# FACTOR MAPS BETWEEN TILING DYNAMICAL SYSTEMS

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Abstract. We show that there is no Curtis-Hedlund-Lyndon Theorem for factor maps between tiling dynamical systems: there are codes between such systems which cannot be achieved by working within a finite window. By considering 1-dimensional tiling systems, which are the same as flows under functions on subshifts with finite alphabets of symbols, we construct a 'simple' code which is not 'local', a local code which is not simple, and a continuous code which is neither local nor simple.

## 1. Introduction

According to the Curtis-Hedlund-Lyndon Theorem [2], every factor map (continuous shift-commuting map) between subshifts (closed shift-invariant subsets of spaces of one- or two-sided sequences on finite alphabets) is a sliding block code, that is to say the central coordinate of the image of any point is determined by a finite range (of fixed size) of central coordinates of the point. Tiling dynamical systems also are based on a finite set of symbols, the prototiles. A tiling is a covering of $\mathbb{R}^d$ by congruent copies, called tiles, of members of the finite set of prototiles which intersect only along their boundaries. Two tilings are considered to be close if within a large neighborhood of the origin the unions of the tile boundaries are close in Hausdorff distance. Then $\mathbb{R}^d$ acts continuously on the space of tilings by translations, and tiling dynamical systems are defined to be closed invariant subsets of the set of all tilings of $\mathbb{R}^d$. Often continuous translation-commuting maps between tiling dynamical systems are constructed as *local codes*: in the image of a tiling, the tile type found at the origin, and the precise location of the origin within that tile, are determined by a fixed-radius neighborhood of the origin in the tiling in the domain. See [8, 9, 11, 12, 13] and the references cited in those sources for background and for examples of tiling dynamical systems and local codes.

Our purpose here is to show that there are (continuous, translation-commuting) factor maps between tiling dynamical systems which are not given by local codes. In fact, such examples can be found for 1-dimensional tiling dynamical systems, which are flows under functions built on subshifts. Mappings between flows under functions commute with translations if and only if they satisfy some cohomological equations (see (2.6) below). Some particularly simple examples of this kind (see (2.1)) will be called *simple* maps. We will make first an example of a simple map which is not local, then a local map which is not simple, and finally a factor

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map which is neither local nor simple. The examples we construct are based on specific symbolic dynamical systems with particular ceiling functions, but only certain properties of the systems and functions are really necessary, so many similar examples can be constructed easily.

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## 2. Tiling systems and factor maps

Let $(X,T)$ and $(Y,S)$ be subshifts on finite alphabets $A$ and $B$. Let $g : X \to (0,\infty)$ and $h : Y \to (0,\infty)$ be continuous functions which depend only on the central entry: $g(x) = g_0(x_0), h(y) = h_0(y_0)$. (If we were given maps $g$ and $h$ that depended on finitely many entries, we could form higher block representations of $X$ and $Y$ to get the dependence down to just the central entry.) We denote by $((X,T)_g,\mathbb{R})$ and $((Y,S)_h,\mathbb{R})$ the flows built under the ceiling functions $g$ and $h$. Recall that $(X,T)_g$, for example, is the quotient space of $(X\times\mathbb{R},\mathbb{R})$ (with the action $(x,s)t = (x,s+t)$) under the equivalence $\sim$ generated by $(x,g(x)) \equiv (Tx,0)$. It is natural to use the notation $[x,s]$ for the equivalence class of a pair $(x,s)$ when $x \in X, 0 \le s < g(x)$, as in [4, 5]. For each equivalence class $\xi \in (X,T)_g$, there are a unique symbolic sequence $\pi_X\xi \in X$ and a unique $\pi_{\mathbb{R}}\xi \ge 0$ such that $0 \le \pi_{\mathbb{R}}\xi < g(\pi_X\xi)$ and $\xi = [\pi_X\xi, \pi_{\mathbb{R}}\xi]$.

*Remark* 2.1. The existence of these maps is special for the one-dimensional situation; in higher-dimensional cases, except for very regular tilings we would obtain a labeled graph rather than a symbolic sequence, and perhaps barycentric coordinates of the origin in the central tile rather than the coordinate $\pi_{\mathbb{R}}\xi$. The maps $\pi_X$ and $\pi_{\mathbb{R}}$ are usually not continuous nor well-behaved with respect to the actions. More properly the notation for these maps would also display their domain, but we rely on the context.

