphabet $\{v(j, 2), v(j, 3), 0_j\}$. In Equation 4, replace each $v(n - 1, i)$ for i not equal to 2 or 3 by 0_{n-1}, arriving at a block $\tilde{C}_{n-1}$ on the alphabet $\{v(n - 1, 2), v(n - 1, 3), 0_{n-1}\}$.

Then in $\tilde{C}_{n-1}$ replace each $v(n - 1, 2)$ by $C_{n-2}(v(n - 1, 2)$, each $v(n - 1, 3)$ by $C_{n-2}(v(n - 1, 3))$, each 0_{n-1} by $(0_{n-2})^{2n-4}$, and finally each $v(n - 2, i)$ for i not equal to 2 or 3 by 0_{n-2}, arriving at a block $\tilde{C}_{n-2}$ on the alphabet $\{v(n - 2, 2), v(n - 2, 3), 0_{n-2}\}$. Noting that $|C_{n-2}(v(n - 1, i))| = 2n - 4$ for all i, we have formed a 1-block factor $\tilde{C}_{n-2}$ of $C_{n-2}(v(n, 1))$ on the alphabet $\{v(n - 2, 2), v(n - 2, 3), 0_{n-2}\}$.

Continue analogously until in the end we have a block $\tilde{C}_2$ on the alphabet $\{v(2, 2), v(2, 3), 0_2\}$ which is a symbol-by-symbol (1-block) factor of $C_2(v(n, 1))$.

If the forward coding of $x_{\min}$ by vertices at level 2 were periodic, then there would be a nonempty block P on the symbols $v(2, m)$ ($m = 1, 2, 3, 4$), and a (possibly empty) prefix Q of P, such that for large enough n we would have $C_2(v(n, 1)) = P^k Q$ for some $k \geq 2$. Then $\tilde{C}_2$ would have the form $\tilde{P}^k \tilde{Q}$ with $\tilde{P}, \tilde{Q}$ 1-block images of P, Q.

But note that for each $j = n - 1, \ldots, 3$ a symbol $v(j, 2)$ in $\tilde{C}_j$ expands to a block $0_{j-1}v(j - 1, 2)v(j - 1, 3)0_{j-1}^{2j-5}$ in $\tilde{C}_{j-1}$, while a symbol $v(j, 3)$ in $\tilde{C}_j$ expands to a block $0_{j-1}v(j - 1, 3)v(j - 1, 2)0_{j-1}^{2j-5}$ in $\tilde{C}_{j-1}$. Thus if we ignore the symbol 0_{j-1} in each $\tilde{C}_j$ we see substrings on $\{a = v(\cdot, 2), b = v(\cdot, 3)\}$ that expand according to the

Prouhet-Thue-Morse (PTM) substitution $a \to ab, b \to ba$.

If $\tilde{C}_2$ were of the form $\tilde{P}^k \tilde{Q}$ as above, with n large enough that $k \geq 2$, then deleting the symbol 0_2 from $\tilde{P}$ and $\tilde{Q}$ would present a long initial block of the PTM sequence in the form $P_0^k Q_0$ with P_0, Q_0 blocks on $\{a, b\}$, and $|P_0| \geq 1$ (because $\tilde{C}_2$ contains symbols other than 0_2). But the PTM sequence $abba\,baab\ldots$ is aperiodic, in fact it cannot begin with BB for any block B. Therefore the forward coding of $x_{\min}$ by vertices at level 2 is not periodic, and hence the 2-coding of $x_{\min}$ is not periodic. $\qquad\square$

Observation 4.5. *In the GJ example, identifying at every level $j > 2$ the vertices other than $v(j, 3)$ produces a sequence of morphisms that forms a recognizable family.*

Proof. Denote by $\mathcal{V}_j$ the set of vertices in our diagram at level j. For each $j \geq 2$ define an alphabet $A_j = \{D_j, E_j\}$ and a (many-to-one) map $\phi_j : \mathcal{V}_j \to A_j$ as follows. Assign to $v(j, 3)$ the symbol $D_j = \phi_j(v(j, 3))$, and to each $v(j, i)$ for $i \neq 3$ assign the symbol $E_j = \phi_j(v(j, i))$. When we expand each symbol (vertex) $v(j, i)$ to $C_{j-1}(v(j, i))$ the effect on the ϕ_j-images $(D_j, E_j, D_{j-1}, E_{j-1})$ at both levels produces the morphism

$$E_j \to E_{j-1}^2 D_{j-1} E_{j-1}^{2j-5}, \qquad D_j \to E_{j-1} D_{j-1} E_{j-1}^{2j-4} \tag{6}$$

for all $j \geq 3$, in the sense of concatenation of blocks.

Denoting as usual by A_j^+ the set of nonempty words on the alphabet A_j, Equation (6) defines a sequence of morphisms $\tau_j : A_j \to A_{j-1}^+$, as (for example) in [2]. By keeping track of the position of D_j in codings of paths by images of vertices at level j under ϕ_j one can prove directly that the sequence is *recognizable*, in the sense that every sequence on the alphabet A_j has at most one *desubstitution*, or *factorization*, on the alphabet A_{j+1}. To see this, suppose that we are given a bisequence on the alphabet A_j. When we see a block $F(j, q) = D_j E_j^q D_j$, since the block $D_{j+1} D_{j+1}$ does not appear in the coding of any path by the images of vertices at level $j + 1$, it must be the case that $F(j, q)$ is a subblock, in a uniquely determined position, of

$$\begin{aligned}
E_{j+1} E_{j+1} &= E_j E_j D_j E_j^{2j-3}\ E_j E_j D_j E_j^{2j-3} & \text{if } q = 2j - 1, \\
E_{j+1} D_{j+1} &= E_j E_j D_j E_j^{2j-3}\ E_j D_j E_j^{2j-2} & \text{if } q = 2j - 2, \text{ or} \\
D_{j+1} E_{j+1} &= E_j D_j E_j^{2j-2}\ E_j E_j D_j E_j^{2j-3} & \text{if } q = 2j.
\end{aligned} \tag{7}$$

Thus the coding of an orbit by D_2, E_2 determines its coding by D_j, E_j for all $j > 2$. This also follows from [2, Theorem 5.1], since each of these alphabets has only two elements and each infinite bisequence generated by the sequence of morphisms is aperiodic, by the arguments in the proof of Observation 4.4. $\qquad\square$

Note that the natural sequence of morphisms for the diagram defined by the coding of vertices at each level j beyond some level j_0 by the vertices at level $j - 1$ *cannot* be recognizable, because the GJ system is nonexpansive. (The coding of any path by vertices at some level would then determine the codings at all subsequent levels and hence the entire path.) Moreover, the BV system corresponding to the sequence of morphisms τ_j is of finite topological rank (as well as expansive). In fact it has topological rank 2: it cannot be 1 (conjugate to an odometer) by [8, Theorem 5.3] and Observation 4.4, since for every $n > 1$ the n-coding of the orbit of any path is not periodic.

