ASYMPTOTIC PRESSURE ON SOME SELF-SIMILAR TREES

KARL PETERSEN AND IBRAHIM SALAMA

Abstract. The vertices of the Cayley graph of a finitely generated semigroup form a set of sites which can be labeled by elements of a finite alphabet in a manner governed by a nonnegative real *interaction matrix*, respecting nearest neighbor adjacency restrictions. To the set of these configurations one can associate a *pressure*, which is defined as the limit, when it exists, of averages of the logarithm of the partition function over certain finite subgraphs. We prove that for shifts of finite type on generalized Fibonacci trees and many primitive interaction matrices, the limit exists and is given by an infinite series. We also show that the limit of any cluster points of the pressure on finite subtrees as the number of generators grows without bound, which we call the *asymptotic pressure*, equals the logarithm of the maximum row sum of the interaction matrix.

1. Introduction

Entropy and its generalization pressure are basic concepts imported from statistical physics into information theory and dynamical systems. Usually the state space has been the set of labelings by elements of a finite alphabet of the vertices of the integer lattice $\mathbb{Z}^d$, which then acts on this set of configurations by shifts in the coordinate directions. There has also been interest in such thermodynamical models related to labelings of other graphs, including trees and Cayley graphs of other groups and semigroups, and even arbitrary countable sets, see for example [6–11, 13, 16, 19, 25, 26, 28]. Burton, Pfister, and Steif [11] gave an infinite series formula for the pressure on the free group or semigroup on two generators and showed that the variational principle for pressure can fail on trees; in fact there is an equilibrium state, which is a Gibbs state, if and only if the interaction matrix has constant row sums. (Although they treated the case of the free group on two generators and the full 2-tree, they mentioned that the results would extend to all homogeneous k-trees with $k \geq 3$.) Piantadosi [23, 24] and the current authors [21, 22], unaware of the work of Burton, Pfister, and Steif [11], obtained infinite series formulas for the topological entropy of the golden mean (also called "hard core" or "hard square") shift of finite type (SFT) on the free group and free semigroup on a finite number k of generators. The current authors [21, 22] also proved the existence of topological entropy for subshifts on trees and, for the golden mean subshift, convergence to it by strip approximations analogous to those of Marcus and Pavlov [17, 20].

Louidor, Marcus, and Pavlov [15] and Meyerovitch and Pavlov [18] studied what they called *limiting entropy* and *independence entropy*, proving that they coincide for

Date: October 3, 2022.

2020 *Mathematics Subject Classification.* 37B10, 37B40, 37B51, 82B20.

Key words and phrases. Tree shift, shift of finite type, entropy, topological pressure, hard square model.

$\mathbb{Z}$ subshifts. Based on numerical evidence, Piantadosi conjectured [23, Conjecture 4.1] that the topological entropy $h^{(k)}$ of the golden mean SFT on the free group on k generators has limiting value $\log 2$ as $k \to \infty$ (see also [24]). The current authors [21] (see also [22]) proved that the topological entropy of a tree shift on the regular k-tree defined by an irreducible $d \times d$ $0,1$ matrix with maximum row sum s has limit $\log s$ as $k \to \infty$, giving a positive answer to Piantadosi's conjecture. We will call this limit the *asymptotic entropy* rather than "limiting entropy", because when they exist entropy and pressure for such systems are already limits or cluster points of functions of sums over finite configurations as the size grows without bound.

Here we extend Piantadosi's observations and our previous result in three ways: the set of sites is allowed to be a tree with branching restrictions, as considered by Ban et al. [1–5], equivalently the Cayley graph of a finitely generated semigroup with certain relations; the labeling is governed by general nearest neighbor shift of finite type (SFT) restrictions, beyond the golden mean model; and beyond entropy we consider asymptotic *pressure*. In Section 3 we follow the plan of [11] to prove the existence of pressure

$$P^{(k)} = \lim_{n\to\infty} P_n^{(k)} \tag{1.1}$$

on any generalized Fibonacci k-tree for many primitive interaction matrices and give an infinite series formula for it (Theorem 3.10).

In Section 4 we show that for a fixed interaction matrix, a fairly general sequence of restricted trees on k generators, and any cluster points $Q^{(k)}$ in n of $P_n^{(k)}$, the *asymptotic pressure* $\lim_{k\to\infty} Q^{(k)}$ of the sequence is the logarithm of the maximum row sum of the interaction matrix. The reason for this seems to be the increasing relative importance of the contribution to the pressure by edges at the boundary of a subtree as the valence (dimension) grows, and that it is possible to find many valid configurations which label the next-to-last row of the subtree with a symbol whose corresponding row in the interaction matrix achieves the maximal row sum (see Lemma 4.1 and Theorem 4.2).

2. Setup

We consider entropy and pressure for sets of configurations on a set of sites. There will be two adjacency matrices involved: a primitive (or more generally irreducible) $0,1$ matrix R that is used to *form* the tree by restricting the ways that it can branch, and a nonnegative $d \times d$ matrix A that will restrict the ways that the tree can be *labeled* by elements of an alphabet of d symbols.. In statistical physics often the set of sites is the k-dimensional integer lattice $\mathbb{Z}^k$. In previous work [21, 22] we have studied entropy on the rooted k-tree, which is the Cayley graph of the free semigroup on k generators $g_1, \dots, g_k$. Each string u on the alphabet $\{g_1, \dots, g_k\}$ corresponds to an element of the semigroup and a vertex of the graph, with the empty string ϵ corresponding to the root. For each string u and $i = 1, \dots, k$ there is a directed edge from u to ug_i.

Here we consider a general setting, considered previously by Ban et al. [1–5] of configurations on the subtree determined by a $k \times k$ $0,1$ *restriction matrix* R:

$$S = S(k,R) = \{0\} \cup \{g_{i_1} \dots g_{i_n} : n \geq 1,\ \text{no } R_{i_j,i_{j+1}} = 0, j = 1, \dots, n-1\}. \tag{2.1}$$

The subtree S can be regarded as the Cayley graph of the semigroup with right absorbing element 0 ($g0 = 0$ for all $g \in S$) generated by $\{g_1, \dots, g_k\}$ and with relations $g_i g_j = 0$ if $R_{ij} = 0$. Vertices of the graph are identified with elements of the semigroup and with strings on the generators that do not reduce to 0, and the absorbing element corresponds to the root of the tree. Alternatively, S can be regarded as the semigroup generated by $\{g_1, \dots, g_k\}$ with relations $g_i g_j = g_i$ if $R_{ij} = 0$ and identity element ϵ at the root, cf. [1–5].

We assume at first that each restriction matrix R is *primitive*, meaning that some positive power has all entries positive.

Example 2.1. For $0 \le r < k$ the matrix $R(k,r)$ defined by $R(k,r)_{ij} = 0$ if and only if each of $i, j > k - r$, otherwise $R(k,r)_{ij} = 1$, defines a "generalized Fibonacci tree". (If $r = 0$, so that $R_{ij} = 1$ for all i, j, S is the free semigroup on k generators and its Cayley graph is the full k-tree.)

