Complexity, disorder, information, chance

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University of Tartu
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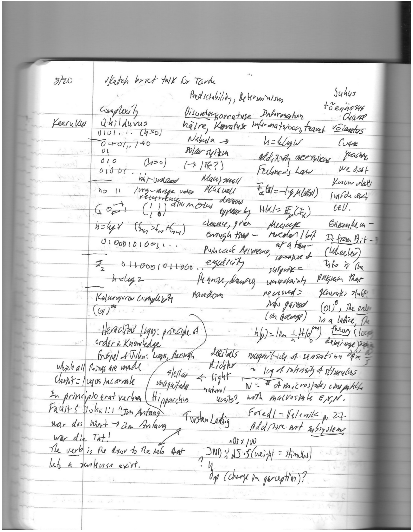
Outline

1. Plan
2. Talk

Plan



8/20 Sketch brief talk for Tartu

Predictability, Determinism

Keerukus — complexity, ühilduvus

0101... ($h=0$)

0+01, 1→0

01

010 ($h=0$)

01001... not ordered

no 11 long-range order, recurrence

$\begin{pmatrix}1&1\\1&0\end{pmatrix}$ alrum o'lus

$h = \log \gamma$ ($f_{n+1} = f_n + f_{n-1}$)

0100010100101...

$\mathbb{Z}_2$ 011000101100...

$h = \log 2$

Kolmogorov complexity

$(01)^\infty$

Disorder, korratuse

häire, korratuse

Nebula → solar system (→ life?)

Maxwell, Maxwell demons

demons appear by chance, given enough time — Poincaré Recurrence, ergodicity

Penrose, drawing

random

Information

informatsioon, teave

$h = k \log W$

additivity

Fechner's Law

$I_\alpha(x) = -\log \mu(\alpha(x))$

$H(\alpha) = E_\mu(I_\alpha)$

Message received / bit at a time — amount of surprise = uncertainty removed = info gained (on average)

$h(\mu) = \lim \frac{1}{n} H(\alpha_0^{n-1})$

juhus

tõenäosus

Chance

võimalus

Cuore

We don't know what's inside each cell.

Gleamlean —

It from Bit — (Wheeler)

Info is the program that generates stuff: $(01)^8$, the order in a lattice, the theory (logos, demiurge)

Heraclitus' logos: principle of order & knowledge

Gospel of John: Logos, through which all things are made

Christ = logos incarnate

In principio erat verbum

Faust: John 1:1 "Im Anfang war das Wort → Im Anfang war die Tat!

The verb is the door to the info that lets a sentence exist.

stellar magnitudes ← light

Hipparchus

decibels

Richter

natural units?

magnitude of sensation dp

~ log of intensity of stimulus

N = # of microstates compatible with macrostate E, V, N.

Friedli – Velenik p. 27

Additive wrt subsystems

Torsten Lahdig

.05 x 100

JND ≈ .05 · S (weight = stimulus)

?

dp (change in perception)?

Complexity

$0, 1$ sequences

The *complexity function* of a sequence, system, or set of configurations is

$$N_n = \text{number of size-}n \text{ “blocks”}.$$

1. Periodic

$$01010101\dots$$

$$h = \lim_{n\to\infty} \frac{\log N_n}{n} = 0$$

2. Substitutions

$$0 \to 01, 1 \to 0$$

$$0$$

$$0 \quad 1$$

$$01 \quad 0$$

$$01 \quad 0 \quad 01$$

$$h = 0$$

$$N_n = n + 1 \quad \text{for all } n \geq 1$$

Not periodic, but long-range order, recurrence.

Exercise: If N_n is bounded for a sequence, then the sequence is eventually periodic: $x = A \quad B^{\infty}$.

