Errata and Comments for *Ergodic Theory*, by Karl Petersen, Cambridge University Press, 1989 paperback (with corrections) edition

I received many notifications about misprints and errors in the original 1983 edition, and I even found a few myself. The most extensive list was compiled by Anzelm Iwanik. All of these corrections were included in the 1989 edition, but somehow new errors continue to appear. Most recently a good bunch were turned up by Uijin Jung and Vladimir Zhuravlev. I thank everyone who has contributed in this way to easing the trouble of future readers.

Corrections are listed by page and line number or other identifier. Quotation marks are omitted with the hope that the intent is clear.

viii. Several people have asked me about the quote on the dedication page. It is from *Kalevipoeg*, the Estonian national epic, which was recorded in the mid-nineteenth century by Fr. R. Kreutzwald from folktales. Loosely translated, it says:

At the direction of the gods
The lines of our lives run
And the waves of chance flow.

I found the references to trajectories and probability (maybe Schrödinger's equation?) relevant for a book on ergodic theory, which is probabilistic dynamics.

12, last. Replace $\phi_{ij}\phi_{jk}$ by $\phi_{jk}\phi_{ij}$

13, G. Replace $\phi^{-1}\mathcal{B} = \hat{\mathcal{B}}$ by $\bigvee_{k=0}^{\infty} \hat{T}^k \phi^{-1}\mathcal{B} = \hat{\mathcal{B}}$

13, G. Replace all $(X_i, \mathcal{B}_i, \mu_i) = (X, T^{-i}\mathcal{B}, \mu)$ by all $(X_i, \mathcal{B}_i, \mu_i) = (X, \mathcal{B}, \mu)$

13, G. Replace $T_i =$ identity by $T_i = T$

16, Th. 4.4. After normalized add $(\mu(0') = 1)$

18, 10. Replace ϕ by $\emptyset$

19, (3). Replace L^2 norm by norm

19, third formula from bottom. Replace L^2 by H

21, -2. replace Lebesgue spaces by Lebesgue probability spaces

27, 3 lines above Th. 2.1. Replace Kolmogorov (1937) by Kolmogorov (1937, see also 1928)

29, first formula. $C_n = B_n' \cup TB_n' \cup \cdots \cup T^{n-1}B_n'$,

33, Ex. 6. Add Hint: Use the proof of the Maximal Ergodic Theorem.

33, 4 and 6. Replace R by $\mathbb{R}$

43, Th. 4.2. In order to stick to Lebesgue spaces, replace Conversely, given a subgroup by Conversely, given a countable subgroup

43, -12. Replace Finally, suppose we are given a subgroup by Finally, suppose we are given a countable subgroup

44, 8. Replace $g = a_\lambda f_\lambda$ by $f = a_\lambda f_\lambda$

47, middle. Replace is an invariant subset of $\tilde{A}$ by is an invariant subset of A^c.

49, first formula. Replace $[(k-n-1)/n]$ by $[(k-n)/n]$

52, second formula. Replace χ_{E_i} by χ_{E_j}

52, third formula should be

$$\begin{aligned} q_{ij} &= \frac{1}{p_i} \int \left[\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{E_i}(x)\chi_{E_j}(\sigma^k(x)) \right] \mathrm{d}\mu(x) \\ &= \frac{1}{p_i} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(E_i \cap \sigma^{-k} E_j) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{ij}^k. \end{aligned}$$

56, Ex. 9. Add Hint: Form a set B of measure less than 1 with $\mu(B \cap T^k B) > 0$ for all k as a union of higher and higher towers.

61, -2. Replace absolutely continuous by equivalent to. (Note: The statement may be all right for Gaussian systems but seems to be an open question in general.)

70, Rem. 6.3. Replace $< \varepsilon$, by $< \varepsilon_1$

71, 1. After "we have" add (for each sequence (a_n) for which the limits on the right side exist)

76, 4. Replace Monotone by Dominated.

76, last. Before the period, add by the Ergodic Theorem or 1.12 and 1.15

94, 4. Replace interval, by interval minus its length,

102, 8. replace $\frac{2}{\lambda}$ by $\frac{4}{\lambda}$

108, Lemma 5.1. Remove and

113, middle. Delete $-f$

127, -4. Replace first entry time of y to A by first entry time of y to A under T^{-1}

138, middle. After therefore $\nu = \mu$. add (Recall from p. 135 that μ is fully supported on X.)