We define the *height* $|g|$ of an element $g \in S$ to be the length of the shortest word that represents it (so that the root has height 0). We will also use $|.|$ to denote the cardinality of a set as well as the sum of the absolute values of all entries of a matrix or vector.

For each $n \ge 0$, for readability suppressing k, we define

$$\Delta_n = \{g \in S : |g| \le n\} \quad \text{and} \quad L_n = \{g \in S : |g| = n\}. \tag{2.2}$$

Then $|L_1| = k$ and, for $n \ge 2$,

$$|L_n| = |R^{n-1}| = \sum_{i,j} (R^{n-1})_{ij}. \tag{2.3}$$

(This holds also for $n = 1$, with $R^0 =$ the $k \times k$ identity matrix.) Further,

$$|\Delta_n| = 1 + \sum_{j=1}^{n} |R^{j-1}|. \tag{2.4}$$

A *configuration* on S is a labeling of the sites (vertices of the subtree) by elements of a finite alphabet $D = \{1, 2, \dots, d\}$, thus an element $x \in X = D^S$. We will be interested in labelings that are allowed by *nearest neighbor* constraints.

Let $A = (a_{ij})$ be a nonnegative real $d \times d$ matrix that we take as specifying *pair interactions* and let $w = (w_j)$ be a positive real vector that we take as specifying *site energies*. By discarding irrelevant states we may assume that A is *nondegenerate* in the sense that no row nor column is identically zero. The idea is that if a vertex h is assigned label j by a configuration $x \in X$, then the "particle" j at site h is given an energy $\log w(j)$ by some ambient field. And if vertices g, with label i, and h, with label j, are joined by an *edge* (so that $h = gg_s$ for some $s = 1, \dots, k$), then they experience an interaction (tension, attraction, or repulsion) determined by a_{ij}. The *interaction matrix*

$$E(i,j) = a_{ij} w_j \tag{2.5}$$

collects these effects. (For improved readability we will sometimes denote matrix or vector indices parenthetically rather than by subscripts.)

In relation to the setup in [11],

$$a_{ij} = e^{\phi(i,j)}, \quad w_j = e^{\chi(j)}; \tag{2.6}$$

but note that we allow $\phi(i,j) = -\infty$, while $\chi(j) \in \mathbb{R}$.

Now we restrict our attention to labelings that conform to the adjacency restrictions provided by the matrix A. Thus our set of configurations will be the (hom—see [12]) *tree shift of finite type* X_A determined by R and A contained in the product space $X = D^S$:

$$\begin{aligned} X_A = \{x \in D^S : A(x(g), x(h)) > 0 \text{ for all } g, h \in S \text{ such that} \\ h = gg_s \text{ for some } s = 1, \dots, k\}. \end{aligned} \tag{2.7}$$

(We ignore every configuration that has an interaction of size 0 between some pair of adjacent sites.) Because we have assumed that A is nondegenerate, X_A is nonempty.

X (as well as X_A) is a compact metric space with distance $d(x,y) = 1/(n+1)$ if $n = \max\{j : x = y \text{ on } \Delta_j\}$. The semigroup S acts continuously on X according to

$$(xg)(h) = x(gh). \tag{2.8}$$

We are interested in (allegedly) physical quantities due to configurations $x \in D^{\Delta_n}$ on finite subtrees Δ_n that are the restrictions of configurations in X_A, and then their limits as $n \to \infty$. Define $X_A^{(n)}$ to be the set of labelings of Δ_n that are restrictions to Δ_n of some element of X_A. The contribution to the pressure from configurations on Δ_n that have symbol $i \in D$ at the root is

$$Z_n(i) = w(i) \sum_{\substack{x \in X_A^{(n)} \\ x(\epsilon)=i}} \prod_{\substack{\gamma = \langle g,h \rangle \\ \text{edge in } \Delta_n}} E(x(g), x(h)). \tag{2.9}$$

We continue to suppress k much of the time when it is fixed to avoid excess notation. The n'th *partition vector* is

$$Z_n = (Z_n(1), \dots, Z_n(d)), \tag{2.10}$$

the *partition function on* Δ_n is

$$|Z_n| = Z_n(1) + \cdots + Z_n(d), \tag{2.11}$$

and the n'th *pressure probability vector* is ρ_n defined by

$$\rho_n(i) = \frac{Z_n(i)}{|Z_n|}, \quad i = 1, \dots, d. \tag{2.12}$$

The *pressure* or *free energy* on Δ_n is

$$P_n^{(k)} = \frac{\log |Z_n|}{|\Delta_n|}. \tag{2.13}$$

The *upper limiting pressure* $\bar{P}^{(k)}$ and *lower limiting pressure* $\underline{P}^{(k)}$ are defined by

$$\bar{P}^{(k)} = \limsup_{n \to \infty} \frac{\log |Z_n|}{|\Delta_n|}, \quad \underline{P}^{(k)} = \liminf_{n \to \infty} \frac{\log |Z_n|}{|\Delta_n|}. \tag{2.14}$$

When A is a $0,1$ matrix and $w_j = 1$ for all j, $|Z_n|$ is the number of configurations on Δ_n that are allowed by the adjacency matrix A (symbols $i, j \in D$ are allowed at vertices connected by an edge if and only if $a_{ij} = 1$). In this case $\lim_{n \to \infty} P_n^{(k)}$, if it exists, is the *topological entropy* of the tree shift of finite type determined by A on the restricted tree. The limit has been proved to exist for any tree shift (not necessarily finite type) on a full k-tree [21, 22]. Ban et al. [1–5] proved existence of the topological entropy for shifts of finite type on classes of restricted trees, including the generalized Fibonacci trees of Example 2.1. In the following section we extend these results to pressure on such trees.

3. Existence of pressure on generalized Fibonacci trees and an infinite series formula for it

To formulate a recursion formula for the partition function in somewhat compressed notation, we use unusual (Hadamard or Schur) coordinatewise products and powers for (usually column) vectors $v = (v_i)$ and matrices $A = (a_{ij})$:

$$(v \times w)_i = v_i w_i, \quad (A^{[n]})_{ij} = a_{ij}^n. \tag{3.1}$$

In this section A is a nonnegative $d \times d$ primitive matrix.

