REINFORCED RANDOM WALKS AND ADIC TRANSFORMATIONS

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ABSTRACT. To a given finite graph we associate three kinds of adic, or Bratteli-Vershik, systems: stationary, symbol-count, and reinforced. We give conditions for the natural walk measure to be adic-invariant and identify the ergodic adic-invariant measures for some classes of examples. If the walk measure is adic-invariant we relate its ergodic decomposition to the vector of limiting edge traversal frequencies. For some particular nonsimple reinforcement schemes we calculate the density function of the edge traversal frequencies explicitly.

1. INTRODUCTION

We define and analyze several kinds of adic, or Bratteli-Vershik, systems [8, 11, 16, 20, 21, 23, 27, 28] related to walks on a finite, directed, strongly connected graph G. Each infinite path on G can also be regarded as a point in a (stationary—see below) adic system. If one wishes to keep track of how many times each edge has been traversed, for example in connection with questions of exchangeability or conservation of particle types, one is led to construct a (nonstationary) symbol-count adic, such as the Pascal (or CCR [3]) system. Suppose that in addition the walker experiences reinforcement, represented by means of nonnegative integers assigned to the edges of G which are incremented according to a fixed rule depending on the edges traversed. For example, in simple positive reinforcement each time that an edge is traversed the number on it is increased by 1, while in simple opposite reinforcement the numbers on all the edges not traversed are increased by 1. Then one is led to consider an adic system in which the edges of the symbol-count adic are ramified according to the reinforcement scheme. For the simplest nontrivial case when G is the graph G_2 that consists of two loops at a single vertex, the stationary adic is the well-known dyadic odometer or von Neumann-Kakutani adding machine, the symbol-count adic is the Pascal system, the simple oppositely reinforced system is the Euler adic [2, 8, 9], and the simple positively reinforced system is the reverse Euler adic [8, 9].

Nonnegative numbers on the edges of G naturally define a probability measure on the set of infinite paths on G by letting the probability that an edge is next traversed be proportional to the number on that edge. The *walk measure* is defined

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by putting the number 1 on each edge. We do not discuss the formulations in terms of urn models or random walks in random environments, our main concern being the three kinds of dynamical systems mentioned above. The initial question of interest is the identification of the ergodic adic-invariant measures (called the "central" measures in the group-representation setting, "states" or "traces" in the C^* context). The walk measure may or may not be adic-invariant or ergodic. For the Pascal adic system, the identification of the ergodic adic-invariant measures as exactly the Bernoulli measures is closely connected with the Hewitt-Savage and de Finetti Theorems (see [23]).

For adic systems arising from reinforced walks on G_2, which are "Pascal based", we state necessary and sufficient conditions for a measure specified by weights on the edges of the Bratteli diagram to be adic-invariant (Proposition 3.1) and in particular for the walk measure on such a system to be adic-invariant (Proposition 3.2). A class of Pascal based adics with *constant diagonals* is especially amenable to analysis (Theorem 3.7). In Section 4 we review how these considerations apply to, and what is so far known about, the main examples based on G_2: the Pascal, reverse Euler, Euler, and Stirling systems. Properties of these systems depend on the combinatorial and number-theoretic properties of the numbers of paths between vertices in the associated Bratteli diagrams, and in fact many of the properties needed to understand the dynamics of these systems more completely remain unknown. The numbers of paths starting from the root vertex in these systems are the binomial coefficients, factorials, Eulerian numbers, and absolute values of Stirling numbers of the first kind, respectively. Section 5 extends the foregoing to higher-dimensional systems, namely walks based on the graph G_d consisting of d loops at a single vertex. In Section 6, we relate the ergodic decomposition of an adic-invariant walk measure to the vector of limiting edge traversal frequencies (Theorem 6.3 and Corollary 6.4). For simple reinforced random walk, Coppersmith and Diaconis [5] stated that this vector has a distribution that is absolutely continuous with respect to Lebesgue measure on the simplex of appropriate dimension and gave a formula for its density function. Keane and Rolles [18] published a proof. For the graph G_d consisting of d loops at a single vertex, arbitrary positive integer vector s of initial numbers on the edges, and a positive reinforcement scheme which increments each edge traversed by a fixed positive integer a, we show in Theorem 7.4, by using some classical analysis, how to obtain the formula explicitly.