We show now how the GJ example can be modified to spoil the strict requirements of the definition of $SSNE$ to produce an example of a nonexpansive system that is not even SNE and is not conjugate to any odometer. (By [6] any such example necessarily has unbounded width.) Further, this example is very well timed (see Definition 5.2).

In [6], Downarowicz and Maass introduced a handy way to visualize paths and their orbits in a BV system by means of 3-sidedly infinite arrays of j-symbols. Every vertex v at level $j \geq 0$ has an associated j-symbol which is also labeled v. The j-symbol is a finite rectangular matrix with $j + 1$ rows consisting of subrectangles (i-symbols for $0 \leq i \leq j$), as described in [6, p. 741]. An array represents the entire orbit of a path and its k-codings for all $k \geq 0$. The path itself is indicated by an arrow pointing to the left edge of its time 0 rectangle at level 0, and this arrow indicates, by extending it vertically downward, all the other rectangles (vertices) through which the path passes at levels $n > 0$. We call this extension the *vertical* that corresponds to the path. Further development and use of these arrays can be seen in [3] and [13].

The family of arrays determined by a diagram has a type of consistency called "agreeable" in [3, Def. 3.2]: each j-symbol with a fixed name v has for its first j rows the same concatenation of $(j-1)$-symbols. Conversely, given an agreeable family of arrays, we can construct its unique associated BV diagram (which may or may not be properly ordered). For each vertex v at level j, the names, in order, of the $(j-1)$-symbols comprising the first j rows of its associated j-symbol specify the edges connecting certain vertices at level $j - 1$ to it in a certain order, possibly with repeats. This concatenation of $(j-1)$-symbols determines the $(j-1)$-basic block $B_{j-1}(v)$ at the vertex v, since it lists in order the paths from the root to v. The Vershik map T on the diagram corresponds to sliding each array one "notch", i.e. level-0 rectangle width, to the right. Thus there is a natural correspondence between BV systems, their diagrams, and agreeable families of arrays; in the following we deal with them interchangeably.

Example 4.6. Denote the system in the GJ example [11, Figure 4, p. 1699] by X.

In their proof in [6], Downarowicz and Maass make use of a modification of the Bratteli diagram and hence of all arrays representing orbits of the system. See [6, pp. 743–744, Figure 4] and [13, p. 204]). We will use a version of their splitting technique to ensure that for every $i \geq 1$ and every $j \geq i + 1$ the two vertices at level j in the i'th Morse component have different 1-basic blocks.

For every $j \geq 2$ and every $i < j$, we replace the j-symbol $v_i = v(j, 2i)$ with two symbols v_i' and v_i'' so that $|B_1(v_i')| = |B_1(v(j - 1, 1))|$ and the concatenation of $B_1(v_i')$ and $B_1(v_i'')$ is $B_1(v_i)$. In particular, $B_1(v_i')$ is a proper prefix of $B_1(v_i)$. In the DM arrays, every occurrence of the j-symbol with label v_i is then replaced by two j-symbols labeled v_i' and v_i'' respectively. We leave the j-symbols for all other vertices at level j unchanged, so their 1-basic blocks are the same in the new diagram as they are in X. The result is that the left vertices in every Morse component of X have been split into two vertices, whereas the right vertices remain intact.

Note that for every $j \geq 2$, the modifications at level j extend vertical bars (i.e rectangle boundaries) at level $j - 1$ in the original diagram for X by only one level. None of these bars is extended further by a modification at level $j + 1$. Hence, no new infinitely long vertical lines consisting entirely of rectangle boundaries are created. This means that after all modifications, the system represented by the

diagram remains in the class of properly ordered systems. Moreover, every orbit still meets every vertex, so the system is simple. Also note that since this new system is conjugate to X, it is not conjugate to any odometer.

The new system has the property that at any level $j \geq 2$ and $i = 1, ..., j - 1$, the 1-basic block at the right vertex $v(j, 2i + 1)$ of the i'th Morse component through level j is longer than the 1-basic blocks at the two new vertices. Hence, the right vertex cannot be 1-equivalent to either of these new vertices. We claim that for every $k \geq 1$ paths that were k-equivalent in the old diagram are no longer k-equivalent in the new diagram. This is because k-equivalent paths in the old diagram are also 1-equivalent in that diagram. By Proposition 4.2, any pair of paths that are 1-equivalent in the original diagram are contained in a Morse component at all levels after which they first differ, so they can no longer be 1-equivalent in the new diagram. Then since k-equivalence implies 1-equivalence, the paths are no longer k-equivalent in the new diagram.

The 1-coding of every path in X is periodic with period length 2. However, at every level $j > 2$ the 2-basic block at $v(j, 3)$ differs from the 2-basic blocks at at all other vertices, and as a result the 2-coding is not periodic (see Observation 4.4). We will exploit this property of 2-basic blocks to show that no new 2-equivalent paths are created by our modifications. It then follows that the new system has no 2-equivalent paths, hence is not SNE.

By Prop 3.9, any 2-equivalent pair in the new system has the same new 2-coding. In fact, it is the image of a pair in X that has the same old 2-coding. This is because there is an invertible mapping between the old and the new codings. Specifically, in both the old and the new system each path in X is represented by a vertical in its array. Row 2 of the array displays the 2-coding of the path and the placement of the dot is specified by the vertical. The splitting of rectangles and relabeling at level 2 is reversible and does not change the position of the vertical. So a pair of 2-equivalent paths in the new diagram is the image of a pair in the diagram for X with the same 2-coding. We now show that any such pair in X is 2-equivalent.

Let x, x' be paths in the diagram for X with the same 2-coding. Using the preceding observations about D_j and E_j (see Observation 4.5), we argue inductively that for every $j \geq 3$, any time the orbit of x is minimal from the root to $v(j, 3)$, the orbit of x' is as well (and vice versa). Then because all j-basic blocks at level j have the same length ($\dim v(j, 1)$), any time the orbit of one of these paths changes vertices at level j, the other one does as well. Since x and x' have the same 2-coding, it follows that they must be 2-equivalent at level j (i.e not only do they have the same 2-basic block at level j but their dot is in the same place).

First note that, since x and x' have the same 2-coding, their orbits must meet $v(2, 3)$ always at the same time; in particular, any time the orbit of one of these paths is minimal from the root into $v(2, 3)$ the orbit of the other is as well.

Next fix $j \geq 2$ and assume that the orbits of x and x' are minimal into $v(j, 3)$ always at the same time. Since the number of paths is the same from the root into any vertex at level j, these orbits must be minimal into each vertex at level j always at the same time.