Consider first the case of the full k-tree ($r = 0$), with symbol $i \in D$ at the root. If $n \geq 2$, then each node of the first row L_1 of Δ_1 can be thought of as the root of a subtree Δ_{n-1} of height $n-1$ and can be assigned any symbol from D that can follow i according to the transitions allowed by A. The labelings of these subtrees are independent of one another, so one may interchange sum and product as in the following.

$$\begin{aligned}
Z_n(i) &= w(i) \sum_{\substack{x \in X_A^{(n)} \\ x(\epsilon)=i}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_n}} E(x(g), x(h)) \\
&= w(i) \sum_{\substack{x \in X_A^{(n)} \\ x(\epsilon)=i}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_n}} A(x(g), x(h)) w(x(h)) \\
&= w(i) \left[\sum_{j=1}^{d} A(i,j) \left(w(j) \sum_{\substack{x \in X_A^{(n-1)} \\ x(\epsilon)=j}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-1}}} E(x(g), x(h)) \right) \right]^k \\
&= w(i) \left[\sum_{j=1}^{d} A(i,j) Z_{n-1}(j) \right]^k = \left(w \times [A Z_{n-1}]^{[k]} \right)(i).
\end{aligned} \tag{3.2}$$

Now consider a *generalized Fibonacci* tree defined by a restriction matrix $R = R(k,r), k \geq 2, 0 \leq r < k$, with $R_{ij} = 0$ if each of $i, j > k - r$, otherwise $R_{ij} = 1$. On row L_1 of Δ_n there are now $k - r$ "free" vertices, each the root of a subtree Δ_{n-1} of height $n-1$, along with r "restricted" vertices, each followed by $k - r$ edges whose terminal vertices are then the roots of subtrees Δ_{n-2} of height $n-2$. These subtrees are labeled independently. Letting

$$Z_0 = w, \quad Z_1 = w \times (A Z_0)^{[k]}, \tag{3.3}$$

and looking at Δ_n and its first few rows, we thus find that for $n \geq 1$,

(3.4)
$$Z_n(i) = w(i)\left(\sum_j A(i,j)Z_{n-1}(j)\right)^{[k-r]}\left[\sum_m A(i,m)w(m)\left(\sum_u A(m,u)Z_{n-2}(u)\right)^{[k-r]}\right]^{[r]}$$
$$= w(i)\left((AZ_{n-1})(i)\right)^{[k-r]}\left[\left(A\left(w \times (AZ_{n-2})^{[k-r]}\right)\right)(i)\right]^{[r]},$$

so that

(3.5)
$$|Z_n| = \left|w \times (AZ_{n-1})^{[k-r]} \times \left[A\left(w \times (AZ_{n-2})^{[k-r]}\right)\right]^{[r]}\right| = |Z_{n-1}|^{[k-r]}|Z_{n-2}|^{[r(k-r)]}g(n),$$

with

$$g(n) = \left|w \times (A\rho_{n-1})^{[k-r]} \times \left[A\left(w \times (A\rho_{n-2})^{[k-r]}\right)\right]^{[r]}\right|. \tag{3.6}$$

More generally, for d-dimensional probability vectors α, β define

$$g(\alpha,\beta) = \left|w \times (A\alpha)^{[k-r]} \times \left[A\left(w \times (A\beta)^{[k-r]}\right)\right]^{[r]}\right|, \tag{3.7}$$

and on the set of pairs of such vectors for which $g(\alpha,\beta) > 0$ define

$$T(\alpha,\beta) = \left(w \times (A\alpha)^{[k-r]} \times \left[A\left(w \times (A\beta)^{[k-r]}\right)\right]^{[r]} / g(\alpha,\beta), \alpha\right). \tag{3.8}$$

Remark 3.1. The function $g(n)$ is bounded above; but it might take values less than 1, and this could cause problems for convergence of the infinite series for pressure (3.33).

(1) In the case of a full k-tree ($r = 0$), $\log g$ is bounded because $w > 0$ and every column of A has a positive entry. (To see this, note that ρ_{n-1} is a probability vector with positive entries, at least one of which, say with index j, is at least $1/d$. We may choose i such that $a_{ij} \geq a = \min\{a_{ij} : a_{ij} > 0\} > 0$. Then $(A\rho_{n-1})(i) \geq a/d$ and $g(n) = |w \times (A\rho_{n-1})^{[k]}| \geq (\inf_i w_i)(a/d)^k$.)
(2) For any generalized Fibonacci tree, if A has a positive row then $\log g$ is bounded. (The statement is clear if for some i all $a_{ij} = 1$, hence also if all $a_{ij} = \delta$ for some $\delta > 0$, hence also if all $a_{ij} \geq \delta$ for some $\delta > 0$.)
(3) If $T^n(\rho_1, \rho_0)$ converges to a fixed point or periodic orbit, then $\log g$ is bounded.
(4) If $g(\alpha,\beta) > 0$ on the closure of the orbit of (ρ_1, ρ_0) under T, then it is bounded below by a positive constant and hence again $\log g$ is bounded.

We show now that if A satisfies a condition apparently stronger than primitive, then $\log g$ is bounded.

Suppose for the moment that we are dealing with a general restricted tree determined by a $k \times k$ restriction matrix R, as in (2.1), and not just a generalized Fibonacci tree. For a fixed $p > 0$, the subtrees rooted at the vertices $v \in L_p$ have different types, depending on the directions $e(v) \in \{1,\dots,k\}$ of their incoming edges to their root vertices v: only initial edges in directions h for which $R(e(v),h) = 1$

are allowed. For $n > p$ denote by $X_A^{(n-p,v)}$ the set of restrictions of configurations in $X_A^{(n)}$ to the restricted subtree $\Delta_{n-p}^{(v)}$ of Δ_n that has root at v, and by

$$Z_{n-p}^{(v)}(j) = \sum_{\substack{x \in X_A^{(n-p,v)} \\ x(\epsilon)=j}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-p}^{(v)}}} E(x(g), x(h)) \tag{3.9}$$

the pressure on the subtree $\Delta_{n-p}^{(v)}$ of Δ_n that has label j at its root, v.

Now consider the set $L_p(i)$ of strings of symbols $J = (j(v), v \in L_p)$ that occur on L_p in configurations $x \in X_A^{(n)}$ on Δ_n that have symbol i at the root: $x(\epsilon) = i, x(v) = j(v)$ for $v \in L_p$. We are interested in matrices A that guarantee that for some p we have $L_p(i) = L_p(j)$ for all $i, j \in \{1, \dots, d\}$.

Definition 3.2. Let $d, k \geq 1$ and let A be a nondegenerate irreducible $d \times d$ nonnegative matrix that determines allowed transitions on trees, as in Equation 2.7. If there is an $n \geq 1$ such that on a tree t with labelings allowed by A we have $L_n(i) = L_n(j)$ for all $i, j \in D = \{1, \dots, d\}$, we will say that A is *n-primitive on t*. We denote the set of n-primitive $d \times d$ matrices on the full k-tree by $P(k, n)$.

If there is an $n \geq 1$ such that for all $i \in D$ we have $L_n(i) = D^{|L_n|}$, we will say that A is *n^*-primitive on t. We denote the set of n^*-primitive matrices on the full k-tree by $P^*(k, n)$.*

Remark 3.3. The following observations are included to provide some familiarity with the condition that $A \in P(k, n)$.

(1) For all k, n we have $P^*(k, n) \subset P(k, n)$.
(2) If $A \in P(k, n)$ for some $k \geq 1$, then A is primitive, and in fact $A^n > 0$.
(3) If $A \in P(k, n)$, then A is n-primitive on every subtree of the full k-tree.
(4) $P(k + 1, n) \subset P(k, n)$.
(5) $P(k, n) \subset P(k, n + 1)$.
(6) The 3×3 matrix $A_2 = (110, 101, 101)$ is in $P(2, 2) \setminus P(2, 1)$.
(7) $A_3 = (110, 101, 100) \in P(2, 3) \setminus P(2, 2)$.
(8) If $A^n > 0$ and A has a positive row, then $A \in P(k, n + 1)$.