With each point $[x,s] \in (X,T)_g$, we associate a tiling of $\mathbb{R}$: we have, for each element $a$ of the alphabet $A$, a prototile which is a closed interval of length $g_0(a)$. The tiling corresponding to $[x,s]$ $(x \in X, 0 \le s < g(x))$ consists of the sequence of tiles specified by the symbolic sequence $x$; and the central tile, of type $x_0$, contains the origin $s$ units from its left endpoint. This identification is a topological conjugacy between the flow under a function and the associated tiling dynamical system.

A *factor map* or *code* $\phi : ((X,T)_g,\mathbb{R}) \to ((Y,S)_h,\mathbb{R})$ is a continuous onto map which commutes with the action of $\mathbb{R}$. We will say that a factor map $\phi$ is *local* if there is $r \ge 0$ such that if $x, x' \in X$ , $x_i = x'_i$ for $|i| \le r$, $0 \le s < g(x)(= g(x'))$, and $\phi[x,s] = [y,t]$ (for some $y \in Y$ and $0 \le t < h(y)$), then also $\phi[x',s] = [y,t]$. Easy examples of local codes arise from subdividing tiles or amalgamating patches of tiles into new tiles. We say that the factor map $\phi$ is *simple* (cf. [4, 5]) if there are a factor map (continuous shift-commuting map) $\pi : (X,T) \to (Y,S)$ and a function $t : X \to \mathbb{R}$ such that

$$\begin{aligned} t(Tx) - t(x) &= g(x) - h(\pi x) \quad \text{for all } x \in X \quad \text{and} \\ \phi[x,s] &= [\pi x, 0](s + t(x)) \quad \text{for all } [x,s] \in ((X,T)_g, \mathbb{R}). \end{aligned} \tag{2.1}$$

Given $g : X \to \mathbb{R}$, define a cocycle $g(x,n)$ on $(X,T)$ by

$$g(x,n) = \begin{cases} \sum_{k=0}^{n-1} g(T^k x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{k=n}^{-1} g(T^k x) & \text{if } n < 0. \end{cases} \tag{2.2}$$

If $g : X \to \mathbb{R}$ is fixed, for $x \in X$ and $s \in \mathbb{R}$ define $n_X(x,s) = n_X^{(g)}(x,s)$ by

$$g(x, n_X(x,s)) \le s < g(x, n_X(x,s) + 1). \tag{2.3}$$

Then the action of $\mathbb{R}$ on $((X,T)_g, \mathbb{R})$ is given by

$$[x,s]u = \left[T^{n_X(x,s+u)} x, s + u - g(x, n_X(x, s+u))\right] \quad (x \in X, 0 \le s < g(x)) \tag{2.4}$$

(note that $0 \le s + u - g(x, n_X(x,s+u)) < g(T^{n_X(x,s+u)} x)$). Similar definitions and formulas apply to $((Y,S)_h, \mathbb{R})$. Further, in $((X,T)_g, \mathbb{R})$ we have

$$\begin{aligned} (x,s) \sim (x',s') \quad &\text{if and only if} \\ \text{there is } n \in \mathbb{Z} \text{ with } T^n x = x' &\text{ and } s' = s - g(x,n). \end{aligned} \tag{2.5}$$

**Proposition 2.2.** *Given functions $\pi : X \to Y$ and $v : X \to \mathbb{R}$ such that*

$$\begin{aligned} S^{n_Y(\pi x, g(x) + v(x))} \pi(x) &= S^{n_Y(\pi T x, v(Tx))}(\pi T x) \quad \textit{and} \\ v(Tx) - v(x) &= g(x) + h(\pi T x, n_Y(\pi T x, v(Tx))) \\ &\qquad - h(\pi x, n_Y(\pi x, g(x) + v(x))), \end{aligned} \tag{2.6}$$

*then putting*

$$\phi[x,s] = [\pi x, 0](s + v(x)) \tag{2.7}$$

*determines a map $\phi : ((X,T)_g, \mathbb{R}) \to ((Y,S)_h, \mathbb{R})$ which commutes with the $\mathbb{R}$ actions and thus is a factor map if it is onto and continuous (which will be the case if $\pi$ and $v$ are continuous). Conversely, given a factor map $\phi : ((X,T)_g, \mathbb{R}) \to ((Y,S)_h, \mathbb{R})$, there are functions $\pi : X \to Y$ and $v : X \to \mathbb{R}$ such that (2.6) holds for all $[x,s] \in ((X,T)_g, \mathbb{R})$.*

*Proof.* The equations (2.6) guarantee that the map defined by (2.7) commutes with the actions. (Clearly $(\phi[x,s])u = \phi([x,s]u)$ as long as $0 \le s + u < g(x)$. The equations (2.6) imply that the commuting is maintained when we move across the boundary of a tile in $((X,T)_g, \mathbb{R})$.)