Now suppose that for some m, $T^m x$ is minimal from the root into $v(j + 1, 3)$, so that $\phi_{j+1}(v_{j+1}(T^m x)) = D_{j+1}$ and $T^{m+\dim(v(j,1))}x$ is minimal to $v(j, 3)$, which maps under ϕ_j to D_j. We claim that since the orbits of x and x' always hit $v(j, 3)$ at the same time, we cannot have $v_{j+1}(T^m x') = v(j + 1, i)$ for some $i \neq 3$, i.e. we cannot have $\phi_{j+1}(v_{j+1}(T^m x')) = E_{j+1}$ rather than D_{j+1}. This follows from looking

at the *next* vertices at level $j + 1$ hit by the orbits of x and x'. Since the block $v(j + 1, 3)v(j + 1, 3)$ cannot appear in the coding of any path by vertices at level $j + 1$, the orbit of x next hits $v(j + 1, i)$ for some $i \neq 3$, which has image E_{j+1}, while the orbit of x' next hits a vertex with image either E_{j+1} or D_{j+1}. As seen in the proof of Observation 4.5, in the block $D_{j+1}E_{j+1}$ (on symbols D_j, E_j) in the coding of the orbit of x (by images of vertices under ϕ_j), consecutive appearances of D_j are separated by a distance $2j$, while in the two possible blocks $E_{j+1}E_{j+1}$ and $E_{j+1}D_{j+1}$ in the coding of the orbit of x' the consecutive appearances of D_j are separated by distance either $2j - 1$ or $2j - 2$. Therefore $v_{j+1}(T^m x') = v(j + 1, 3)$.

Since $T^{m+\dim(v(j,1))}x$ is minimal to $v(j, 3)$ and the orbits of x and x' always hit $v(j, 3)$ at the same time, we must have that $T^{m+\dim(v(j,1))}x'$ is also minimal to $v(j, 3)$. Since the dot for both $\phi_j(T^{m+\dim(v(j,1))}x)$ and $\phi_j(T^{m+\dim(v(j,1))}x')$ is at the beginning of an appearance of D_j, and D_j appears only once in each $\phi_j(v(j+1, 3))$, applying $T^{-\dim((v(j,1))}$ shows that both $T^m x$ and $T^m x'$ are minimal from the root into $v(j + 1, 3)$.

5. Nonexpansive systems are conjugate to either well timed or untimed systems.

In this section we define several classes of nonexpansive BV systems $(W, W_0, DM2, H2, U, U_0, U_1,$ and $U_2)$ according to various possibilities for the existence of pairs of paths with cuts. Recall that all systems under consideration are nonexpansive, properly ordered, and simple. This work is necessary in order to establish Theorem 5.15, which states that every such system is conjugate to a system in exactly one of W ("well timed") or U ("untimed").

Definition 5.1. For any class S of systems, we will denote the class of systems conjugate to some system in S by CS:

$$CS = \{X : \text{there is } Y \in S \text{ such that } X \text{ is conjugate to } Y\}. \tag{8}$$

We denote the class of systems not conjugate to any system in S by NCS, and the class of systems not in S by $\neg S$. Note that it is not true that if $X, Y \in CS$ then X must be conjugate to Y.

In their proof that every bounded width system is either expansive or conjugate to an odometer, Downarowicz and Maass [6] considered a class of systems that they called *Case (2)*, which we denote here by $DM2$. In [13] Hoynes' Case (2) is a slightly weaker condition, which we call $H2$, apparently still sufficient for the proof to succeed. Downarowicz and Maass as well as Hoynes actually used stronger properties, opposite to well timed, which here we call *very untimed (U_0)* and U_2. Here is a list of some relevant classes of systems, obtained by varying quantifiers.

Definition 5.2. We say that a depth k pair of paths *has long cuts* if for every $n > k$ the pair has an n cut.

Definition 5.3. We define the following classes of systems:
(1) We say that a system is *well timed* if for every $k \geq 1$ for every $j > k$ there is a depth k pair with a j cut. We denote the class of well timed systems by W.
(2) We say that a system is *very well timed* if for every $k \geq 1$ there exists a depth k pair with long cuts. We denote the class of very well timed systems by W_0.
(3) $DM2$: For infinitely many k there is a $j(k) > k$ such that no depth k pair has a $j(k)$ cut. The smallest such $j(k)$ is called a k *cutoff*.

(4) $H2$: For infinitely many k for every depth k pair x, x' there is a $j(k, x, x') > k$ such that x, x' has no $j(k, x, x')$ cut. The smallest such $j(k, x, x')$ could be called a *k pair cutoff for the pair x, x'*.

(5) U (untimed): For every k there is a k cutoff. (I.e., for every $k \geq 1$ there is a $j(k) > k$ such that no depth k pair has a $j(k)$ cut.)

(6) U_0 (very untimed): For every k no depth k pair has a $k + 1$ cut. (I.e., for every k, $k + 1$ is a k cutoff).

(7) U_2: For every k there is a depth k pair with no $k + 1$ cut.

(8) U_1: For infinitely many k there is a depth k pair x, x' and there is a $j(k, x, x') > k$ such that the pair x, x' has no $j(k, x, x')$ cut. (I.e., for infinitely many k there is a depth k pair with a pair cutoff.)

Recall that the GJ example is SNE and is very well timed. The modified GJ example (Example 4.6) is not SNE but it is very well timed, because the only changes we made to the DM arrays were to add additional vertical bars, which will not destroy the existing cuts.

Remark 5.4. In [6] the proof of the main theorem in Case (2) begins by telescoping any system in $DM2$ so that the result is in U_0. Hoynes [13, Remark 4.4] does not see why this should always be possible, but suggests that changing the universal quantifier to existential, i.e. replacing $DM2$ with U_1, does allow one to telescope such a system to one in U_2, and that should be enough to let the proof proceed.

The proof in [13] assumes $H2$, telescopes so that for every $k \geq 1$ there is a depth k pair, and then telescopes to obtain a system in U_2. The proof of Sublemma 4.1, though, applies $H2$ to possibly ineligible pairs $y_i, y_{i'}$, because $H2$ is not closed under telescoping. Because the telescoped system is in U_2, for every i there is a pair x_i, x_i' with no $i + 1$ cut. But the pair $y_i, y_{i'}$ of the proof could be depth i' and without a cutoff, if it were the image under the telescoping of a pair of a depth other than one of the infinitely many "good" k in the definition of $H2$ (see Proposition 5.7).

Indeed, in Example 5.13 we present a system that is in both classes $DM2$ (vacuously) and WW (see Definition 5.8, below) and for which telescoping to any strictly increasing sequence of levels takes it out of the class $H2$ (and hence out of $DM2$). Such a system cannot be telescoped into U_0, because then it would be in both CU and $WW = CW$ (see Proposition 5.10), but by Theorem 5.14 these classes are disjoint.