We return now to a general restricted tree and strings $J = (j(v), v \in L_p) \in L_p(i)$ on row p of Δ_n in the tree. For such a string J define

$$F_n(i, J) = \prod_v Z_{n-p}^{(v)}(j(v)), \tag{3.10}$$

to be the sum over all configurations on $\cup_{v \in L_p} \Delta_{n-p}$ that have J on L_p of the products of the edge weights (the product of sums is a sum of products—see (3.18) below), and let

$$M_n(i) = \max_{\text{allowable } J} F_n(i, J). \tag{3.11}$$

Choose one allowable string $J_0 = (j_0(v), v \in L_p)$ that achieves the maximum value M_n of $F_n(i, J)$ over all i and J.

Remark 3.4. If $A \in P(k, p)$ for some p, then for all choices of symbol i at the root and all $n \geq p$, we have $J_0 \in L_p(i)$ and $M_n(i) = M_n$.

Proposition 3.5. *Suppose that $p \geq 1$ and $A \in P(k,p)$, so that $A^p > 0$. For a general restricted tree, there are positive constants η_p and ξ_p such that for all $n > p$ there is $M_n > 0$ such that for each $i = 1, \dots, d$,*

$$\xi_p M_n \leq Z_n(i) \leq \eta_p M_n. \tag{3.12}$$

Proof. We form some special configurations on Δ_n. Given $i \in \{1, \dots, d\}$, because $J_0 \in L_p(i)$ we can choose a labeling $x \in X_A^{(p)}$ that has i at the root and assigns labels $j_0(v)$ to the vertices $v \in L_p$ (see Remark 3.4). Let

$$\xi_p(i,x) = w(i) \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_p}} E(x(g), x(h)) \tag{3.13}$$

denote the product of $w(i)$ and the weights determined by x on the edges in Δ_p, and let

$$\xi_p = \inf\{\xi_p(i,x) : x \in X_A^{(p)}, x(\epsilon) = i, i = 1, \dots, d\}. \tag{3.14}$$

Now fix $i \in \{1, \dots, d\}$ and choose any $x \in X_A^{(p)}$ with $x(\epsilon) = i$ and $x(v) = j_0(v)$ for all $v \in L_p$. We label Δ_p by x, and then for each Δ_{n-p} with a vertex on L_p we assign label $j_0(v)$ to its root v and any labeling allowed by A to the rest of its vertices. These labelings of the Δ_{n-p} are assigned independently of each other.

The sum over *all* configurations on Δ_n with i at the root of the products of the weights on the edges in Δ_n is greater than or equal to the sum over just these special configurations (that have x on Δ_p and J_0 on L_p), so (recall Remark 3.4)

$$Z_n(i) \geq \xi_p(i) F_n(i, J_0) \geq \xi_p M_n, \tag{3.15}$$

proving the left-hand inequality.

For the right-hand inequality, we decompose each configuration $x \in X_A^{(n)}$ into a configuration x_1 on Δ_p and a configuration x_2 on $\cup_{v \in L_p} \Delta_{n-p}^{(v)}$. These configurations are not independent, since they overlap in L_p; this causes the first inequality below. Denote by η_p the sum, over all allowed configurations x on Δ_p, of the product of w at the root and the weights determined by x on the edges in Δ_p, multiplied by d:

$$\eta_p = d \sum_{x \in X_A^{(p)}} w(x(\epsilon)) \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_p}} E(x(g), x(h)). \tag{3.16}$$

FIGURE 1. Two small trees

Then

(3.17)

$$
\begin{aligned}
Z_n(i) &= w(i) \sum_{\substack{x \in X_A^{(n)} \\ x(\epsilon)=i}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_n}} E(x(g),x(h)) \\
&\leq w(i) \sum_{\substack{x_1 \in X_A^{(p)} \\ x_1(\epsilon)=i}} \sum_{\substack{x_2 \text{ on} \\ \cup_{v\in L_p}\Delta_{n-p}^{(v)}}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_n}} E(x(g),x(h)) \\
&= w(i) \sum_{\substack{x_1 \in X_A^{(p)} \\ x_1(\epsilon)=i}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_p}} E(x(g),x(h)) \sum_{\substack{x_2 \text{ on} \\ \cup_{v\in L_p}\Delta_{n-p}^{(v)}}} \prod_{v\in L_p} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-p}^{(v)}}} E(x(g),x(h)) \\
&\leq \frac{\eta_p}{d} \sum_{\substack{x_2 \text{ on} \\ \cup_{v\in L_p}\Delta_{n-p}^{(v)}}} \prod_{v\in L_p} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-p}^{(v)}}} E(x(g),x(h)) = \frac{\eta_p}{d} \prod_{v\in L_p} \sum_{\substack{x_3 \text{ on} \\ \Delta_{n-p}^{(v)}}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-p}^{(v)}}} E(x(g),x(h)) \\
&= \frac{\eta_p}{d} \prod_{v\in L_p} \sum_{j=1}^{d} \sum_{\substack{x_3 \text{ on} \\ \Delta_{n-p}^{(v)} \\ x_3(v)=j}} \prod_{\substack{\gamma=\langle g,h\rangle \\ \text{edge in } \Delta_{n-p}^{(v)}}} E(x(g),x(h)) = \frac{\eta_p}{d} \prod_{v\in L_p} \sum_{j=1}^{d} Z_{n-p}^{(v)}(j) \\
&= \frac{\eta_p}{d} \sum_{j=1}^{d} \prod_{v\in L_p} Z_{n-p}^{(v)}(j) \leq \frac{\eta_p}{d} d \prod_{v\in L_p} Z_{n-p}^{(v)}(j_0(v)) = \eta_p M_n.
\end{aligned}
$$

□

The interchange of sum and product in the fourth line of the above calculation (and elsewhere) takes some thought, but it seems to be correct, since the configurations on each $\Delta_{n-p}^{(v)}$ are assigned independently. The figure with two small trees and nodes labeled by a, b, c, d, e taking values in $\{1, \dots, d\}$ is meant to give the idea: by taking out a common factor we have

(3.18)

$$
\begin{aligned}
\sum_{a,b,c,d,e} E(a,b)E(a,c)E(d,e) &= \sum_{a,b,c} \sum_{d,e} E(a,b)E(a,c)E(d,e) \\
&= \left(\sum_{d,e} E(d,e) \right) \left(\sum_{a,b,c} E(a,b)E(a,c) \right).
\end{aligned}
$$

Corollary 3.6. *Suppose that $p \geq 1$ and $A \in P(k,p)$. Then for a general restricted tree, for each $i, j = 1, \dots, d$ and $n > p$,*

(3.19)
$$\frac{Z_n(i)}{Z_n(j)} \geq \frac{\xi_p}{\eta_p}.$$

For convenience we write $k_1 = k - r$ and $k_2 = r$.