Conversely, given a factor map $\phi : ((X,T)_g, \mathbb{R}) \to ((Y,S)_h, \mathbb{R})$, we define

$$\pi x = \pi_Y(\phi[x,0]), \quad v(x) = \pi_{\mathbb{R}}(\phi[x,0]).$$

Then

$$\begin{aligned} \phi[x,s] &= (\phi[x,0])s = [\pi x, v(x)]s \quad (\text{with } 0 \le v(x) < h(\pi x)) \\ &= [\pi x, 0](v(x) + s), \end{aligned} \tag{2.8}$$

and then $\pi$ and $v$ necessarily satisfy (2.6). □

In the case of a simple map, $\pi : (X,T) \to (Y,S)$ is a factor map and we take $v(x) = t(x)$. Then Equations (2.6) follow from (2.1).

## 3. Examples

3.1. **Example 1.** We first construct an example of a simple map which is not a local code.

Let $(X,T)$ be a Sturmian subshift which codes translation $R_\alpha(t) = \langle t+\alpha \rangle = t+\alpha \mod 1$ by an irrational $\alpha$ on $[0,1]$: define $\omega(n) = \chi_{[0,1/2)}\langle n\alpha \rangle$ for all $n \in \mathbb{Z}$ and let $X$ be the orbit closure of $\omega$ under the shift transformation $T$. Then $(X,T)$ is a minimal, uniquely ergodic topological dynamical system, and there is a factor map $\rho : (X,T) \to ([0,1), R_\alpha)$ which is one-to-one except on a countable set on which it is two-to-one (the union of the orbits of 0 and 1/2 under $R_\alpha$).

Let $(Y,S) = (X,T)$ and let the factor map $\pi : (X,T) \to (Y,S)$ be the identity. We will specify $h$ and $t$ first and then *define* $g$ so that Equation (2.1) is satisfied. Fix $\gamma \in (\mathbb{Z}\alpha \mod 1) \cap (0, 1/4)$ and $\eta_1, \eta_2 > 5$ and define $h$ by letting

$$h(y) = \begin{cases} \eta_1 & \text{if } \rho y \in (0,\gamma) \\ \eta_2 & \text{if } \rho y \in (\gamma, 1) \end{cases} \tag{3.1}$$

and extending $h$ by continuity to the four points in $\rho^{-1}\{0\} \cup \rho^{-1}\{\gamma\}$. (The choices of 1/4 and 5 are not completely arbitrary: they guarantee that $g$ is positive and easy to graph.) Since $h$ is continuous on $Y$ and takes only finitely many values, by recoding we may assume that $h$ is a function of just the central coordinate. Let

$$t(x) = \rho(x) \text{ if } 0 < \rho(x) < 1, \tag{3.2}$$

and extend $t$ by continuity to the two points in $\rho^{-1}\{0\}$. (If $x^{(n)} \to x$ in $X$, then eventually $(x^{(n)})_0$ stabilizes at either 0 or 1, so that for large enough $n$ either all $\rho(x^{(n)}) \in [1/2, 1)$ or $\rho(x^{(n)}) \in [0, 1/2)$.) Then

$$t(Tx) - t(x) = \begin{cases} \alpha & \text{if } 0 < \rho(x) < 1-\alpha \\ -1+\alpha & \text{if } 1-\alpha < \rho(x) < 1. \end{cases} \tag{3.3}$$

Finally, define

$$g(x) = t(Tx) - t(x) + h(\pi x), \tag{3.4}$$

thereby obtaining a positive continuous function on $X$ which takes only finitely many values and also automatically satisfying (2.1). Further, (3.4) implies that $\phi(x,s) = (\pi x, 0)(s + t(x))$ is constant on $\sim$-equivalence classes; therefore it defines a continuous map $((X,T)_g, \mathbb{R}) \to ((Y,S)_h, \mathbb{R})$.