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2. Adic systems related to graphs, walks, and reinforced walks

Let G be a finite, directed, strongly connected graph (so that between each pair of vertices there exists a directed path). We regard the edge set $\mathcal{E}(G)$ of G also as a finite alphabet of symbols, $\mathcal{E}(G) = A = \{0, 1, \dots, d-1\}$, and we assume that $d \geq 2$. As usual, each edge e has a *source* or *initial vertex* $i(e)$ and a *target* or *terminal vertex* $t(e)$. Let $p = e_1 \dots e_n$ be a path in G, so that the e_i are edges, and the path is allowed, in the sense that $t(e_i) = i(e_{i+1})$ for all $i = 0, \dots, n-1$. We define $i(p) = i(e_1)$ and $t(p) = t(e_n)$. We will associate with G three kinds of adic systems: a *stationary adic* $\mathcal{S}(G)$, a *symbol-count adic* $\mathcal{C}(G)$, and, when given a reinforcement scheme, a *ramified adic* $\mathcal{R}(G)$.

An *adic or Bratteli-Vershik system* consists of the space X of infinite paths starting at the root in a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ and a successor map T defined from the set of nonmaximal paths in X to X. In more detail, a *Bratteli diagram* is a directed graded graph $(\mathcal{V}, \mathcal{E})$. The set $\mathcal{V}$ of vertices is the disjoint union of finite nonempty sets $\mathcal{V}_n, n = 0, 1, 2, \dots$. $\mathcal{V}_0$ consists of a single vertex v_0, called the *root*. For each n, we identify $\mathcal{V}_n$ with the set $\{(n, k) : k = 0, 1, \dots |\mathcal{V}_n| - 1\}$. The edge set $\mathcal{E}$ is also the disjoint union of finite sets $\mathcal{E}_n, n = 1, 2, \dots$. Each edge in $\mathcal{E}_n$ has initial vertex in $\mathcal{V}_{n-1}$ and terminal vertex in $\mathcal{V}_n$. We assume that every vertex is the initial vertex of at least one edge, and every vertex other than v_0 is the terminal vertex of at least one edge. For each vertex $v \neq v_0$, denote by $I(v)$ the set of edges E into v, i.e., those for which $t(E) = v$. We assume that each set $I(v)$ is linearly ordered. When drawing diagrams, we label the vertices of $\mathcal{V}_n$ by $(n, k), k = 0, 1, \dots, |\mathcal{V}_n| - 1$, and if possible we assume that the edges in each $I(v)$ are ordered from left to right. The *dimension* of a vertex (n, k) is defined to be the number of paths from the root to (n, k) and denoted by $\dim(n, k)$. The set X consists of infinite paths in the graph $(\mathcal{V}, \mathcal{E})$ which begin at the root v_0. These paths can be specified either as a sequence of vertices indexed by $\mathbb{Z}_+$ or a sequence of edges indexed by $\mathbb{N}$: $x = x_1 x_2 \dots$, with $x_k \in \mathcal{E}_k$ for all $k = 1, 2, \dots$. For notational purposes, if x passes through vertex (n, k), we can denote this vertex by $(n, k_n(x))$. We say that two paths $x = x_1 x_2 \dots$ and $y = y_1 y_2 \dots$ are at distance less than or equal to $1/2^n$ if $x_i = y_i$ for all $i = 1, \dots, n$. This definition makes X a compact totally disconnected metric space. A *cylinder set* is a set of paths determined by fixing finitely many edges (usually an initial segment of the path, in which case we say that the cylinder set is *anchored*). Cylinder sets are open and closed. We may identify cylinders with the paths and edge strings that define them. Denote by $X_{\max}$ the set of those paths in X which include only maximal edges: x_k is the maximal element of $I(t(x_k))$ for all $k = 1, 2, \dots$. The set $X_{\min}$ of minimal paths is defined similarly. There is a partial order $<$ on X defined by agreeing that if $x, y \in X$ are paths specified as sequences of edges, then $x < y$ if there is an N such that $x_N < y_N$ as elements of $I(t(x_N)) = I(t(y_N))$ while $x_n = y_n$ for all $n > N$. Then the *adic transformation* $T : X \setminus X_{\max} \to X \setminus X_{\min}$ is defined by setting Tx equal to the successor in X of x, that is, the smallest y that satisfies $y > x$. T is a homeomorphism from $X \setminus X_{\max}$ to $X \setminus X_{\min}$.