Corollary 3.7. *For a generalized Fibonacci tree with $A \in P(k,p)$ and*

$$G_m = w \times (A\rho_{m-1})^{[k_1]} \times [A\left(w \times (A\rho_{m-2})^{[k_1]}\right)]^{[k_2]}, \tag{3.20}$$

there is a constant $c > 0$ such that

$$g(m) = |G_m| \geq c > 0 \quad \text{for all } m > p+2. \tag{3.21}$$

Proof. By Equation 3.12, each $\rho_m(i) \geq 1/d|Z_p|$. Let $a_i = \inf_i \sum_j a_{ij}$ and $b = \inf_i w_i$. Then for each i,

$$G_m(i) \geq b\left(\frac{a_1\xi_p}{d\,\eta_p}\right)^{k_1}\left[a_1\left(b\left(\frac{a_1\xi_p}{d\,\eta_p}\right)\right)^{k_1}\right]^{k_2} > 0, \tag{3.22}$$

independently of m. □

Remark 3.8.
(1) The conclusion of Corollary 3.7 holds also for some matrices A that are not in $P(k,n)$ for any n. For example, if $d = 3$, $w(i) = 1$ for all $i = 1,\dots,d$, and $A_5 = (011, 101, 110)$, then, because A_5 has constant row sums, $\rho_n(i) = 1/d$ for all n and i, and hence $g(n)$ is the same positive constant for all n. For general positive w, we may bound $g(n)$ from below by replacing w by the constant vector $\tilde{w}(i) = \inf_j w(j) > 0$ for all i.
(2) Maybe the conclusion holds for all primitive matrices A.
(3) Numerical calculations indicate that if A is assumed only to be irreducible, there might not be a positive lower bound for $\rho_n(i)$, yet there is one for $g(n)$.
(4) We do not see how to deduce directly from Equation 3.12 that $g(n) = |Z_n|/(|Z_{n-1}|^{[k-r]}|Z_{n-2}|^{[r(k-r)]})$ is bounded below by a positive constant for all general restricted trees, without using the particular form of the recurrences (3.5) and (3.6) for Z_n and ρ_n for generalized Fibonacci trees.

We proceed now to establish an infinite series formula for pressure on generalized Fibonacci trees, which also proves existence of the limit that defines it.

In order to iterate 3.5, let us abbreviate

$$|Z_0| = a, |Z_1| = b, u = k - r, v = ru. \tag{3.23}$$

Thus
(3.24)

$$\begin{aligned}
|Z_2| &= a^v b^u g(1)\\
|Z_3| &= a^{uv} b^{u^2+v} g(1)^u g(2)\\
|Z_4| &= a^{u^2v+v^2} b^{u^3+2uv} g(1)^{u^2+v} g(2)^u g(3)\\
|Z_5| &= a^{u^3v+2uv^2} b^{u^4+3u^2v+v^2} g(1)^{u^3+2uv} g(2)^{u^2+v} g(3)^u g(4)\\
|Z_6| &= a^{u^4v+3u^2v^2+v^3} b^{u^5+4u^3v+3uv^2} g(1)^{u^4+3u^2v+v^2} g(2)^{u^3+2uv} g(3)^{u^2+v} g(4)^u g(5)
\end{aligned}$$

etc.

Let us define $c(n), d(n), e(n)$ for $n \geq 0$ by

$$|Z_n| = a^{c(n)} b^{d(n)} g(1)^{e(n-1)} \dots g(n-2)^{e(2)} g(n-1)^{e(1)}, \tag{3.25}$$

starting with $c(1) = 0, d(1) = 1, e(0) = 0, e(1) = 1$. Then

$$\begin{bmatrix} c(n) \\ d(n) \end{bmatrix} = \begin{bmatrix} 0 & v \\ 1 & u \end{bmatrix}^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} e(n) \\ e(n-1) \end{bmatrix} = \begin{bmatrix} u & v \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.26}$$

The following Proposition shows that $c(n), d(n), e(n)$ and $|L_n|$ all have the same exponential growth rate.

Proposition 3.9. *The matrices $R(k, r)$ and*

$$\begin{gathered} M_1 = \begin{bmatrix} 0 & v \\ 1 & u \end{bmatrix} = \begin{bmatrix} 0 & r(k-r) \\ 1 & k-r \end{bmatrix}, M_2 = \begin{bmatrix} u & v \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} k-r & r(k-r) \\ 1 & 0 \end{bmatrix}, \text{ and} \\ M_3 = \begin{bmatrix} u & u \\ r & 0 \end{bmatrix} = \begin{bmatrix} k-r & k-r \\ r & 0 \end{bmatrix} \end{gathered} \tag{3.27}$$

have the same maximal eigenvalue.

Proof. M_1, M_2, M_3 all have the same characteristic polynomial $-x(u-x) - v$ and eigenvalues

$$\beta = \frac{1}{2}(u + \sqrt{u^2 + 4v}), \quad \alpha = \frac{1}{2}(u - \sqrt{u^2 + 4v}). \tag{3.28}$$

To see that $R(k, r)$ also has the same maximal eigenvalue, note that the shifts of finite type determined by the matrices M_3 and $R(k, r)$ are topologically conjugate by state splitting or amalgamation [14, Theorem 2.4.10, p. 54] and hence have the same maximal eigenvalue.

Alternatively, for each $n \geq 0$ we can count on L_n the number u_n of "free" vertices $g_{i_1} \dots g_{i_n}$ with $i_n \in \{1, \dots, k - r_k\}$ and the remaining number v_n of "restricted" vertices. Then

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} k - r_k & k - r_k \\ r_k & 0 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} \quad \text{for } n \geq 0. \tag{3.29}$$

The powers of these two matrices both count the number of vertices on the n'th row of Δ_n, which has exponential growth rate given by the maximal eigenvalue. □

Abbreviate $s = \sqrt{u^2 + 4uv}$. The recurrences in (3.26) have solutions

$$\begin{aligned} c(n) &= \frac{1}{2^{n+1}s} \left[(u+s)(u-s)^n + (s-u)(u+s)^n \right], \\ d(n) &= \frac{1}{2^n s} \left[(u+s)^n - (u-s)^n \right], \\ e(n) &= \frac{1}{s} \left[(\frac{u+s}{2})^n - (\frac{u-s}{2})^n \right] = \frac{1}{s}(\beta^n - \alpha^n). \end{aligned} \tag{3.30}$$

We also solve the recurrence for the number of vertices on row n of the restricted tree:

$$\begin{aligned} |L_n| &= |R^{n-1}| = k|L_{n-1}| + v|L_{n-2}|, \quad L_0 = 1, L_1 = k, L_2 = ku + v, \dots \\ |L_n| &= \frac{1}{2s} \left[\beta^n (2k + s - u) - \alpha^n (2k - s - u) \right]. \end{aligned} \tag{3.31}$$

Thus

$$|\Delta_n| \sim \frac{\beta}{\beta-1}|L_n| \sim \frac{\beta}{\beta-1}\frac{2k-u+s}{2s}\beta^n. \tag{3.32}$$