Because $\phi$ satisfies (2.1), it commutes with the $\mathbb{R}$ actions. Because $((Y,S)_h, \mathbb{R})$ has a dense orbit (in fact all orbits are dense), $\phi$ is onto. Also, $\phi$ is one-to-one. For if $\phi[x,s] = \phi[x', s']$, then $(x, s+t(x)) \sim (x', s'+t(x'))$ (they determine the same points in $((Y,S)_h,\mathbb{R})$), so there is $d\in\mathbb{Z}$ with $T^d x = x'$ and $s'+t(x') = s+t(x)-h(x,d)$. But (2.1) implies that

$$h(x,d)-g(x,d) = t(x)-t(T^d x), \tag{3.5}$$

so that

$$x' = T^d x, \quad s' = s-g(x,d),$$

and hence $(x,s)\sim(x',s')$ in $((X,T)_g,\mathbb{R})$.

Since $t(x)$ depends on the *entire sequence* $x$ of tile types, $\phi[x,s] = [\pi x, 0](s+t(x))$ cannot be determined from a finite window: the central coordinate of $\pi x$ can be, so that we can determine from a finite window what tile in $\phi[x,s]$ is at the origin, but we cannot tell exactly where in this tile to place the origin without knowing the full sequence $x$ of tile types.

*Remark* 3.1. The pair of maps $(T,\pi)$ in this example provides an action of $\mathbb{Z}^2$ on $X$, and (2.1), when expressed as

$$(t+g)(x) = t(Tx)+g(\pi x), \tag{3.6}$$

is a sort of $\mathbb{Z}^2$ cocycle equation. See [4, Prop. 1.2], [5, p. 352].

3.2. **Example 2.** If one desires a simple map which is not a local code and, unlike the preceding example, is not a topological conjugacy, it is easy to modify Example 3.1 to accomplish that. Let $(X,T)$ be the subshift of $\{0,1,2,3\}^{\mathbb{Z}}$ determined by coding $([0,1),R_\alpha)$ according to entries to the four intervals $[0,1/4)$, $[1/4,1/2)$, $[1/2,3/4)$, $[3/4,1)$. $(Y,S)$ is the coding of $([0,1),R_{2\alpha})$ by the partition $\{[0,1/2),[1/2,1)\}$. Now the factor map $\pi:(X,T)\to(Y,S)$ is determined by the 1-block code that sends the symbols 0 and 2 to 0, and 1 and 3 to 1. (Under the isomorphisms of $X$ and $Y$ with $[0,1)$, $\pi$ corresponds to $s\to 2s$ on $[0,1)$.) Then $h$ is defined as before except with $2\gamma$ replacing $\gamma$, and $t$ and $g$ are determined as before. We arrive at a simple code which is not local, and which this time is not one-to-one.

3.3. **Example 3.** With slightly more care in the choice of the parameters in Example 3.1, we can guarantee that *no local code exists* between $((X,T)_g,\mathbb{R})$ and $((Y,S)_h,\mathbb{R})$, even though there is a simple code $\phi:((X,T)_g,\mathbb{R})\to((Y,S)_h,\mathbb{R})$. In the definition of the ceiling function $h$ over $(Y,S)$, let the division point $\gamma = 1-\alpha$, and choose the heights $\eta_1$ and $\eta_2$ so that $1,\alpha,\eta_1$, and $\eta_2$ are linearly independent over $\mathbb{Z}$. Then the ceiling function $g$ over $(X,T)$ takes values $\eta_1+\alpha$ (over $(0,1-\alpha)$) and $\eta_2+\alpha-1$ (over $(1-\alpha,1)$).

Suppose that $\phi:((X,T)_g,\mathbb{R})\to((Y,S)_h,\mathbb{R})$ is a local code. Select $x\in X$ and let $[y,u]=\phi[x,0]$. Now the central tile $y_0$ of $[y,u]$ and the position $u$ of the origin in it are supposed to be completely determined by a finite window in $[x,0]$, that is to say, by a finite block $C = x_{-r}\dots x_r$ of entries in $x$ (since the position of the origin in $x_0$ is fixed at the left edge). Whenever $C$ appears in $x$, say the translated tiling $[x,0]s$ agrees exactly with $[x,0]$ on its central range of $2r+1$ tiles, we should see in the corresponding position of $[y,u]$ the same tile $y_0$ as before, with its left edge $u$ units to the left of that of the central tile in $C$:

$$(\pi_Y([y,u]s))_0 = y_0, \quad \pi_{\mathbb{R}}([y,u]s) = u. \tag{3.7}$$

This forces a sum of lengths of tiles in $((X,T)_g,\mathbb{R})$ to equal a sum of lengths of tiles in $((Y,S)_h,\mathbb{R})$, which is impossible because of the linear independence.