We say that a Borel probability measure μ on X is *adic-invariant* (or *T-invariant*) if $\mu(T^{-1}E) = \mu(E)$ for each Borel set $E \subset X \setminus X_{\min}$. To avoid degenerate situations, we assume that the path space X is homeomorphic to the Cantor set and we consider only nonatomic measures.

The *stationary adic system* $\mathcal{S}(G)$ associated to a finite directed graph G consists of the Bratteli-Vershik system defined as follows. At level 0 there is a single vertex, the root, with edges to each vertex at level 1. For each $n \geq 1$, the vertex set $\mathcal{V}_n$ at level n is a copy of the vertex set of G. For each edge in G, say from g_1 to g_2, there is an edge in each $\mathcal{E}_n$ from the vertex in V_{n-1} that corresponds to g_1 to the edge in V_n that corresponds to g_2. For each $g \in G$ the set $I(g) = t^{-1}\{g\}$ of edges entering

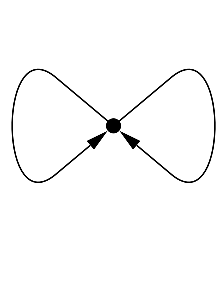

FIGURE 1. G_2 and the associated stationary adic system.

g is linearly ordered, and this same order is lifted to each $I(v)$ for the vertices $v \in \mathcal{V}$ that correspond to g. In case G is the graph G_2 which consists of a single vertex with two loops based there, $\mathcal{S}(G)$ is the 2-odometer.

Next we define an adic system $\mathcal{C}(G)$ which keeps track of the possible symbol counts (the numbers of times that each edge has been traversed) accumulated by a walker on the graph G. The set $\mathcal{V}_1$ of vertices at level 1 is a copy of the edge set of G, and there is an edge from the root to each vertex in $\mathcal{V}_1$. For a path $p = e_1 \dots e_n$ in G we define the *symbol count vector* $C(p) \in \mathbb{Z}_+^d$ by $C(p)_j = |\{i = 1, \dots, n : e_i = j\}|$ for $j = 0, \dots, d-1$. We say that two paths p and q are *count and terminal equivalent* if $C(p) = C(q)$ and $t(p) = t(q)$. The set of vertices $\mathcal{V}_n$ at each level $n > 1$ of the Bratteli diagram for $\mathcal{C}(G)$ is then defined to be the set of equivalence classes of paths in G of length n. There is an edge E from $u \in \mathcal{V}_n$ to $v \in \mathcal{V}_{n+1}$ if and only if there are a path p with $t(p) = u$, a path q with $t(q) = v$, and an edge e of G such that $C(pe) = C(q)$. Note that for each edge E in $\mathcal{C}(G)$ there is at most one such edge $e = e(E)$ in G, since the symbol counts for $u \in v_n$ and $v \in v_{n+1}$ are known. If $v \in \mathcal{V}_n$ and $E, E' \in I(v)$, we define $E < E'$ if and only if $e(E) < e(E')$.

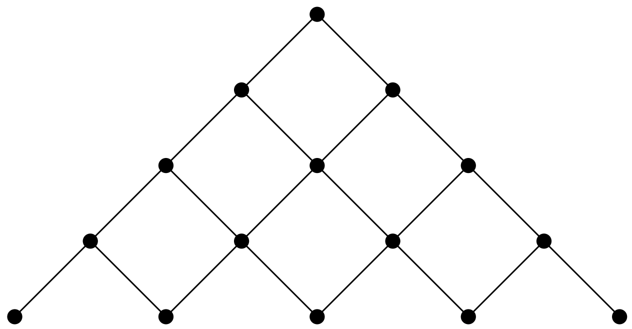

FIGURE 2. The Pascal adic as $\mathcal{C}(G_2)$.

If we want to take into account possible restrictions on transitions between edges of G specified by a 0,1 matrix M indexed by the edge set A of G (in addition to the restrictions inherent in the graph G itself), we may vary the above definition, replacing count and terminal equivalence by count and last edge equivalence: two paths p and q are equivalent in this sense if $C(p) = C(q)$ and they have the same final edges (and hence the same terminal vertices). We denote the resulting adic system by $\mathcal{C}(G, A)$.