3. Shifts of finite type (topological Markov chains)

No 11:

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad 0100010100100\ldots$$

$$h = \log \lambda_{\max} > 0$$

Exponential growth rate of the number of paths in the graph.

$$N_{n+1} = N_n + N_{n-1}$$

Full shift, Σ_2:

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

$0110001011101001111\ldots$

$h = \log 2$

Kolmogorov complexity

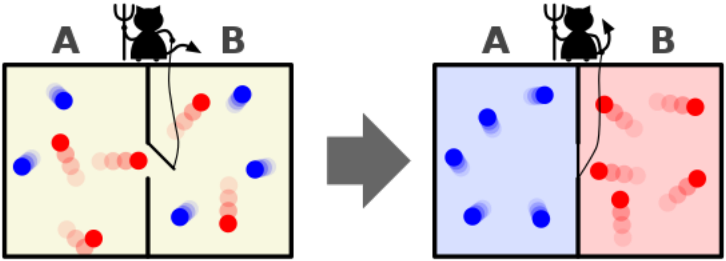
length of the shortest program (for a universal Turing machine) that produces the (finite) string

$(01)^\infty$ is an instruction that produces the infinite sequence $0101010101\ldots$

The program is like a *theory* of observed phenomena–

a *compression* of *data*.

Disorder


A
B
A
B

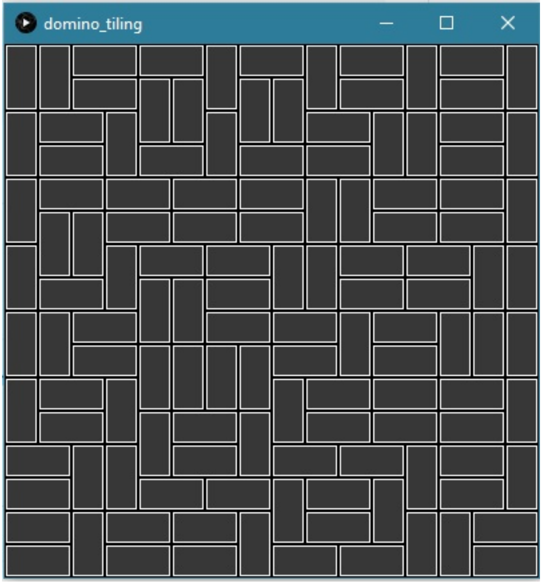
Given enough time, many small Maxwell demons can appear by chance and, using ambient energy, divert parts of a system toward unlikely states.

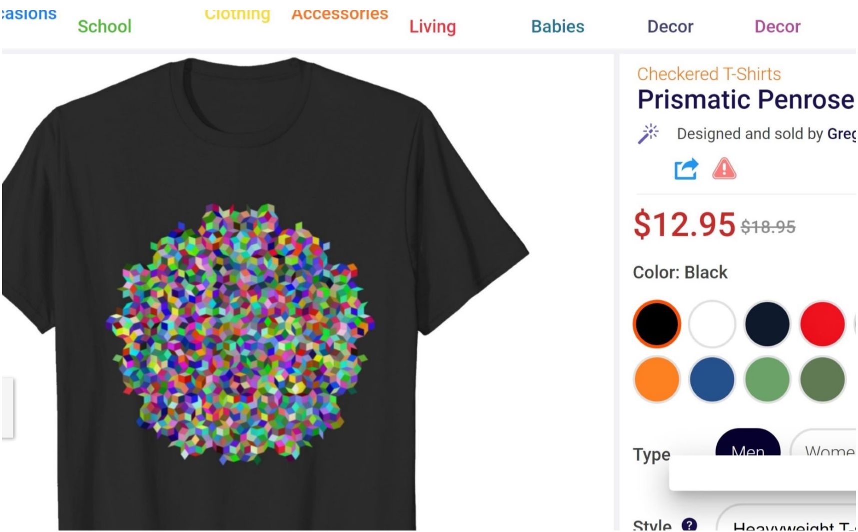
A nebula can contract under gravity, but it is unlikely to have total angular momentum zero.

Rotation, gravity, and random particle collisions cause flattening to a disk with fairly uniform angular velocities.

Random irregularities sweep up particles by collisions and gravity, leading to stars and planets.

??


domino_tiling


School
Living
Babies
Decor
Decor
Checkered T-Shirts
Prismatic Penrose
$12.95 $18.95
Color: Black
Type
Style

Information

Claude Shannon, 1948:

When we are told that an event A that has probability $P(A)$ has occurred, the amount of *information gained* is

$$-\log_2 P(A) \quad \textit{bits}.$$

This equals the *uncertainty removed*.

Information is measured in the number of binary digits (answers to yes-no questions) needed to convey it.