3.4. **Example 4.** Next we note that it is easy to construct local codes which are not simple. We will produce in fact a local code between two tiling dynamical systems between which no simple code exists. Let $(X,T)$ be a Sturmian symbolic dynamical system as in Example 3.1, with rotation $R_\alpha$ on $[0,1)$ coded by entries to the interval $[0,1/2)$. This time use the ceiling function $g(x) \equiv 1$ on $X$, so that the associated tiling dynamical system $((X,T)_g,\mathbb{R})$ has two tile types (0 and 1), each of length 1. Define a local code onto another tiling system $((Y,S)_h,\mathbb{R})$ by splitting each tile labeled 1 into two tiles, each of length 1/2, both still labeled 1. The underlying symbolic dynamical system $(Y,S)$ is the primitive (induced) transformation over $(X,T)$ formed by doubling 1's in all sequences in $X$, and the ceiling function $h$ on $Y$ takes the value 1 on the cylinder over 0 (at time 0) and 1/2 on the cylinder over 1.

This local code is a topological conjugacy between the two tiling dynamical systems, both of which are minimal. But $(Y,S)$ is topologically weakly mixing [3, 6] while $(X,T)$ has many continuous eigenfunctions (every integer power of $\exp(2\pi i\alpha)$ is an eigenvalue), so there is no factor map from $(Y,S)$ to $(X,T)$, and hence there is no simple code from $((Y,S)_h,\mathbb{R})$ to $((X,T)_g,\mathbb{R})$. Further, there can be no simple code from $((X,T)_g,\mathbb{R})$ to $((Y,S)_h,\mathbb{R})$, since there is no factor map from $(X,T)$ to $(Y,S)$. This is because $([0,1),R_\alpha)$ and $(Y,S)$ are disjoint ([1], see also [7]), so that the existence of factor maps $\rho : X \to [0,1)$ and $\psi : X \to Y$ would imply the existence of a factor map $\phi : X \to [0,1) \times Y$ with $\rho = \pi_{[0,1)}\phi$ and $\psi = \pi_Y\phi$; but $\rho^{-1}\{u\}$ is a singleton for some (even many) $u \in [0,1)$, so $Y$ must consist of just one point. (Alternatively, for certain $\alpha$, and some $\beta$ replacing 1/2, the uniquely ergodic system $(Y,S)$ is measure-theoretically weakly mixing [3] and hence cannot be a (topological, hence measure-theoretic) factor of the uniquely ergodic, purely discrete spectrum system $(X,T)$.)

3.5. **Example 5.** Finally, we construct a factor map between tiling dynamical systems which is neither simple nor local, and in fact is such that no simple or local code exists between the two tiling dynamical systems.

Let $(X,T)$ and $(Y,S)$ be as in Example 3.4. $(X,T)$ is a Sturmian symbolic dynamical system based on rotation modulo 1 by $\alpha$, coded via entries to an interval $[0,\beta)$ with $\beta < 1-\alpha$ and $\beta \notin \mathbb{Z}\alpha \mod 1$. $(Y,S)$ is the induced transformation over $(X,T)$ with a 'second floor' over $X_1 = [1]$ = the cylinder set determined by fixing the central entry to be 1. If $X_0 = [0]$ is the cylinder set consisting of those points with central entry 0, then $X = X_0 \cup X_1$, while $Y = X_0 \cup X_1 \cup X_1'$, where $X_1'$ is a homeomorphic copy of $X_1$ under a map $\theta : X_1 \to X_1'$. Moreover,

$$Sy = \begin{cases} Ty & \text{if } y \in X_0 \\ \theta y & \text{if } y \in X_1 \\ T\theta^{-1} y & \text{if } y \in X_1'. \end{cases} \tag{3.8}$$

Define a continuous, not shift-commuting map $\pi : Y \to X$ which is one-to-one on part of $Y$ and two-to-one on part by collapsing each block 11 to a single 1; in terms of the induced transformation,

$$\pi y = \begin{cases} y & \text{if } y \in X_0 \cup X_1 \\ \theta^{-1} y & \text{if } y \in X_1', \end{cases} \tag{3.9}$$

so that $\pi$ identifies $[1] \subset Y$ and the second floor over it and is the identity on $[0]$.