Sublemma 4.1 of [13] could be proved as follows. Assuming that $H2$ is satisfied nonvacuously, let i_0 and all the other i's mentioned in the argument be good k's according to the definition of $H2$. They may not fill up an interval in the integers, but given L one can pick a sequence of them of length L and proceed to write the same argument, being careful to choose the pairs x_i, x_i' so that their cutoffs $j(i, i')$ interleave the levels with good k's.

Remark 5.5. (1) $U_0 \subset U \subset DM2 \subset H2$, and $U_2 \subset U_1$.

(2) Each of the classes W, W_0, U, U_0 is closed under telescoping. We will later (in Remark 5.12) provide a proof of this for W and W_0. To see that the very untimed property persists under telescoping, note that if in a telescoping to levels $\{n_l, l \geq 0\}$ of a very untimed system we found a depth j pair with a $j + 1$ cut, that pair would correspond to a pair in the original system of some depth $k \geq n_j$ with an n_{j+1} cut, and hence with a $k + 1$ cut—cf. Proposition 5.7, (3) and (4).

Example 5.6. Every odometer presented with one vertex at every level is in U_0 (very untimed). To see this, suppose that x, x' are depth $k \geq 1$, so that at some

time m in their orbits they follow different edges from level k to level $k + 1$, i.e. $(T^m x)_k \neq (T^m x')_k$.

Because $(T^j x)_k$ and $(T^j x')_k$, $j \in \mathbb{Z}$, follow the same periodic sequence of edges as j varies, we have that, for all $j \in \mathbb{Z}$, $(T^j x)_k \neq (T^j x')_k$. In particular, x and x' cannot have a $k + 1$ cut, since this would require that for some j they both follow the unique minimal edge from level k to level $k + 1$.

Note that in this example for every $k \geq 1$ there is a depth k pair, but that is not a requirement in the definition of U_0.

We used here a general principle that applies in any system: If level $k + 1$ is strongly uniformly ordered with respect to level k (see Definition 3.1) and x, x' follow edges with different ordinal labels from level k to level $k + 1$, so do $T^j x$ and $T^j x'$ for all $j \in \mathbb{Z}$.

On the other hand, every odometer can be presented (up to conjugacy) with at least two vertices per level after the root and all levels strongly uniformly ordered. We claim that for such a system for every $k \geq 1$ there is a depth k pair with a $k + 1$ cut, but no depth k pair can have a $k + 2$ cut. Thus every such system is in $U \setminus U_0$, with a k cutoff of $k + 2$ for every $k \geq 1$.

At each level $n \geq 1$, the system has vertices $v(n, 1), \ldots, v(n, q_n)$ for some $q_n \geq 2$. Edges with source $v(n, i)$ have ordinal label i.

Given $k \geq 1$, let x be a path that is minimal from the root to $v(k + 1, 1)$, and let x' be a path that is minimal from the root to $v(k, 1)$ at level k and then follows the edge (labeled 1) to $v(k + 1, 2)$ at level $k + 1$. Because $T^j x, T^j x'$ are at strongly uniformly ordered vertices at level $k + 1$ for all $j \in \mathbb{Z}$, they follow edges with the same ordinal label from level k to level $k + 1$. Thus x, x' have the same k-coding and hence the pair is depth k. The paths x, x' also follow minimal edges to level $k + 1$, so they have a $k + 1$ cut. Thus the system is not in U_0.

We show now that if x, x' is a depth k pair, then it cannot have a $k + 2$ cut, so the system is in U, with k cutoff equal to $k + 2$. For suppose that x, x' is a depth k pair. Applying a power of T if necessary, we may assume that these paths follow different edges from level k to level $k + 1$. Because level $k + 1$ is strongly uniformly ordered with respect to level k, for all $j \in \mathbb{Z}$ the paths $T^j x$ and $T^j x'$ follow different edges from level k to level $k + 1$, and hence they are at different vertices at level $k + 1$:

$$v_{k+1}(T^j x) \neq v_{k+1}(T^j x'). \tag{9}$$

Thus the edges downward from these vertices to level $k + 2$ always have different ordinal labels, precluding existence of a $k + 2$ cut.

We aim to show that $CW \subset NCU$ and $NCW \subset CU$, so that the family of simple, perfectly ordered nonexpansive BV systems is the disjoint union of CW and CU;

$$W_0 \subset W \subset CW = NCU \subset NCU_0. \tag{10}$$

For this purpose we need to know how pairs of some depth and cuts in a system relate to those in a telescoping of that system. If $\tilde{X}$ is a telescoping of X, we will call X a *lift* of $\tilde{X}$. The following Proposition says, informally, that pairs of some depth in one of $X, \tilde{X}$ telescope (lift) to pairs of a related depth in the other, as do cuts for those pairs. One consequence is that nonexistence of cutoffs is preserved under telescoping and lifts.

Proposition 5.7. *Let (X, T) be a system and $(\tilde{X}, \tilde{T})$ be another system obtained by telescoping X. The following statements hold for every $k \geq 1$:*

(1) *There exists an $\tilde{i}(k) \leq k$ such that the image of any depth k pair in X under the telescoping is depth $\tilde{i}(k)$ in $\tilde{X}$. Furthermore, $\tilde{i}(k) \to \infty$ as $k \to \infty$.*

(2) *For all sufficiently large j there exists $\tilde{J}(j) \leq j$ such that if a depth k pair in X has a j cut then the depth $\tilde{i}(k)$ image of that pair in $\tilde{X}$ has a $\tilde{J}(j)$ cut. Furthermore, $\tilde{J}(j) \to \infty$ as $j \to \infty$.*

(3) *For every depth k pair $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ in $\tilde{X}$ there exists an $i(\tilde{x}^{(k)}, \tilde{y}^{(k)}) \geq k$ such that $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ is the image under the telescoping of a depth $i(\tilde{x}^{(k)}, \tilde{y}^{(k)})$ pair in X.*

(4) *For any $j > k$ there exists a $J(j) \geq j$ such that if $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ is a depth k pair in $\tilde{X}$ with a j cut, then that pair is the image under the telescoping of a depth $i(\tilde{x}^{(k)}, \tilde{y}^{(k)})$ pair in X with a $J(j)$ cut.*

Proof. Suppose that $\tilde{X}$ is a telescoping of X to levels $(n_l, l \geq 0)$ (with the root being at level $n_0 = 0$).

Proof of (1) and (2):

If $k < n_1$, then after telescoping from the root to level n_1, the image of any pair of paths that is depth k in (X, T) is a pair of paths with different 1-codings in $(\tilde{X}, \tilde{T})$, i.e., $\tilde{i}(k) = 0$. So assume $k \geq n_1$. In other words, there exists $l_k \geq 1$ such that $n_{l_k} \leq k < k + 1 \leq n_{l_k+1}$. In particular, $l_k \to \infty$ as $k \to \infty$.