Now we define an adic system $\mathcal{R}(G)$ which takes into consideration a reinforcement scheme (s, R) for numbers on the edges of G. For each infinite path $p = e_1 e_2 \dots$ in G (with each $e_i \in A$), at each time $n = 0, 1, 2, \dots$ we will have a vector $N(p, n) \in \mathbb{Z}_+^d$ of nonnegative numbers indexed by the edges of G. A *start vector* $s \in \mathbb{Z}_+^d$ is given, and for each p we put $N(p, 0) = s$: the initial number on edge j is $s_j, j = 0, 1, \dots, d-1$. The *reinforcement scheme* consists of s and a function $R : A \to \mathbb{Z}_+^d$. The idea is that when a walker traverses edge $e \in A$, the numbers on all the edges are incremented by $R(e)$. Thus, for $n \geq 0$ we put $N(p, n+1) = N(p, n) + R(e_{n+1})$.

Note that if $p = e_1 \dots e_n$ and $q = e'_1 \dots e'_n$ are paths in G which are count and terminal equivalent, then they determine the same numbers on edges of G leaving $t(p) = t(q)$: $N(p, n+1) = N(q, n+1)$; this is a consequence of the commutativity of vector addition. Then the Bratteli diagram for $\mathcal{R}(G)$ is obtained from the one for $\mathcal{C}(G)$ as follows. The vertex sets of $\mathcal{R}(G)$ and $\mathcal{C}(G)$ are the same, but we multiply the edges of $\mathcal{C}(G)$ according to the reinforcement scheme as follows. Suppose that in $\mathcal{C}(G)$ we have an edge E from $u \in \mathcal{V}_n$ to $v \in \mathcal{V}_{n+1}$, so that as above there are paths $p \in u$ and $q \in v$ and an edge e of G such that $pe = q$. Then E is replaced in $\mathcal{R}(G)$ by an ordered set of $N(p, n)_e$ edges (recall that $e \in \{0, 1, \dots, d-1\}$). If $E < E'$ in $\mathcal{C}(G)$ as edges entering the same vertex, we agree that in $\mathcal{R}(G)$ each of the edges replacing E is less than each of the edges replacing E'.

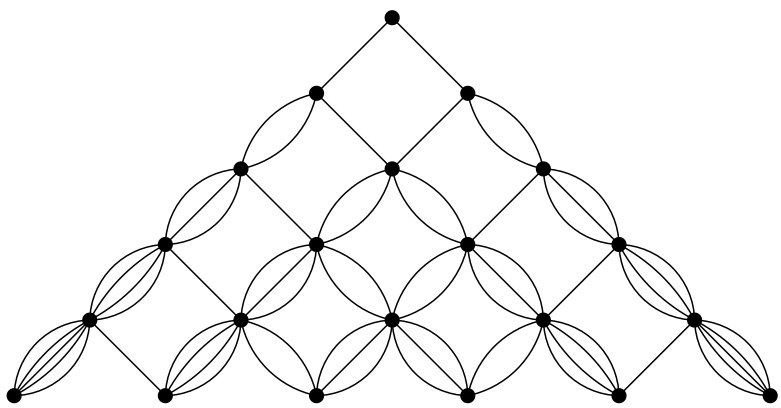

FIGURE 3. $\mathcal{R}(G_2)$ where $R(0) = (1, 0)$ and $R(1) = (0, 1)$.

3. Measures on adic systems

A Borel probability measure can be defined on the path space X of a Bratteli diagram by specifying nonnegative weights $w(E)$ on the edges $E \in \mathcal{E}$ of the Bratteli diagram in such a way that (1) if E and E' have the same initial and terminal vertices, then $w(E) = w(E')$; and (2) for each vertex $V \in \mathcal{V}$, the sum of the weights on the edges leaving V is 1:

$$\sum_{i(E)=V} w(E) = 1. \tag{3.1}$$

Then for each anchored cylinder set $C = [E_1 \dots E_n] = \{x \in X : x_1 = E_1, \dots, x_n = E_n\}$, we put $\mu(C) = w(E_1) \dots w(E_n)$. This defines a countably additive probability measure on the semialgebra of anchored cylinder sets, and so by the Carathéodory extension theorem μ extends to a Borel probability measure on X. Every adic-invariant measure arises in this way, but not all measures so defined are invariant. The *walk measure* η is defined by putting equal weights on the edges leaving each vertex V:

$$w(E) = \frac{1}{|i^{-1}V|} \quad \text{for all } E \in i^{-1}V. \tag{3.2}$$

Continue to denote by G_2 the graph consisting of two loops, labeled 0 and 1, based at a single vertex. The symbol-count adic $\mathcal{C}(G)$ in this case is the *Pascal adic*, studied for example in [1, 20, 21, 23, 27](see Figure 2). By applying various reinforcement schemes $R : A \to \mathbb{Z}_+^d$ to this system, we arrive at some interesting adic systems. In order for the walk measure η to be adic invariant, the reinforcement schemes will have to be restricted. And in order for us to be able to identify all the adic-invariant ergodic measures, the Bratteli diagrams, and hence reinforcement schemes, will have to be restricted further.

Let us say that a Bratteli diagram, and the adic system that it defines, is *Pascal based* if it has the same vertex set as that of the Pascal system, and its edge set arises from that of the Pascal just by adding edges in some places where the Pascal system has just one edge: its edge set contains that of the Pascal, and there are no edges between vertices which are not connected by edges in the Pascal system. (Examples of such systems will arise from reinforcement schemes for walks on G_2.) Thus in a Pascal based system for each $n = 0, 1, \dots$ and $k = 0, 1, \dots, |V_n| - 1$ there are edges from vertex (n, k) only to vertex $(n + 1, k)$ and vertex $(n + 1, k + 1)$; suppose that there are $l(n, k)$ of the first kind and $r(n, k)$ of the second kind, for each n and k. Denote by $t(n, k) = l(n, k) + r(n, k)$ the total number of edges leaving vertex (n, k).

Proposition 3.1 ((Diamond Law)). *Let $(\mathcal{V}, \mathcal{E})$ be a Pascal based Bratteli-Vershik system with adic transformation T, and let μ be a measure on the path space X of the system defined by weights as above, with $\lambda(n, k)$ the weight on each edge from (n, k) to vertex $(n + 1, k)$ and $\rho(n, k)$ the weight on each edge from (n, k) to vertex $(n + 1, k + 1)$. Then μ is T-invariant if and only if $\lambda(n, k)\rho(n + 1, k) = \rho(n, k)\lambda(n + 1, k + 1)$ for all $n = 0, 1, \dots$ and $k = 0, 1 \dots, n - 1$.*

Proof. Since T and T^{-1} interchange anchored cylinder sets terminating at the same vertex, all such cylinder sets must be assigned the same measure by any T-invariant measure. Conversely, since the set of anchored cylinders generates the topology, if every anchored cylinder set C has the property that $\mu(T^{-1}C) = \mu(C)$, then μ is T-invariant. For non-minimal anchored cylinders, C and $T^{-1}(C)$ have the same terminal vertex. Any minimal anchored cylinder can be decomposed into a union

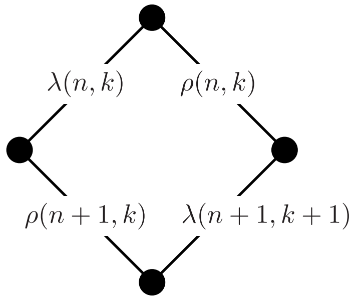

FIGURE 4. The Diamond Law seen graphically.

of a countable number of non-minimal anchored cylinders and at most a countable number of minimal infinite paths. Therefore, μ is T-invariant if and only if all anchored cylinders terminating in the same vertex have the same measure.

First assume that μ is T-invariant. Let C be any anchored cylinder terminating at vertex (n, k). Let E_{r_i} be an edge connecting $(n+i, k)$ to $(n+i+1, k+1)$ for $i = 0, 1$. Let E_{l_i} be an edge connecting $(n + i, k + i)$ to $(n + i + 1, k + i)$. Define $CE_{r_0}E_{l_1}$ to be the cylinder C extended by edge E_{r_0} and then by E_{l_1}. Likewise, define $CE_{l_0}E_{r_1}$ in a similar fashion. Then both $CE_{r_0}E_{l_1}$ and $CE_{l_0}E_{r_1}$ terminate in vertex $(n+2, k+1)$. Since μ is T-invariant, $\mu(CE_{l_0}E_{r_1}) = \mu(CE_{r_0}E_{l_1})$. Also $\mu(CE_{r_0}E_{l_1}) = \mu(C)\rho(n, k)\lambda(n + 1, k + 1)$ and therefore $\mu(CE_{l_0}E_{r_1}) = \mu(C)\lambda(n, k)\rho(n + 1, k)$. Therefore $\rho(n, k)\lambda(n + 1, k + 1) = \lambda(n, k)\rho(n + 1, k)$.