152, last. Replace discontinuity by continuity

162, Theorem 3.1. Replace $,j = 1, \dots r,$ by , $1 \le j \le r$,

165, -3. Replace $X = \text{cl}\{\sigma_i^n \chi_S : n \in \mathbb{Z}, i = 1, 2, \dots, r\}$ by $X = \text{cl}\{\sigma_1^{m_1} \dots \sigma_r^{m_r} \chi_S : m_i \in \mathbb{Z}, i = 1, 2, \dots, r\}$

172, -12, in Th. 3.16. Replace k_m by k_r and $1 \le i \le m$ by $1 \le i \le r$

181, -2. Replace f_1 by f_i

182, -8, displayed formula in (2). Add integral sign before first f

188, -12. Replace we put $\tilde{\phi}(E) = \phi(\chi_E)$ for $E \in \mathcal{B}$ by for $E \in \mathcal{B}$ we put $\tilde{\phi}(E)$ equal to the set whose characteristic function is $\phi(\chi_E)$

199, 11. Replace N_E by n_E

214, -12, in (2). Replace Then by then

216-217. Replace Lemma 5.4 by:

Lemma 5.4 $(\overline{\mathcal{O}(x)}, \sigma)$ is uniquely ergodic if and only if there is a sequence of integers $k_n \nearrow \infty$ such that each block B appears in every k_n-block in x with uniform limiting frequency: Given a block B there is $c_B \ge 0$ such that for each $\epsilon > 0$ there is N such that if $n \ge N$ and C is any k_n-block in x, then

$$|\frac{N(B,C)}{k_n} - c_B| < \epsilon.$$

Proof. Suppose that $(\overline{\mathcal{O}(x)}, \sigma)$ is uniquely ergodic with invariant measure μ. For each block B, the *cylinder set*

$$[B] = \{y : y_0 y_1 \dots y_{l(B)-1} = B\}$$

is open and closed, hence its characteristic function is continuous. By 2.8,

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_{[B]}(\sigma^k y) \to \mu[B]$$

uniformly for $y \in \overline{\mathcal{O}(x)}$. The expression on the left-hand side is the frequency of B in the initial n-block of y, which could be any n-block in x. Thus the frequency of appearance of B in any sufficiently long n-block in x is close to $\mu[B] = c_B$. (Note that this could be 0.) Clearly the condition in the statement of the Lemma is satisfied.

Conversely, suppose that the condition is satisfied and μ is an ergodic invariant Borel probability measure for $(\overline{\mathcal{O}(x)}, \sigma)$. Let B be any block. Then, by the Ergodic Theorem, for μ-almost all $y \in \overline{\mathcal{O}(x)}$,

$$\frac{N(B, y_0 y_1 \dots y_{k_n-1})}{k_n} \to \mu[B].$$

But since $y_0 y_1 \dots y_{k_n-1}$ are blocks of length k_n that appear in x, the above fractions also converge to c_B. The same would apply to any other ergodic invariant measure ν, showing that $\nu = \mu$, since the cylinder sets generate the full σ-algebra. □

218. Replace Proposition 5.6 by:

Proposition 5.6 (Ω, σ) is uniquely ergodic.

Proof. Given a block B, note first that B appears with a well-defined limiting frequency in the A_j as $j \to \infty$. For choose m so that B appears in A_m. There is q_m with $0 \le q_m \le 2l(B)$ such that

$$N(B, A_{m+1}) = 3N(B, A_m) + q_m,$$

since $A_{m+1} = A_m A_m 1 A_m$, B appears $N(B, A_m)$ times in each of the three A_m's, and B can appear no more than $l(B)$ times across each juncture $A_m A_m$ or $A_m 1 A_m$. Repeating the argument and inducting,

$$N(B, A_{m+k}) = 3^k N(B, A_m) + q_m(3^{k-1} + 3^{k-2} + \dots + 1),$$

so that (using the above formula for l_{m+k})

$$\frac{N(B, A_{m+k})}{l_{m+k}} \to 2\frac{N(B, A_m)}{3^{m+1}} + \frac{q_m}{3^{m+1}} = c_B.$$

Now if C is any block in x with $l(C) = l(A_{m+k})$, then C appears in $A_{m+k+1} = A_{m+k} A_{m+k} 1 A_{m+k}$, and thus C appears in either $A_{m+k} A_{m+k}$ or $A_{m+k} 1 A_{m+k}$. As we move our attention from A_{m+k} to C, appearances of B at the beginnings or ends of the A_{m+k} or across the junctures can be either disrupted or created, but no more than $l(B)$ in each case. Therefore

$$|N(B, C) - N(B, A_{m+k})| \le 3l(B),$$

and hence given $\epsilon > 0$ we will have

$$|\frac{N(B, C)}{l_{m+k}} - c_B| < \epsilon$$

for all large enough k. The conclusion then follows from Lemma 5.4. □

219, -12 to -1. Replace last paragraph by the following.

The actual argument applies Lusin's Theorem to find, given $\epsilon > 0, \epsilon \ll 1/100$, a set F with $\mu(F) > 1 - \epsilon$ such that if $f(x) = \exp 2\pi i \Theta(x)$, then $\Theta(x)$ is uniformly continuous on F. Choose n large enough that $\mu[A_n] = 2/3^{n+1} \ll \epsilon$ and so that if x, x' agree on their central $l_n/3$-blocks then $|\Theta(x) - \Theta(x')| < \epsilon$.