Let $x^{(k)}, y^{(k)}$ be a depth k pair of paths in X. Since these paths agree to level k and differ at level $k + 1$, they have the same n_{l_k} coding, but not the same n_{l_k+1} coding. This means row n_{l_k} in the array of j-symbols for $x^{(k)}$ is identical to row n_{l_k} in the array for $y^{(k)}$ and is met in the same position by the verticals (see the discussion preceding Example 4.6) for both paths, whereas the same is not true for row n_{l_k+1}. For every $l \geq 1$, the telescoping removes all rows of the arrays between rows n_l and n_{l+1}, so that what used to be row n_l becomes row l. Afterwards, row l_k in the array for the image of $x^{(k)}$ is identical to row l_k in the array for the image of $y^{(k)}$ and is met in the same position by the verticals for both paths, whereas the same is not true for row l_k+1. Hence, the image of the pair $x^{(k)}, y^{(k)}$ under the telescoping is depth l_k. Letting $\tilde{i}(k) = l_k$, (1) is proved.

Suppose that $j > k$ and the paths $x^{(k)}$ and $y^{(k)}$ have a j cut. This cut appears as a pair of vertical segments in the arrays for the two paths that begin in the same position in row 0, end in row j, and consist entirely of rectangle boundaries. Find l_j such that $n_{l_j} \leq j < n_{l_j+1}$. After removing the rows for the telescoping, these vertical segments in the arrays for $x^{(k)}$ and $y^{(k)}$ extend from level 0 to level l_j and still consist entirely of rectangle boundaries. Hence both represent minimal paths in $\tilde{X}$ from the root to level l_j. It follows that the image of $x^{(k)}, y^{(k)}$ has an l_j cut. Letting $\tilde{J}(j) = l_j$, (2) is proved.

Proof of (3) and (4):

Now let $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ be a depth k pair of paths in $\tilde{X}$. This means row k in the array for $\tilde{x}^{(k)}$ is identical to row k in the array for $\tilde{y}^{(k)}$ and is met in the same position by the verticals for both paths, whereas the same is not true for row $k + 1$. In the original diagram, there is a pair of paths whose image under the telescoping is $\tilde{x}^{(k)}, \tilde{y}^{(k)}$. If we restore the rows in their respective arrays that were removed by the telescoping, rows k and $k + 1$ in the arrays for $\tilde{x}^{(k)}$ and $\tilde{y}^{(k)}$ become rows n_k and n_{k+1}. Hence, for some $i(\tilde{x}^{(k)}, \tilde{y}^{(k)}) \in [n_k, n_{k+1})$, it must be the case that these arrays are now the same from row 0 to row $i(\tilde{x}^{(k)}, \tilde{y}^{(k)})$ with the vertical in the same position, and that the same is not true for row $i(\tilde{x}^{(k)}, \tilde{y}^{(k)}) + 1$. It follows that $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ is the image of a depth $i(\tilde{x}^{(k)}, \tilde{y}^{(k)})$ pair of paths in X under the telescoping. This proves (3).

For (4), it is important to note that when we reinsert rows into an array for $\tilde{X}$ that were removed during the telescoping to get an array for some path in X, any vertical segment bounding a rectangle in row $l \geq 1$ of the former becomes a rectangle boundary in row n_l of the new array. Moreover, the vertical line containing this rectangle boundary at level n_l must include rectangle boundaries from row n_l all the way up to row 0. So any j cut for a depth k pair in $\tilde{X}$ just becomes longer when we reinsert into their respective arrays rows that were removed by the telescoping. Specifically, any j cut for a depth k pair of paths $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ in $\tilde{X}$ corresponds to an n_j cut for the preimage of $\tilde{x}^{(k)}, \tilde{y}^{(k)}$ before the telescoping. Letting $J(j) = n_j$, (4) is proved. $\square$

Some of the systems that we shall encounter while proving our main results will have the property encapsulated in the following definition.

Definition 5.8. We say that a system is *weakly well timed* if for infinitely many k for every $j > k$ there is a depth k pair with a j cut. WW denotes the class of systems with this property.

By definition, $W \subset \neg DM2 \subset WW$. We will want to know what happens to the well timed and WW properties under microscoping and telescoping.

It is not necessarily the case that if $(\tilde{X}, \tilde{T})$ is a telescoping of (X, T) and is well timed, then (X, T) is well timed. For example, suppose we telescope X to even levels. It could happen that (X, T) has no pairs with odd depth, and yet for every $k \geq 1$ and every $j > k$ there exists a depth k pair in $(\tilde{X}, \tilde{T})$ with a j cut whose lift to (X, T) is depth $2k$. In this case, $(\tilde{X}, \tilde{T})$ is well timed while (X, T) is not.

Lemma 5.9. *Let (X, T) be a system and $(\tilde{X}, \tilde{T})$ be another system obtained by telescoping X to levels $(n_l, l \geq 0)$.*
(1) If $(X, T) \in W$, then $(\tilde{X}, \tilde{T}) \in W$. Likewise, if $(X, T) \in W_0$, then $(\tilde{X}, \tilde{T}) \in W_0$.
(2) If $(X, T) \in WW$, then $(\tilde{X}, \tilde{T}) \in WW$.
(3) If $(\tilde{X}, \tilde{T}) \in WW$, then $(X, T) \in WW$.

Proof. Proof of (1):

Suppose (X, T) is well timed. Fix $k \geq 1$ and let $j > k$. There exists in X a depth n_k pair with an n_j cut. As shown in the proof of Proposition 5.7, parts (1) and (2), after the telescoping this yields a depth k pair in $\tilde{X}$ with a j cut.

If (X, T) is very well timed, then for every $k \geq 1$ there exists a depth n_k pair with an n_j cut for all $j > k$. The above argument shows that this yields a depth k pair with long cuts after the telescoping.

Proof of (2):

Suppose $(X, T) \in WW$. In other words, there exists an increasing sequence (m_k) such that for every k and every $j > m_k$ there is a depth m_k pair in (X, T) with a j cut.

By Proposition 5.7 (1), for every $k \geq 1$ there exists $\tilde{i}(m_k) \leq m_k$ such that the image of any depth m_k pair in X under the telescoping is depth $\tilde{i}(m_k)$ in $\tilde{X}$. Let $\tilde{m}_k = \tilde{i}(m_k)$. Since $\tilde{i}(k) \to \infty$ as $j \to \infty$, the sequence $(\tilde{m}_k)$ is increasing.