Now assume that $\lambda(n, k)\rho(n + 1, k) = \rho(n, k)\lambda(n + 1, k + 1)$ for all $n = 0, 1, \ldots$ and $k = 0, 1 \ldots, n - 1$. We will use induction to show that all anchored cylinders terminating in the same vertex have the same measure. Clearly $\lambda(0, 0)\rho(1, 0) = \rho(0, 0)\lambda(1, 1)$ implies that up to level two anchored all cylinders with the same terminal vertex have the same measure. Now assume that up to level $n - 1$, all anchored cylinders with the same terminal vertex have the same measure. Let C_1 and C_2 be anchored cylinders into vertex (n, k). We will show that $\mu(C_1) = \mu(C_2)$. If both C_1 and C_2 pass through the same vertex at level $n-1$, we are done. Without loss of generality, assume C_1 passes through vertex $(n - 1, k - 1)$ and C_2 passes through vertex $(n - 1, k)$. Let C be an anchored cylinder terminating at vertex $(n - 2, k - 1)$. By the invariance at level $n - 1$, all anchored cylinders terminating at vertex $(n - 1, k - 1)$ have the same measure regardless of the vertices through which they pass at level $n - 2$. The same can be said for all anchored cylinders terminating at vertex $(n-1, k)$. Therefore $\mu(C_1) = \mu(C)\lambda(n-2, k-1)\rho(n-1, k-1)$ and $\mu(C_2) = \mu(C)\rho(n - 2, k - 1)\lambda(n - 1, k)$. Then $\lambda(n - 2, k - 1)\rho(n - 1, k - 1) = \rho(n - 2, k - 1)\lambda(n - 1, k)$ implies that $\mu(C_1) = \mu(C_2)$. □

Proposition 3.2. *The walk measure on a Pascal based Bratteli-Vershik system is adic-invariant if and only if for each* $n = 0, 1, 2, \ldots,$ *the total number* $t(n, k)$ *of edges with source vertex* (n, k) *is constant in* k*.*

Proof. From Proposition 3.1 we know that the walk measure is adic-invariant if and only if $\lambda(n, k)\rho(n + 1, k) = \rho(n, k)\lambda(n + 1, k + 1)$. In the case of the walk measure this equation becomes:

$$\frac{1}{t(n,k)t(n+1,k)} = \frac{1}{t(n,k)t(n+1,k+1)}$$

and holds if and only if $t(n+1,k) = t(n+1,k+1)$. There is only one vertex at level 0, so this implies that $t(n,k)$ must be independent of k for all $n = 0,1,\dots$. □

Corollary 3.3. *The walk measure on the ramified adic $\mathcal{R}(G_2)$ with $s = (s_0, s_1)$ is adic-invariant if and only if there are $a, b, t \in \mathbb{Z}_+$ such that $R(0) = (a, t-a)$ and $R(1) = (b, t-b)$.*

Proof. Assume $\mathcal{R}(0) = (l_1, l_2)$ and $\mathcal{R}(1) = (r_1, r_2)$. The walk measure is adic invariant if and only if $t(n,k)$ is independent of k. We have

$$\begin{aligned} t(n,k) &= l(n,k) + r(n,k) \\ &= (n-k)l_1 + kr_1 + s_0 + (n-k)l_2 + kr_2 + s_1 \\ &= n(l_1 + l_2) + k(r_1 + r_2 - l_1 - l_2) + s_0 + s_1. \end{aligned}$$

Then $t(n,k)$ is independent of k if and only if $r_1 + r_2 = l_1 + l_2$, which is equivalent to saying $\mathcal{R}(0) = (a, t-a)$ and $\mathcal{R}(1) = (b, t-b)$. □

Definition 3.4. We say that a Pascal based adic system $(\mathcal{V}, \mathcal{E})$ has *constant diagonals* if there are sequences (a_n) and (b_n) in $\mathbb{N}$ such that $l(n,k) = a_{n-k}$ and $r(n,k) = b_k$ for all $n = 0,1,\dots$ and $k = 0,1,\dots,n$.