Theorem 3.10. *For a generalized Fibonacci tree, assume that* $\log g(n)$ *is bounded (which is the case on the full* k*-tree, or if the matrix* $A \in P(k,p)$ *for some* $p \geq 1$*, or if* $g(\alpha,\beta) > 0$ *on the closure of the orbit of* (ρ_1, ρ_0) *under* T*). Then the limit below exists, and the pressure on the restricted tree is*

(3.33)

$$P^{(k)} = \lim_{n\to\infty} \frac{\log|Z_n|}{|\Delta_n|} = \frac{\beta-1}{\beta(2k+s-u)}\left[(s-u)\log|Z_0| + 2\log|Z_1| + 2\sum_{i=1}^{\infty}\frac{\log g(i)}{\beta^i}\right].$$

Proof. Looking at (3.25), we have

$$\frac{\log|Z_n|}{|\Delta_n|} = \frac{c(n)}{|\Delta_n|} + \frac{d(n)}{|\Delta_n|} + \frac{1}{|\Delta_n|}\sum_{i=1}^{n-1} e(n-i)\log g(i). \tag{3.34}$$

When in the first two terms we expand $c(n)$ and $d(n)$ using (3.30), as $n \to \infty$ we can ignore the terms involving $(u-s)^n$, since $s+u > s-u > 0$.

For the third term,

$$\frac{1}{|\Delta_n|}\sum_{i=1}^{n-1} e(n-i)\log g(i) = \frac{1}{s|\Delta_n|}\sum_{i=1}^{n-1}(\beta^{n-i} - \alpha^{n-i})\log g(i), \tag{3.35}$$

split the sum into two series, one involving powers of β and one involving powers of α. If $|\alpha| < 1$, then the series involving powers of α converges, and when we divide by $|\Delta_n|$ and let $n \to \infty$ this part contributes nothing to the limit.

If $|\alpha| \geq 1$, the sum in the third term involving powers of α is bounded by

$$|\alpha|^n \sum_{i=1}^{n-1} \frac{\log g(i)}{\alpha^i} \leq |\alpha^n|(n-1)||g||_\infty, \tag{3.36}$$

so when we divide by $|\Delta_n|$, which is of order $\beta^n \gg \alpha^n$, again this contribution vanishes.

Since $\log g(i)$ is bounded and as $n \to \infty$

$$\begin{aligned} \frac{c(n)}{|\Delta_n|} &\sim \frac{\beta-1}{\beta}\frac{s-u}{2k-u+s}, \\ \frac{d(n)}{|\Delta_n|} &\sim 2\frac{\beta-1}{\beta}\frac{s-u}{2k-u+s}, \quad \text{and} \\ \frac{e(n)}{|\Delta_n|} &\sim 2\frac{\beta-1}{\beta}\frac{s-u}{2k-u+s}, \end{aligned} \tag{3.37}$$

the conclusion follows. □

We specialize to the Fibonacci tree, when $k = 2$ and $r = 1$. Denote by λ the golden mean, which is the maximal eigenvalue of the restriction matrix $R(2,1)$.

Corollary 3.11. *For the Fibonacci tree and matrix $A \in P(k,p)$ for some $p \geq 1$, the limit below exists, and*

$$\lim_{n\to\infty} \frac{\log|Z_n|}{|\Delta_n|} = \frac{1}{\lambda^5}\log|Z_0| + \frac{1}{\lambda^4}\log|Z_1| + \frac{1}{\lambda^4}\sum_{i=1}^{\infty}\frac{\log g(i)}{\lambda^i}. \tag{3.38}$$

4. Asymptotic pressure on restricted trees

Suppose that for each dimension $k \geq 2$ we have a primitive $k \times k$ $0,1$ matrix R_k that defines a subtree $S(k,R_k)$ of the full rooted k-tree as above (see 2.1). We denote by $\Delta_n^{(k)}$ the subtree of $S(k,R_k)$ consisting of all vertices of height no more than n and by λ_k the maximal (Perron-Frobenius) eigenvalue of R_k.

Lemma 4.1. *For each $k > 1$ the limit as $n \to \infty$ of the ratio of the size of the last row of $\Delta_n^{(k)}$ to the number of vertices in $\Delta_n^{(k)}$ is*

$$\lim_{n\to\infty}\frac{|L_n^{(k)}|}{|\Delta_n^{(k)}|} = \lim_{n\to\infty}\frac{|R_k^{n-1}|}{1+\sum_{m=1}^n |R_k^{m-1}|} = \frac{\lambda_k - 1}{\lambda_k}. \tag{4.1}$$

Proof. By the Perron-Frobenius Theorem (see for example [27, Theorem 1.2, p. 9] and [14, Theorem 4.5.12, p.130]), if a nonnegative primitive matrix M has maximal eigenvalue λ and corresponding left and right eigenvectors l and r, normalized so that $l \cdot r = 1$, then for each $i,j = 1,\dots,k$ and $n \geq 0$ there are $\epsilon_{ij}(n)$ which tend to 0 as $n \to \infty$ such that

$$(M^n)ij = \lambda^n(r_i l_j + \epsilon_{ij}(n)). \tag{4.2}$$

Thus for fixed k (suppressing the dependence of ϵ on k), and letting $\epsilon(n) = \sum_{i,j}\epsilon_{ij}(n)$,

$$\frac{|R_k^{n-1}|}{1+\sum_{m=1}^n |R_k^{m-1}|} = \frac{\lambda_k^{n-1}(1+\epsilon(n-1))}{1+\sum_{m=1}^n \lambda_k^{m-1}(1+\epsilon(m-1))}. \tag{4.3}$$

The terms involving the $\epsilon(n)$ and $\epsilon(m)$ can be ignored as $n \to \infty$, because

$$\frac{1}{\lambda_k^{n-1}}\sum_{m=1}^n \lambda_k^{m-1}\epsilon(m-1) \to 0 \quad \text{as } n \to \infty. \tag{4.4}$$

(To see this, given $\delta > 0$, choose n_0 so that $n \geq n_0$ implies that $\epsilon(n) < \delta$. Then choose $n_1 \geq n_0$ so that for $n \geq n_1$,

$$\frac{1+\epsilon(1)\lambda_k + \epsilon(2)\lambda_k^2 + \cdots + \epsilon(n_0-1)\lambda_k^{n_0-1}}{\lambda_k^{n-1}} < \delta.) \tag{4.5}$$

This shows that for $n \geq n_1$,

$$\frac{|R_k^{n-1}|}{1+\sum_{m=1}^n |R_k^{m-1}|} \approx \frac{\lambda_k^{n-1}}{1+\sum_{m=1}^n \lambda_k^{m-1}} \approx \frac{\lambda_k - 1}{\lambda_k}. \tag{4.6}$$

□

Suppose now that we have also a $d \times d$ nonnegative pair interaction matrix $A = (a_{ij})$ and a d-dimensional positive vector w. Recall that then the interaction matrix is $E(i,j) = a_{ij}w_j$.