Given $k \geq 1$ and $j > m_k$, find in (X, T) a depth m_k pair with a j cut. By Proposition 5.7 (2), there exists a $\tilde{J}(j) \leq j$ such that the depth $\tilde{m}_k$ image of this pair in $(\tilde{X}, \tilde{T})$ has a $\tilde{J}(j)$ cut. Since $\tilde{J}(j) \to \infty$ as $j \to \infty$, it follows that $(\tilde{X}, \tilde{T}) \in WW$.

Proof of (3):

Suppose that $(\tilde{X}, \tilde{T}) \in WW$. There exists an increasing sequence $(\tilde{m}_k)$ such that for every k and every $j > \tilde{m}_k$ there is a depth $\tilde{m}_k$ pair $\tilde{x}^{(\tilde{m}_k)}, \tilde{y}^{(\tilde{m}_k)}$ in $\tilde{X}$ with a j cut.

Fix $k \geq 1$. As we vary j, the pair $\tilde{x}^{(\tilde{m}_k)}, \tilde{y}^{(\tilde{m}_k)}$ may change, and hence $i(\tilde{x}^{(\tilde{m}_k)}, \tilde{y}^{(\tilde{m}_k)})$ may vary. However, it was shown in the proof of Proposition 5.7 (3) that for every depth $\tilde{m}_k$ pair $\tilde{x}^{(\tilde{m}_k)}, y^{(\tilde{m}_k)}$, we have $n_{\tilde{m}_k} \leq i(\tilde{x}^{(\tilde{m}_k)}, \tilde{y}^{(\tilde{m}_k)}) < n_{\tilde{m}_k+1}$. Hence, there exists $m_k \in [n_{\tilde{m}_k}, n_{\tilde{m}_k+1})$ such that for infinitely many $j > \tilde{m}_k$, the corresponding pair $\tilde{x}^{(\tilde{m}_k)}, \tilde{y}^{(\tilde{m}_k)}$ lifts to a depth m_k pair, which by Proposition 5.7 (4) has a $J(j) > j$ cut. Since $J(j) \to \infty$ as $j \to \infty$ (and the m_k are all distinct), it follows that $(X, T) \in WW$. $\qquad \square$

Proposition 5.10. *For a system (X, T) the following statements are equivalent:*
(1) (X, T) is weakly well timed.
(2) (X, T) has a telescoping that is well timed.
(3) (X, T) is conjugate to a well timed system.
Thus $WW = CW$.

Proof. We show first that (1) implies (2). Suppose there are infinitely many k for which for every $j > k$ there exists a depth k pair in X with a j cut. In other words, there exists an increasing sequence (m_k) such that for all k and for every $j > m_k$ there is a depth m_k pair in X with a j cut. Let $\tilde{X}$ be the telescoping of X to levels (m_k).

Given $k \geq 1$ and $j > m_k$, find in (X, T) a depth m_k pair with a j cut. By Proposition 5.7 (1) and (2), the image of this pair is depth k with a $\tilde{J}(j)$ cut. Since $\tilde{J}(j) \to \infty$ as $j \to \infty$, it follows that $\tilde{X}$ is well timed.

That (2) implies (3) is clear, since conjugacy of BV systems is the equivalence relation that corresponds to the one for diagrams that is generated by telescoping and isomorphism [12, Section 4]. So if (X, T) has a telescoping that is well timed then it is conjugate to that well timed system.

To prove that (3) implies (1), suppose that (X, T) is conjugate to a well timed system (Y, S). As remarked above, then there are a system Z and telescopings $\tilde{X}$ of X and $\tilde{Y}$ of Y such that Z telescopes on even levels $0, 2, 4, \ldots$ to $\tilde{X}$ and on odd levels $0, 1, 3, \ldots$ to $\tilde{Y}$. By Lemma 5.9 (1), any telescoping of a well timed system is also well timed. So we may assume that $\tilde{Y} = Y$.

Since $W \subset WW$, Y is also weakly well timed. Hence by Lemma 5.9 (3), $Z \in WW$. Next consider the telescoping of Z to $\tilde{X}$. By Lemma 5.9 (2), $(\tilde{X}, \tilde{T}) \in WW$.

Applying part (3) of the lemma again, we conclude that $(X, T) \in WW$. $\qquad \square$

With appropriate adjustments the foregoing Proposition and its proof adapt to very well timed systems.

Proposition 5.11. *For a system (X, T) the following statements are equivalent:*
(1) There are infinitely many k for which there exists a depth k pair with long cuts.
(2) (X, T) has a telescoping that is very well timed.
(3) (X, T) is conjugate to a very well timed system.

Remark 5.12. Part (3) of Proposition 5.11 shows that parts (1) and (2) persist under telescoping.

Example 5.13. As promised in Remark 5.4, we show that $WW \cap DM2 \neq \emptyset$. We modify the diagram for the GJ example X by changing every other (even-numbered)

Morse component so that all its vertices are uniformly ordered: for all $j \geq 4$ and all even i such that $1 \leq i < j - 1$ we change the ordering at $v(j, 2i + 1)$ so that it is left to right. Let Y denote the new diagram and system, with the same vertices $v(j, i)$ and "Morse components" $MC(k)$ as in X. Then for every odd k, there is still a depth k pair with long cuts (these are just the same as they were in the GJ example, see Proposition 4.2 and Corollary 4.3), so the new system is in WW .

(In more detail, as in Corollary 4.3, let x and x' be paths that agree to level $k + 1$ and pass through $v(k+1, 2k)$ and $v(k+1, 2k+1)$, respectively, and such that the ordinal edge label for every edge in x and in x' after level $k + 1$ is $2k$. In other words, x and x' enter the Morse component $MC(k)$ at its top and then follow its two sides all the way down. By Proposition 4.2, x and x' are depth k. It is easily verified that for every $j > k$, there exists a time $m < 0$ such that $T^m x$ and $T^m x'$ agree to level $j - 1$ and are minimal from the root into $v(j, 2k)$ and $v(j, 2k + 1)$, respectively. Therefore x and x' are depth k and have long cuts.)

We show now that for all even k there are no longer any depth k pairs, so the new system is (vacuously) in $DM2$ (and hence in $H2$). The idea is that, as in X, the only candidates for depth k pairs are paths down $MC(k)$, but the changed edge orders cause these pairs to have different 2-codings, so they have become depth 1.

All changes to X were at levels 4 and higher. Moreover, for every $j \geq 2$, the coding of $v(j, 3)$ by vertices at level $j - 1$ remains the same in Y as it was in X; and at vertices in Y where this coding is different than it was in X, it is now the same as the coding at $v(j, 1)$. Hence, if as in Observation 4.5 we expand each symbol (vertex) $v(j, i)$ to $C_{j-1}(v(j, i))$, the effect on the ϕ_j-images (strings on $D_j, E_j, D_{j-1}, E_{j-1}$) produces the same morphisms as before. The argument made in Example 4.6 that paths in X with the same 2-coding are 2-equivalent can then be applied to show the same holds true in Y. Hence, the proof of Proposition 4.2 (4) still works to show that for any $k \geq 2$ any depth k pair in the new system represented by Y is k-equivalent but not $(k + 1)$-equivalent.