Claude Shannon

Signals can be *compressed*, subject to how much average information per symbol they contain, which is called the *entropy of the source*:

$$
\begin{aligned}
h(\mathcal{X}) &= \lim_{n\to\infty}\left[-\frac{1}{n}\sum_{B\in D^n} P(B)\log_2 P(B)\right] \\
&= \lim_{n\to\infty}\left[-\frac{1}{n}\log_2 P(x_0x_1\ldots x_{n-1})\right].
\end{aligned}
$$

Why $I(A) = -\log_2 P(A)$?

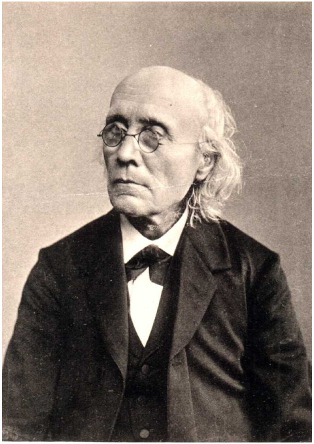
Additive for independent events.

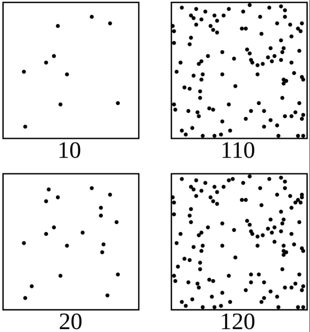
Zero for certain events, large for improbable events.

Fechner's Law, 1860: The magnitude of the sensation produced by some stimulus is proportional to the logarithm of the intensity of the stimulus.

(In what *units* are sensation strength and stimulus intensity being measured?)

Gustav Theodor Fechner





An illustration of the Weber–Fechner law. On each side, the lower square contains 10 more dots than the upper one. However the perception is different: On the left side, the difference between upper and lower square is clearly visible. On t ...

More details

Д.Ильин: vectorization - File:Weber-Fechner law demo - dots.png by MrPomidor

An illustration of the Weber-Fechner law. On each side, the lower square contains 10 more dots than the upper one. However the perception is different: On the left side, the difference between upper and lower square is clearly visible. On the right side, both squares look almost the same.

File: Weber-Fechner law demo - dots.svg

Created: 2 January 2022

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Boltzmann, c. 1875:

$$S = k \log W$$

The entropy of a macrostate $X = (E, V, N)$ is proportional to the number W of compatible microstates.

W is a measure of our ignorance about the “inside” of the macrostate X, its complexity, its degree of disorder.

(When all microstates are assumed to be equally likely, each has probability $P(A) = 1/W$.)

Ludwig Boltzmann

Entropy (Macquom Rankine 1850, Rudolf Clausius 1865) can be defined to be a function of the values of the variables that specify a system that is *additive* with respect to subsystems:

$$S(X_1 \sqcup X_2) = S(X_1) + S(X_2).$$

It is postulated that when two systems are put in contact, the values of the variables for each system adjust so that at equilibrium S is at a maximum.

It follows that if they have different inverse temperatures

$$\beta = \left(\frac{\partial S}{\partial E}\right)_{N,V}$$

and are allowed to exchange energy but not to exchange particles, to reach equilibrium they equalize their temperatures.

Boltzmann sought to reduce thermodynamics to statistical mechanics.

It from bit

John Archibald Wheeler, 1989: *“It from bit.”*

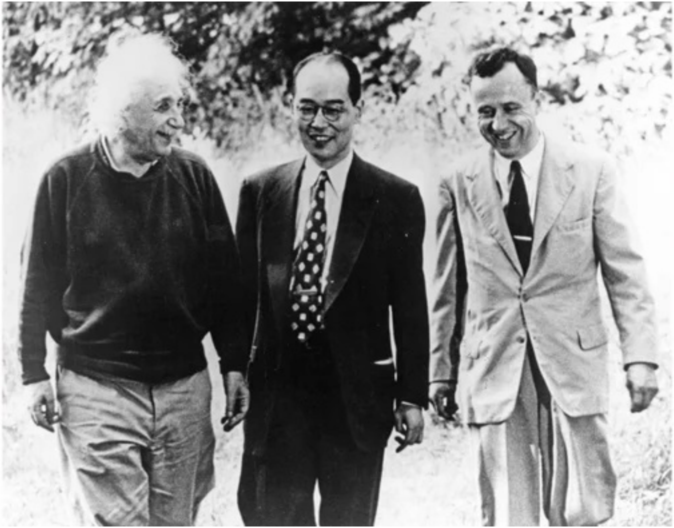
Information encapsulates the program that generates stuff.