Theorem 4.2. *Denote by* $X_A^{(k)}$ *the tree SFT on the subtree* $S(k, r)$ *determined by the primitive restriction matrix* R_k *and nondegenerate pair interaction matrix* A*; by* $|Z_n^{(k)}|$ *its partition function on* $\Delta_n^{(k)}$*; by* $P_n^{(k)}$ *the pressure or free energy on* $\Delta_n^{(k)}$*; and by* $\bar{P}^{(k)} = \limsup_{n\to\infty} P_n^{(k)}$ *and* $\underline{P}^{(k)} = \liminf_{n\to\infty} P_n^{(k)}$ *the upper and lower limiting pressures. Denote the maximal eigenvalue of* R_k *by* λ_k *and assume that*

$$\lim_{k\to\infty} \lambda_k = \infty. \tag{4.7}$$

Denote by s *the maximum row sum of the interaction matrix* E*:*

$$s = \max\{\sum_{j=1}^{d} E(i, j) : i = 1, \dots, d\}. \tag{4.8}$$

Then the asymptotic pressure *of the sequence of restricted tree shifts (all with the same interaction matrix) as the dimension tends to infinity is*

$$P^{(\infty)} = \lim_{k\to\infty} \bar{P}^{(k)} = \lim_{k\to\infty} \underline{P}^{(k)} = \log s. \tag{4.9}$$

Proof. The argument at the end of [21] adapts to this generalized context. Fix $k \geq 2$. When considering the possible values of $|Z_n^{(k)}|$ due to the various configurations $x \in X_A$, we see that the root ϵ can be assigned any of the values $w_1, \dots, w_d$, and the edges entering any row with the same label i at their initial vertices can independently be assigned values $E(i, j)$ $(j = 1, \dots, d)$. Each of these products is bounded by a power of the maximal row sum. Thus

$$|Z_n^{(k)}| \leq |w| s^{|L_1|} s^{|L_2|} \dots s^{|L_n|}, \tag{4.10}$$

so that

$$\frac{\log |Z_n^{(k)}|}{|\Delta_n|} \leq \frac{\log |w| + \sum_{m=1}^{n} |L_m| \log s}{1 + \sum_{m=1}^{n} |L_m|} \to \log s \quad \text{as } n \to \infty, \tag{4.11}$$

and hence

$$\bar{P}^{(k)} \leq \log s. \tag{4.12}$$

For the lower estimate, let $i \in \{1, \dots, d\}$ be an index for which

$$\sum_{j=1}^{d} E(i, j) = s, \tag{4.13}$$

and consider configurations $x \in X_A$ for which $x(g) = i$ for all $g \in L_{n-1}$.

We complete each configuration x by next working up from row L_{n-1} to the root. On row $n - 2$ assign at every vertex any allowed predecessor j of i (i.e., $a_{ji} > 0$), then on row $n - 3$ assign at each vertex any allowed predecessor of j, etc.

Finally, on row n assign independently to each vertex any of the allowed successors of i. This way we produce a large set $\mathcal{C}$ of allowed configurations on Δ_n, each of which extends to a legal $x \in X_A$.

Let

$$b = \inf\{E(i, j) : E(i, j) > 0\} \tag{4.14}$$

and, for each i, j,

$$\tilde{E}(i, j) = E(i, j)/b, \quad \text{so that all } \tilde{E}(i, j) \geq 1. \tag{4.15}$$

Let $\tilde{s} = s/b$ denote the maximum row sum of the matrix $\tilde{E}$ and w_* the minimum entry of w. Then

$$
\begin{aligned}
|Z_n^{(k)}| &\geq \sum_{x \in \mathcal{C}} w(x(\epsilon)) \prod_{\gamma=\langle g,h\rangle \text{ edge in } \Delta_n} E(x(g),x(h)) \\
&= \sum_{x \in \mathcal{C}} w(x(\epsilon)) \prod_{\gamma=\langle g,h\rangle \text{ edge in } \Delta_n} b\tilde{E}(x(g),x(h)) \\
&\geq w_* b^{|\Delta_n|-1} \sum_{j_1,\dots,j_{|L_n|}} \tilde{E}(i,j_1)\tilde{E}(i,j_2)\cdots\tilde{E}(i,j_{|L_n|}) \\
&= w_* b^{|\Delta_n|-1}[\tilde{E}(i,1)+\tilde{E}(i,2)+\cdots+\tilde{E}(i,d)]^{|L_n|} \\
&= w_* b^{|\Delta_n|-1}\tilde{s}^{|L_n|} = w_* b^{|\Delta_n|-1-|L_n|} s^{|L_n|}.
\end{aligned} \tag{4.16}
$$

Therefore

$$
\frac{\log |Z_n^{(k)}|}{|\Delta_n|} \geq \frac{\log w_* + (|\Delta_n| - 1 - |L_n) \log b + |L_n| \log s}{|\Delta_n|}, \tag{4.17}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\log w_* + (|\Delta_n| - 1 - |L_n) \log b + |L_n| \log s}{|\Delta_n|} &= \frac{\log b}{\lambda_k} + \frac{\lambda_k - 1}{\lambda_k} \log s \\
\leq \liminf_{n\to\infty} \frac{\log |Z_n^{(k)}|}{|\Delta_n|} &= \underline{P}^{(k)} \leq \bar{P}^{(k)} \leq \log s.
\end{aligned} \tag{4.18}
$$

The conclusion follows by letting $k \to \infty$. □

Corollary 4.3. *The previous theorem holds as well if the hypothesis that the restriction matrices R_k be primitive is relaxed to require only that they be irreducible.*

Proof. An irreducible nonnegative matrix M has an associated strongly connected directed graph G whose vertices are the indices of M. They share a minimal period p. M can be replaced by its canonical form produced by grouping indices into classes that are permuted cyclically when following the walk that M, regarded as an adjacency matrix, defines on G. Then M^p has square primitive matrices on its diagonal and 0 entries elsewhere. See [14, Sections 2.2 and 4.5] and [27, Section 1.3]. We apply this to the matrices R_k.

As before denote by λ_k the maximal eigenvalue of the irreducible nonnegative matrix R_k and by r, l the associated right and left positive eigenvectors. To simplify the notation we suppress most of the dependence on k. By [27, Theorems 1.3 and 1.4, pp. 18 and 21] and [14, Exercise 4.5.14, p. 134], for each $i, j = 1, \dots, d$ there is a unique integer $t(i,j) \in [0, p-1]$ such that

$$
(R^m)_{ij} > 0 \quad \text{implies } m \equiv t(i,j) \mod p, \text{ and, for all large enough } n, \tag{4.19}
$$

$$
0 < (R^{np+t(i,j)})_{ij} = (r_i l_j + \epsilon_{ij}(n))\lambda_k^{np+t(i,j)}, \tag{4.20}
$$

where each $\epsilon_{ij}(n) \to 0$ as $n \to \infty$. Thus the proof of Lemma 4.1 goes through as before. □

Example 4.4. Let us see how this works for the generalized Fibonacci trees defined by the restriction matrices $R(k, r_k)$ of Example 2.1 (2). For each $k \geq 1$ and $r_k \in [0, k-1]$, assuming that $\log g$ is bounded, denote by $P^{(k)}$ the pressure on the (possibly) restricted tree shift determined by the tree restriction matrix

$R_k = R(k, r_k)$ and the labeling matrix A (the limit is known to exist by Theorem 3.10). If $r_k = 0$, we have the full k-tree.