This time the code will go in the direction $\phi : ((Y,S)_g, \mathbb{R}) \to ((X,T)_h, \mathbb{R})$—we alter the choice of letters a little. As in Examples 3.1 and 3.3, let $\gamma = 1 - \alpha$ and define $h$ on $X$ to be the continuous extension of the function that takes the value $\eta_1$ on $(0, \gamma)$ and $\eta_2$ on $(\gamma, 1)$. We assume that $\eta_1$ and $\eta_2$ are positive and not too small, say at least 5. Define $v_0 : Y \to \mathbb{R}$ by extending continuously to all of $Y$ the function

$$v_0(y) = \rho(\pi y) = \begin{cases} \rho(y) & \text{if } y \in X_0 \cup X_1, \rho(y) \neq 0 \\ \rho(\theta^{-1} y) & \text{if } y \in X_1', \rho(\theta^{-1}y) \neq 0. \end{cases} \tag{3.10}$$

Then $v_0(Sy) - v_0(y)$ is continuous on $Y$ and takes only finitely many values ($0, \alpha$, and $-1+\alpha$). Define

$$v(y) = v_0(Sy).$$

Then $v(Sy) - v(y)$ still takes only the values $0, \alpha$, and $-1 + \alpha$, and it takes the value $\alpha$ on $X_1$.

Having defined $\pi, h$, and $v$, we must now define $g$ on $Y$ so as to satisfy the relations (2.6), which in the current setting take the form

$$\begin{aligned} T^{n_X(\pi y, g(y) + v(y))} \pi y &= T^{n_X(\pi S y, v(Sy))}(\pi S y), \\ v(Sy) - v(y) &= g(y) + h(\pi S y, n_X(\pi S y, v(Sy))) \\ &\quad - h(\pi y, n_X(\pi y, g(y) + v(y))). \end{aligned} \tag{3.11}$$

First note that if

$$m(y) = \chi_{X_0 \cup X_1'}(y), \tag{3.12}$$

then

$$\pi(Sy) = T^{m(y)}(\pi y). \tag{3.13}$$

Further, $n_X(x, v(y)) = 0$ for all $x, y$, since $0 \le v(y) \le 1$ for all $y$ and $h \ge 5$. Therefore

$$h(\pi S y, n_X(\pi S y, v(Sy))) = 0 \quad \text{for all } y, \tag{3.14}$$

simplifying (3.11).

Define

$$g(y) = v(Sy) - v(y) + h(\pi y, m(y)). \tag{3.15}$$

Then $g$ is a continuous function on $Y$ that assumes finitely many positive values (since $v(Sy) - v(y) = \alpha > 0$ on $X_1$, where $m(y) = 0$ so $h(\pi y, m(y)) = 0$). If we can verify that

$$n_X(\pi y, g(y) + v(y)) = m(y) \quad \text{for all } y, \tag{3.16}$$

then (3.11) will follow.

Now $h(\pi y, m(y))$ is either 0 or $h(\pi y)$, depending on whether $m(y) = 0$ or 1. Thus

$$g(y) + v(y) = v(Sy) + h(\pi y, m(y)), \tag{3.17}$$

and $v$ takes values in $[0,1)$, so the index $n_X(\pi y, g(y) + v(y))$, which gives the value of $n$ for which

$$\sum_{k=0}^{n-1} h(T^k \pi y) \leq g(y) + v(y) < \sum_{k=0}^{n} h(T^k \pi y), \tag{3.18}$$

must coincide with $m(y)$.

As in Example 3.4, since there can be no factor map between the underlying symbolic dynamical systems, for this example there can be no simple code between the tiling dynamical systems. The tiles in $((X,T)_h, \mathbb{R})$ have lengths $\eta_1$ and $\eta_2$, while those in $((Y,S)_g, \mathbb{R})$ have lengths $\alpha, -1+\alpha+\eta_1, \eta_2, \eta_2+\alpha$, and $-1+\alpha+\eta_2$. If $1, \alpha, \eta_1$, and $\eta_2$ are chosen to be linearly independent over $\mathbb{Z}$, then no nonzero sum of tile lengths in $((X,T)_h, \mathbb{R})$ can equal a sum of tile lengths in $((Y,S)_g, \mathbb{R})$, unless only tiles of length $\eta_2$ are used in both cases. But since the systems $(X,T)$ and $(Y,S)$ are minimal, any finite block $C = x_{-r} \dots x_r$ of entries in a sequence $x \in X$ will appear separated by (arbitrarily long) blocks in which every symbol appears. Thus the argument of Example 3.3 applies to show that no local code can exist between these tiling dynamical systems.

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