“We seek to understand all of physics in terms of information.”

For example:
$(01)^\infty$
the *long-range order* in a substitution sequence or Penrose tiling
the *theory* that explains a set of observations
an impulse that *generates* the realities that it describes?

Logos? Demiurge?

Einstein, Yukawa, Wheeler



John Archibald Wheeler

In ancient Greek philosophy, *logos* was the principle of order and knowledge. (Heraclitus, Stoics)

John 1:1. *En arche ēn ho Logos, In principio erat verbum.* It is logos through which all things are made. Logos is an idea or plan in the mind of God.

Christ and Mohammed are seen as logos incarnate, intermediaries or messengers between humans and higher entities.

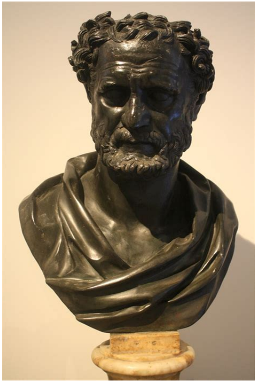
Faust: *Geschrieben steht: "Im Anfang war das Wort". ... der Sinn ... die Kraft ... Mir hilft der Geist! Auf einmal seh' ich Rat und schreibe getrost: Im Anfang war die Tat!*

Torsten Larbig: The verb is the door to the information that lets a sentence exist.

Heraclitus

Nürnberg Chronicle, 1493

Heraclitus



From the Villa of the Papyri, Herculaneum
National Archaeological Museum, Naples

$$\text{idea} \xrightarrow{\text{word}} \text{action}$$

$$\begin{aligned} \text{logos} &\longrightarrow \text{world} \\ \text{program} &\longrightarrow \text{data} \\ \text{theory} &\longrightarrow \text{experiment} \end{aligned}$$

Science seeks to reverse the process.

Science is *knowledge*, not just of a collection of facts, but of a *design* (theory) that "explains" the facts by giving them coherence.

Chance

Chance

A possible source of chance is coarse-graining.

Our imperfect perception and measuring capabilities necessarily leave us ignorant of the properties of an occurring microstate inside of our observed macrostate.

A photon encountering a beam splitter can either be reflected or not, and we don't know which happens until it hits a detector.

So we say that it is in a mixture of states, each with probability $1/2$. Ignorance also comes from quantum entanglement.

When we are about to roll a die and don't know what number might come up, we say that each has probability $1/6$.

Such statements are supported by the evidence of repeated trials. (Frequentism vs. subjectivism.)

Conclusion

Second Law of Thermodynamics: Entropy tends to increase.

John von Neumann to Claude Shannon: “You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more importantly, no one really knows what entropy really is, so in a debate you will always have the advantage.”

Complexity and disorder both promote (or arise from) *ignorance* (lack of information=entropy).

So does chance.

Complexity, disorder, information, chance are deep and slippery ideas.

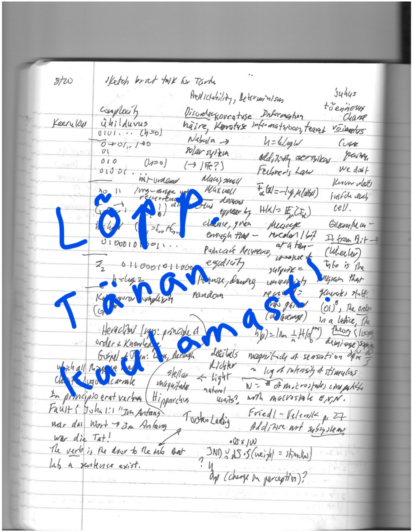
But they are subject to study by mathematics.

Sometimes progress can be made by *separating* structure and randomness and analyzing the effects of each.

Examples: Furstenberg's proof of Szemerédi's Theorem (every positive density set of natural numbers contains arbitrarily long arithmetic progressions), ergodic theorems, Austin's recent proof of the Weak Pinsker Conjecture, etc.

Intricacy (Sporns, Tononi, Edeleman 1994): Maximum effectiveness is achieved by *combining* (balancing) freedom and order.

Brain, society, etc.


Lõpp.
Tänan kuulamast!