Now let x, x' be a pair of paths in Y that for some $k \geq 2$ is depth k in the corresponding system. Since Observation 4.1 (2) still holds for Y, we can argue as in the proof of Proposition 4.2 (3) that x and x' in Y have the same sequence of edge labels, enter $MC(k)$ at the level at which they first differ and are contained in $MC(k)$ at all subsequent levels. At any level of Y only pairs of edges inside one of the odd Morse components in Y can have the same label and distinct sources. Hence, k must be odd, and so there are no pairs of paths with even depth in the new system.

Telescoping the new diagram to a strictly increasing sequence of levels j_n will produce a system that is not in $H2$ (or $DM2$). This is because given any n there are an odd integer $k \in [j_n, j_{n+1})$ and a depth k pair with long cuts. After the telescoping this pair will be depth n (by the proof of Proposition 5.7 (1)) and will still have long cuts (by Proposition 5.7 (2)), so the telescoped system is in W_0. By definition, $W_0 \cap H2 = \emptyset$, so the class $H2$ is not closed under telescoping.

Remark 5.4 mentioned that $H2$ not being closed under telescoping impacts the proofs in [6, 13]. If in some system for every odd k there were a depth k pair with a cutoff (so the system is nonvacuously in $H2$, unlike our example), and for every even k there were a depth k pair with long cuts (so the system is also in WW), the same argument as above would show that telescoping to any strictly increasing sequence of levels produces a system that is not in $H2$. If the levels were chosen to produce a system in U_2, the result would be in $U_2 \setminus H2$. The proofs in [6, 13]

could be clarified by ruling out this case or dealing with it, maybe along the lines we suggest in Remark 5.4.

The following Theorem subsumes Proposition 3.12.

Theorem 5.14. *No system that is conjugate to a well timed system can also be conjugate to an untimed system: $CW \subset NCU$.*

Proof. Suppose that X is a system that is conjugate to a well timed system and is also conjugate to an untimed system Y. As mentioned above, by [12, Theorem 4.7], then there are a system Z and telescopings $\tilde{X}$ of X and $\tilde{Y}$ of Y such that Z telescopes on even levels $0, 2, 4, \ldots$ to $\tilde{X}$ and on odd levels $0, 1, 3, \ldots$ to $\tilde{Y}$.

By Proposition 5.10, $X \in WW$. Repeated application of Lemma 5.9 gives $\tilde{X} \in WW$, then $Z \in WW$, then $\tilde{Y} \in WW$, and finally $Y \in WW$. This is a contradiction, since by definition no system can be both untimed and weakly well timed. $\square$

Theorem 5.15. *The family of nonexpansive systems is the disjoint union of those conjugate to well timed systems and those conjugate to untimed systems:*

$$NE = CW \sqcup CU. \tag{11}$$

Proof. By Theorem 5.14, $CW \subset NCU$, so it remains only to show that $NCW \subset CU$. By Proposition 5.10, if $(X, T) \in NCW$, then there is a k_0 such that for all $k \geq k_0$ there is a k cutoff $j(k) > k$: no depth k pair has a $j(k)$ cut. Let us telescope to levels $n_l, l \geq 0$, with $n_1 > k_0$ to produce the system $(\tilde{X}, \tilde{T})$. Suppose that $\tilde{k} \geq 1$ and $\tilde{x}, \tilde{y}$ is a depth $\tilde{k}$ pair in $\tilde{X}$ with a $\tilde{j}$ cut. Proposition 5.7 (3) and (4) tell us that then $\tilde{x}, \tilde{y}$ is the image under the telescoping of a pair x, y in X of depth $i(\tilde{x}, \tilde{y}) \in [n_{\tilde{k}}, n_{\tilde{k}+1})$ with a $J(\tilde{j}) = n_{\tilde{j}}$ cut. Every $i \in [n_{\tilde{k}}, n_{\tilde{k}+1})$ has a cutoff $j(i)$ in X, in particular $J(\tilde{j}) < j(i(\tilde{x}, \tilde{y}))$. Since

$$\tilde{j} \leq J(\tilde{j}) < j(i(\tilde{x}, \tilde{y})) \leq \max\{j(i) : i \in [n_{\tilde{k}}, n_{\tilde{k}+1})\}, \tag{12}$$

$\tilde{j}$ is bounded, over all pairs in $\tilde{X}$ of depth $\tilde{k}$. Thus in $\tilde{X}$ for every $\tilde{k}$ there is a $\tilde{k}$ cutoff. This shows that telescoping past k_0 produces a system in U, and hence $(X, T) \in CU$. $\square$

6. **Describing untimed systems.** Now we take a few steps towards determining exactly which BV systems are very untimed. If a system (X, T) is very untimed and not conjugate to an odometer, then, by [6], it must have infinite "topological rank". At the moment we do not have an example of a very untimed system that has unbounded width.

Lemma 6.1. *If in a BV diagram level $n + 1$ is uniformly ordered and has more than one vertex, then in the BV system there is a depth n pair of paths with an $n + 1$ cut.*

Proof. Choose two paths x, x' that are minimal into distinct vertices at level $n + 1$. Since level $n + 1$ is uniformly ordered, the n-basic block at each vertex at level $n + 1$ is periodic with shortest repeating block P. (By this we mean that for each vertex v at level $n + 1$, there is an integer n_v such that the n-basic block at v is P^{n_v}). Thus the n-factor is a rotation on $|P|$ points, and the n-coding of every path is the two-sided sequence P^∞, with a choice of the center position. Since x and x' are minimal into level $n + 1$, their "dot" is at the beginning of an explicit appearance of P in the n-basic blocks at $v_{n+1}(x)$ and $v_{n+1}(x')$ respectively, so x, x' have the same n-coding. Hence x and x' are depth n with an $n + 1$ cut. $\square$

Proposition 6.2. *Every bounded width very untimed system has only finitely many levels with more than one vertex.*

Proof. By [6], bounded width and nonexpansive implies conjugate to an odometer. By [8] (and also mentioned in [11]) there is a telescoping that has infinitely many uniformly ordered levels. Let $n + 1$ ($n > 0$) be a uniformly ordered level in the telescoped system, which is still very untimed (see Remark 5.5, (2)). Because of the Lemma and because no depth n pair can have an $n + 1$ cut, this level $n + 1$ must have just one vertex. Returning to the original (not telescoped) system, the level m corresponding to level $n + 1$ in the telescoped system has just one vertex. Any level that follows a level with just one vertex is uniformly ordered, and so, by the Lemma, level $m + 1$ in the original system must also have just one vertex. Thus every level in the original system from level $m + 1$ on has just one vertex.   □

Example 6.3. So one way to produce very untimed systems is to begin with levels with more than one vertex with edges ordered so that all the edges downward from each vertex have different ordinal labels. Exiting these levels downward there can be no cuts. Then continue forever with single-vertex levels connected by multiple edges. Any such system is conjugate to an odometer, upon telescoping from the root to the first single-vertex level.