The sets $[A_n], \sigma[A_n], \ldots, \sigma^{l_n-1}[A_n]$ have equal measures, are pairwise disjoint, and, together with $\sigma^{l_n}[A_n]$, cover Ω. This implies that there is $k \in [l_n/3, 2l_n/3]$ with $\mu(\sigma^k[A_n] \cap F) \geq (1-6\epsilon)\mu(\sigma^k[A_n])$. Let $E = \sigma^k[A_n]$ and $G = \sigma^k[A_{n+1}]$.

We want to show that

$$G \cap F \cap \sigma^{-l_n} F \cap \sigma^{-(2l_n+1)} F \neq \emptyset.$$

Since $\mu(F^c \cap G) \leq \mu(F^c \cap E) \leq 6\epsilon\mu(E)$, we have

$$\begin{aligned}\mu(F \cap G) \geq \mu(G) - 6\epsilon\mu(E) &= (\frac{1}{3} - 6\epsilon)\mu(E) \\ &= (1-18\epsilon)\mu(G).\end{aligned}$$

Similarly, since $\sigma^{l_n} G \subset E$, so that $\mu(\sigma^{l_n}(F \cap G) \cap F^c) \leq \mu(E \cap F^c) \leq 6\epsilon\mu(E)$, we have

$$\begin{aligned}\mu(\sigma^{l_n}(F \cap G) \cap F) &\geq \mu(\sigma^{l_n}(F \cap G)) - 6\epsilon\mu(E) \\ &\geq (1-18\epsilon)\mu(G) - 6\epsilon\mu(E) = (1-36\epsilon)\mu(G).\end{aligned}$$

Finally, using $\sigma^{l_n+1}\sigma^{l_n} G \subset E$, so that $\mu(\sigma^{l_n+1}[\sigma^{l_n}(F \cap G) \cap F] \cap F^c) \leq \mu(E \cap F^c) \leq 6\epsilon\mu(E)$, we find that

$$\begin{aligned}\mu(\sigma^{l_n+1}[\sigma^{l_n}(F \cap G) \cap F] \cap F) &\geq \mu(\sigma^{l_n+1}[\sigma^{l_n}(F \cap G) \cap F]) - 6\epsilon\mu(E) \\ &\geq (1-36\epsilon)\mu(G) - 18\epsilon\mu(G) = (1-54\epsilon)\mu(G).\end{aligned}$$

Applying σ^{-2l_n-1} shows that there is at least one point

$$x \in G \cap F \cap \sigma^{-l_n} F \cap \sigma^{-(2l_n+1)} F.$$

Then $x, \sigma^l_n x, \sigma^{2l_n+1} x$ are all in $\sigma^k[A_n] \cap F$, so they agree on their central $l_n/3$-blocks and therefore differ by no more than ϵ in the values assigned to them by Θ. We have

$$\Theta(\sigma^{l_n} x) = l_n\lambda + \Theta(x), \quad \Theta(\sigma^{2l_n+1} x) = (l_n+1)\lambda + \Theta(\sigma^{l_n} x) \qquad \text{mod } 1,$$

so that

$$\Theta(\sigma^{2l_n+1} x) - \Theta(\sigma^{l_n} x) = \lambda + \Theta(\sigma^{l_n} x) - \Theta(x) \qquad \text{mod } 1.$$

Now both the left side and $\Theta(\sigma^{l_n} x) - \Theta(x)$ are less than ϵ, so we must have $\lambda = 0$.

222, 8. delete $\in \Omega$

222, 18. Replace ϕ would have by σ^k would have

228, -6. Remove superscript on E'

228, -11. Add minus sign before N

237, -6. replace $[\log_2$ by $\log_2[$

238, Formula $5'$. Replace a by α

238, -2. Replace a by α

239, -4. Put brackets around first fraction $\mu(A \cap B)/\mu(B)$

246, 13. Delete the first minus sign

258, 6. Replace E by α

258, 16. Replace $H\{E_n\}$ by $H(\{E_n\})$

265, -8. Replace . after $(\mathcal{U})_{-n}^{n}$ by ,

265, -7. Replace As by as

269, -3. Replace $\mathcal{U}_1$ by $(\mathcal{U}_1)_0^{n-1}$

269, last. Replace $\mathcal{U}_0^{n-1}$ by $\mathcal{U}$

270, 1. Replace $\mathcal{U}_0^{n-1}$ by $\mathcal{U}$

270, 1. Replace $\mathcal{U}, n$ by $\mathcal{U}_1, n$

270, 1. Replace X, by X seen in $\mathcal{U}$,

270, 4. Replace $\mathcal{U}, n$ by $\mathcal{U}_1, n$

270, 6 Replace $\mathcal{U}, n$-names, by $\mathcal{U}_1, n$-names seen in $\mathcal{U}$,

273, 1. Replace different αs by different α's and μ's

273, 3. Add $\sup_\mu$ at beginning of line

274, 1. Replace 1976). by 1976)).

277, 16. Replace the index k by i

279, -2. Replace The hard-sphere gas in a rectangular box by Some hard-sphere gas models

316, -13. Replace 117 by 177

327, 4, under Morse sequence. Replace 210 by 209

328, 30, under toeplitz. Replace toeplitz by Toeplitz

328, -1. Replace stong by strong