The Bratteli diagram depicted in Figure 3 has constant diagonals where $a_i = b_i = i+1$.

Proposition 3.5. *The ramified adic $\mathcal{R}(G_2)$ with $s = (s_0, s_1)$ has constant diagonals if and only if there are $l, r \in \mathbb{Z}_+$ such that $R(0) = (l, 0)$ and $R(1) = (0, r)$.*

Proof. Assume $\mathcal{R}(0) = (l_1, l_2)$ and $\mathcal{R}(1) = (r_1, r_2)$. For a ramified adic $l(n,k) = (n-k)l_1 + kr_1 + s_0$ and $r(n,k) = (n-k)l_2 + kr_2 + s_1$. Then $l(n,k)$ is dependent only on $(n-k)$ if and only if $r_1 = 0$ and $r(n,k)$ is dependent only on k if and only if $l_2 = 0$. □

Corollary 3.6. *The ramified adic $\mathcal{R}(G_2)$ with $s = (s_0, s_1)$ has constant diagonals and adic-invariant walk measure if and only if there is $a \in \mathbb{Z}_+$ such that $R(0) = (a, 0)$ and $R(1) = (0, a)$.*

For an infinite path x and a cylinder set C, we define $\dim(C, x_n)$ to be the number of infinite edge paths in X which belong to C and agree with x after level n. Recall that $\dim(n,k)$ denotes the number of paths from the root to the vertex (n,k). A theorem of Vershik (see [27–29]) which follows from the Ergodic Theorem says that for any ergodic, adic-invariant measure μ on X, for any cylinder set C, and for μ-a.e. $x \in X$

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(C, x_n)}{\dim(n, k_n(x))}.$$

Theorem 3.7. *Let $(\mathcal{V}, \mathcal{E})$ be a Pascal based adic system with constant diagonals as in Definition 3.4. Then the set of adic-invariant ergodic measures on the path space X of the system consists of exactly those measures μ for which there is $\alpha \in [0,1]$ such that $\mu = \mu_\alpha$ is determined by the weights $\lambda(n,k) = \alpha/l(n,k)$ on each edge from vertex (n,k) to vertex $(n+1,k)$ and $\rho(n,k) = (1-\alpha)/r(n,k)$ on each edge from vertex (n,k) to vertex $(n+1,k+1)$, for all $n = 0,1,\ldots$ and $k = 0,1,\ldots,n$.*

Proof. We extend previous arguments for the case of Pascal adic, described in the sources cited below. First, assume that μ is an ergodic invariant measure. We will show it must be one of the ones described in the theorem. Let C be an anchored cylinder in X terminating at vertex (n_0, k_0). Then for μ-a.e. $x \in X$,

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(C, x_n)}{\dim(n, k_n(x))}$$

$$(3.3) \qquad = \lim_{n\to\infty} \frac{\left(\prod_{j=n_0-k_0+1}^{n-k_n(x)} a_j\right)\left(\prod_{i=k_0+1}^{k_n(x)} b_i\right)\binom{n-n_0}{k_n(x)-k_0}}{\left(\prod_{j=0}^{n-k_n(x)} a_j\right)\left(\prod_{i=0}^{k_n(x)} b_i\right)\binom{n}{k_n(x)}}$$

$$= \frac{1}{\left(\prod_{j=0}^{n_0-k_0} a_j\right)\left(\prod_{i=0}^{k_0} b_i\right)} \lim_{n\to\infty} \frac{\binom{n-n_0}{k_n(x)-k_0}}{\binom{n}{k_n(x)}}.$$

The remaining limit is the same as the limits determining the measures on the Pascal graph (see, for example, [15, 20, 23]). Consider a cylinder set C' defined by extending the path defining C by an edge to the vertex (n_0+1, k_0+1). Then

$$(3.4) \qquad \frac{\mu(C')}{\mu(C)} = \lim_{n\to\infty} \frac{\binom{n-n_0-1}{k_n(x)-k_0-1}}{\binom{n-n_0}{k_n(x)-k_0}} = \lim_{n\to\infty} \frac{k_n(x)}{n}$$

exists a.e. $d\mu$. Hence, as in the sources just cited, the limit in (3.3) is $\alpha^{k_0}(1-\alpha)^{n_0-k_0}$ and μ is as stated in the theorem.