Definition 6.4. A *BV* system is *deterministic* if for every $n \geq 1$ the edges downward from each vertex at level n have different ordinal labels.

The following Proposition provides more detail about the possible form of deterministic diagrams such as the one in Example 6.3.

Proposition 6.5. *Let* (X, T) *be a deterministic system. For each* $n \geq 1$ *let* k_n *denote the number of vertices at level* n. *Then* $k_1 \geq k_2 \geq \ldots$, *and eventually all* $k_n = 1$. *(So the diagram has the form of the previous Example.)*

Proof. If $k_n < k_{n+1}$ for some $n \geq 1$, then the label 1 must be repeated on edges between levels n and $n + 1$, so $k_n \geq k_{n+1}$ for all n.

If eventually all $k_n = k$, we claim that there would be k minimal paths, and so we must have $k = 1$. This is because from each low (late) enough level $n + 1$ there are k minimal edges (labeled 1) upward to k different vertices at level n (because each vertex at level n has a single edge downward that is labeled 1), and hence k minimal paths from level $n + 1$ up to the root. Looking at larger and larger values of n, we can see k distinct paths that follow only minimal edges.   □

Example 6.6. Not every bounded width very untimed system must be deterministic, as Figure 3 shows. The small numbers are ordinal edge labels, and the strings in parentheses are 1-basic blocks. In this example the only possible cuts are for pairs of paths among $x = 0auA...$ (thick edges), $x' = 0avB \ldots$ (dashed edges) and $x'' = 0bwB \ldots$ (or pairs in the orbits of such pairs). While x and x' have a 2 cut, they are not depth 1; x and x'' have 2 and 3 cuts, but they are not depth 1 or 2; and x', x'' have a 2 cut, but they are not depth 1. In fact there are no depth 1 or depth 2 pairs.
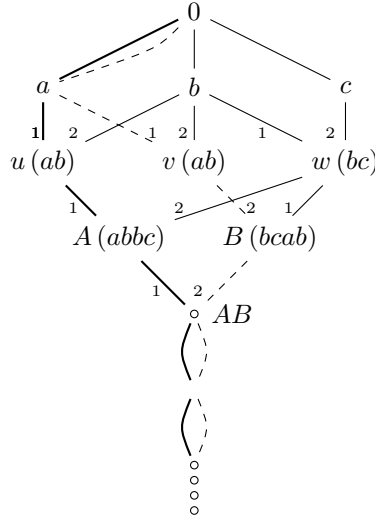
FIGURE 3. A nondeterministic bounded width very untimed system

7. **Conclusion and questions.** The family of simple, properly ordered Bratteli-Vershik systems is the disjoint union of the expansive systems, the systems conjugate to well timed systems, and the systems conjugate to untimed systems:

$$BV = E \sqcup CW \sqcup CU. \tag{13}$$

The foregoing suggests several questions:

1. Is there an example of an untimed (or even very untimed) system that has unbounded width? Is there an untimed system that has infinite topological rank, equivalently is not conjugate to any odometer? Is the class U closed under conjugacy?

2. Is every well timed system conjugate to an $SSNE$ (or SNE) system? If so, one could regularize nonexpansiveness by modifying any member of this class W of well timed NE systems so that it becomes $SSNE$.

3. There are many questions about the relations among the classes of systems that we have defined, and others that could be defined. Exactly which classes can overlap, exactly what inclusions are there among them, exactly which of these inclusions are strict, etc.? Is every odometer (by which we here mean all levels strictly uniformly ordered, cf. Example 5.6) in U? Is every system conjugate to an odometer in U? Is SNE contained in W? Can $H2$ and W intersect? (We know $DM2 \cap W = \emptyset$.) Is $U_1 \cap W \neq \emptyset$? Is $U_2 \cap W \neq \emptyset$?

REFERENCES

[1] T. Adams, S. Ferenczi and K. Petersen, Constructive symbolic presentations of rank one measure-preserving systems, *Colloq. Math.*, **150** (2017), 243-255.

[2] V. Berthé, W. Steiner, J. M. Thuswaldner and R. Yassawi, Recognizability for sequences of morphisms, *Ergodic Theory Dynam. Systems*, **39** (2019), 2896-2931, arXiv:1705.00167v1.

[3] S. Bezuglyi, J. Kwiatkowski and K. Medynets, Aperiodic substitution systems and their Bratteli diagrams, *Ergodic Theory Dynam. Systems*, **29** (2009), 37-72.

[4] S. Bezuglyi, J. Kwiatkowski and R. Yassawi, Perfect orderings on finite rank Bratteli diagrams, *Canad. J. Math.*, **66** (2014), 57-101.

[5] S. Bezuglyi and R. Yassawi, Perfect orderings on Bratteli diagrams II: General Bratteli diagrams, (2014), arXiv:1306.1788v2.

[6] T. Downarowicz and A. Maass, Finite-rank Bratteli-Vershik diagrams are expansive, *Ergodic Theory Dynam. Systems*, **28** (2008), 739-747.

[7] F. Durand, Combinatorics on Bratteli diagrams and dynamical systems, *Combinatorics, Automata and Number Theory*, Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, **135** (2010), 324-372.

[8] S. Frick, K. Petersen and S. Shields, Dynamical properties of some adic systems with arbitrary orderings, *Ergodic Theory Dynam. Systems*, **37** (2017), 2131-2162.

[9] S. Frick, K. Petersen and S. Shields, Periodic codings of Bratteli-Vershik systems, *Math. Scand.*, **126** (2020), 298-320.

[10] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, **469** (1995), 51-111.

[11] R. Gjerde and Ø. Johansen, Bratteli-Vershik models for Cantor minimal systems: Applications to Toeplitz flows, *Ergodic Theory Dynam. Systems*, **20** (2000), 1687-1710.

[12] R. H. Herman, T. F. Putnam and C. F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, *Internat. J. Math.*, **3** (1992), 827-864.

[13] S.-M. Høynes, Finite-rank Bratteli-Vershik diagrams are expansive—a new proof, *Math. Scand.*, **120** (2017), 195-210.

[14] A. M. Vershik, Uniform algebraic approximation of shift and multiplication operators, *Dokl. Akad. Nauk SSSR*, **259** (1981), 526-529.

[15] A. M. Vershik, A theorem on Markov periodic approximation in ergodic theory, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **115** (1982), 72-82, 306.