$R(k, r_k)$ has maximal eigenvalue $\lambda(k, r_k) = (k - r_k + \sqrt{(k - r_k)(k + 3r_k)})/2$ (equal to the maximal eigenvalue of M_3, above). Each $\lambda(k, r) \to \infty$, in fact uniformly in $\{(k, r_k) : r_k \in [0, k-1]\}$.

Thus for such a sequence of (possibly) restricted trees, with fixed pair interaction matrix A, fixed site energy vector w for labeling sites, and fixed interaction matrix $E(i, j) = a_{ij} w_j$ with maximal row sum s, we have that the asymptotic pressure is

$$\lim_{k\to\infty} P^{(k)} = \log s, \quad \text{uniformly in } \{(k, r_k) : r_k \in [0, k-1]\}. \tag{4.21}$$

References

[1] Jung-Chao Ban and Chih-Hung Chang, *Characterization for entropy of shifts of finite type on Cayley trees* (2017), available at `arXiv:1705.03138v1[math.DS]9May2017`.

[2] Jung-Chao Ban, Chih-Hung Chang, and Nai-Zhu Huang, *Entropy bifurcation of neural networks on Cayley trees* (2017), available at `arXiv:1706.09283v1[math.DS]27Jun2017`.

[3] Jung-Chao Ban and Chih-Hung Chang, *Coloring Fibonacci-Cayley tree: An application to neural networks* (2017), available at `arXiv:1707.02227v1[math.DS]5Jul2017`.

[4] Jung-Chao Ban, Chih-Hung Chang, Yu-Liang Wu, and Yu-Ying Wu, *Stem and topological entropy on Cayley trees* (2021), available at `arXiv:2110.08960v1[math.DS]18Oct2021`.

[5] Jung-Chao Ban, Chih-Hung Chang, and Yu-Hsiung Huang, *Complexity of shift spaces on semigroups* (2017), available at `arXiv:1808,o4925:v1[math.DS]14Aug2018`.

[6] Itai Benjamini and Yuval Peres, *Markov chains indexed by trees*, Ann. Probab. **22** (1994), no. 1, 219–243. MR1258875

[7] Toby Berger and Zhong Xing Ye, *Entropic aspects of random fields on trees*, IEEE Trans. Inform. Theory **36** (1990), no. 5, 1006–1018, DOI 10.1109/18.57200. MR1066588

[8] Zhongxing Ye and Toby Berger, *Information Measures for Discrete Random Fields*, Science Press Beijing, Beijing, 1998. MR1672220

[9] Graham R. Brightwell and Peter Winkler, *Hard constraints and the Bethe lattice: adventures at the interface of combinatorics and statistical physics*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 605–624. MR1957564

[10] ———, *A second threshold for the hard-core model on a Bethe lattice*, Random Structures Algorithms **24** (2004), no. 3, 303–314, DOI 10.1002/rsa.20006. MR2068871

[11] R. M. Burton, C.-E. Pfister, and J. E. Steif, *The variational principle for Gibbs states fails on trees*, Markov Process. Related Fields **1** (1995), no. 3, 387–406. MR1403088

[12] Nishant Chandgotia and Brian Marcus, *Mixing properties for hom-shifts and the distance between walks on associated graphs*, Pacific J. Math. **294** (2018), no. 1, 41–69, DOI 10.2140/pjm.2018.294.41. MR3743365

[13] T.P. Eggarter, *Cayley trees, the Ising problem, and the thermodynamic limit*, Phys. Rev. B **9** (1974), 2989–2992.

[14] Douglas Lind and Brian Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995. MR1369092

[15] Erez Louidor, Brian Marcus, and Ronnie Pavlov, *Independence entropy of $\mathbb{Z}^d$-shift spaces*, Acta Appl. Math. **126** (2013), 297–317, DOI 10.1007/s10440-013-9819-2. MR3077954

[16] Jean Mairesse and Irène Marcovici, *Uniform sampling of subshifts of finite type on grids and trees*, Internat. J. Found. Comput. Sci. **28** (2017), no. 3, 263–287, DOI 10.1142/S0129054117500174. MR3652917

[17] Brian Marcus and Ronnie Pavlov, *Approximating entropy for a class of $\mathbb{Z}^2$ Markov random fields and pressure for a class of functions on $\mathbb{Z}^2$ shifts of finite type*, Ergodic Theory Dynam. Systems **33** (2013), no. 1, 186–220, DOI 10.1017/S0143385711000824. MR3009110

[18] Tom Meyerovitch and Ronnie Pavlov, *On independence and entropy for high-dimensional isotropic subshifts*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 4, 921–945, DOI 10.1112/plms/pdu029. MR3273488

[19] E. Müller-Hartmann and J. Zittartz, *New type of phase transition*, Phys. Rev. Lett. **33** (1974), 893–897.

[20] Ronnie Pavlov, *Approximating the hard square entropy constant with probabilistic methods*, Ann. Probab. **40** (2012), no. 6, 2362–2399, DOI 10.1214/11-AOP681. MR3050506

[21] Karl Petersen and Ibrahim Salama, *Tree shift topological entropy*, Theoret. Comput. Sci. **743** (2018), 64–71, DOI 10.1016/j.tcs.2018.05.034. MR3842480

[22] ———, *Entropy on regular trees*, Discrete Contin. Dyn. Syst. **40** (2020), no. 7, 4453–4477, DOI 10.3934/dcds.2020186. MR4097549

[23] Steven T. Piantadosi, *Symbolic Dynamics on Free Groups*, Honors Project, University of North Carolina at Chapel Hill, 2006.

[24] ———, *Symbolic dynamics on free groups*, Discrete Contin. Dyn. Syst. **20** (2008), no. 3, 725–738, DOI 10.3934/dcds.2008.20.725. MR2373212

[25] Christopher J. Preston, *Gibbs States on Countable Sets*, Cambridge University Press, London-New York, 1974. Cambridge Tracts in Mathematics, No. 68. MR0474556

[26] L.K. Runnels, *Phase transition of a Bethe lattice gas of hard molecules*, J. Math. Phys. **8** (1967), 2081–2087.

[27] E. Seneta, *Non-negative Matrices and Markov Chains*, Springer Series in Statistics, Springer, New York, 1981.

[28] Frank Spitzer, *Markov random fields on an infinite tree*, Ann. Probability **3** (1975), no. 3, 387–398. MR0378152

Department of Mathematics, CB 3250 Phillips Hall, University of North Carolina, Chapel Hill, NC 27599 USA

Email address: petersen@math.unc.edu

School of Business, North Carolina Central University, Durham, NC 27707 USA

Email address: isalama@nccu.edu