Conversely, let μ_α be a measure as described in the statement of the theorem. Because the diagram has constant diagonals, the Diamond Law is satisfied and so μ_α is adic-invariant. Putting $k_0(x) = 0$, notice that $Y_j(x) = k_j(x) - k_{j-1}(x)$ for $j \geq 1$ is an i.i.d. process with $\mu_\alpha\{Y_j = 0\} = \alpha$ for all j. By the Strong Law of Large Numbers, $k_n(x)/n \to 1-\alpha$ a.e. $d\mu_\alpha$. Thus for each cylinder set C, the limit in (3.3) is a.s. constant in x. Using either the Ergodic Theorem or Reverse Martingale Theorem as in the sources just cited, this implies that μ_α is ergodic. (Alternatively, one may use the "collision argument" of [2, 17] as in the proof below of Theorem 4.1.) □

4. Examples based on the two-loop graph G_2

We list the possibilities arising from Corollary 3.6 for constant-diagonal ramified adics with adic-invariant walk measure when $G = G_2$, $s = (1, 1)$, and $a = 0$ or 1. For these examples it makes sense to identify all the ergodic invariant measures and to determine the ergodic decomposition of the walk measure.

4.1. **The Pascal adic.** As mentioned above, the Pascal adic is the symbol count adic $\mathcal{C}(G_2)$ and thus is also the ramified adic $\mathcal{R}(G_2)$ for the reinforcement scheme as above (in Corollary 3.6) with $a = 0$. The dimension of each vertex (n, k) is the binomial coefficient $C(n, k)$. The ergodic measures are given as in Theorem 3.7, with $l(n, k) = r(n, k) = 1$ for all n, k. The identification of the ergodic measures for the Pascal adic as just the Bernoulli measures for the i.i.d. sequence of left and right turns has been known for a long time and can be seen as a combination of the Hewitt-Savage and de Finetti theorems. There are also proofs using direct computation, the Ergodic Theorem, the Reverse Martingale Theorem, and the isotropy of the graph—see [15,20,23,27]. The walk measure is the Bernoulli $1/2, 1/2$ measure μ_α of Theorem 3.7 with $\alpha = 1/2$.

It is known that for each of its nonatomic ergodic measures the Pascal adic has no roots of unity among its eigenvalues besides 1 [23] and is loosely Bernoulli [4]. Weak mixing is still open, but the subshift produced by coding by the first edge is topologically weakly mixing [21].

4.2. **The reverse Euler adic.** Simple positive reinforcement for a walk on the two-loop graph G_2 starting with the number 1 on both edges (see [5,18]) is modeled by a reinforcement scheme as above with $s = (1, 1), R(0) = (1, 0)$, and $R(1) = (0, 1)$, as can be seen in Figure 3. In this scheme each edge of G_2 has 1 added to its number each time that the walker uses that edge. The resulting ramified adic again has constant diagonals: $a_n = b_n = n + 1$ for all $n = 0, 1, \dots$. The dimension of vertex (n, k) is $n!$ for all n, k. By Theorem 3.7, the ergodic invariant measures are again a one-parameter family indexed by $\alpha \in [0, 1]$. For each nonatomic invariant measure, the system has all roots of unity as eigenvalues, which is easily seen since the dimension of every vertex on level n is $n!$. The walk measure is *not* ergodic. Its ergodic decomposition is discussed below.

4.3. **The Euler adic.** The Euler adic is the result of simple opposite reinforcement for a walker on G_2: $s = (1, 1), R(0) = (0, 1), R(1) = (1, 0)$, so that at each step the number on the edge of G_2 not taken is increased by 1. The resulting ramified system does not have constant diagonals. The numbers of edges down from each vertex to the left and right, respectively, are $l(n, k) = k+1$ and $r(n, k) = n-k+1$ for all n, k. The dimension of vertex (n, k) is the *Eulerian number* $A(n, k)$. These are defined by $A(0, 0) = 1$, $A(n, k) = 0$ for $k \notin \{0, 1, \dots, n\}$, and $A(n, k) = (n - k + 1)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$ for $n = 1, 2, \dots, k = 0, 1, \dots, n$. The walk measure is adic invariant and has been proved to be ergodic [2]. In fact, the walk measure is the only fully-supported adic-invariant measure for this system

[9, 13, 24]. This adic system, with the walk measure, has been proved to have no roots of unity among its eigenvalues besides 1 and to be loosely Bernoulli [8].

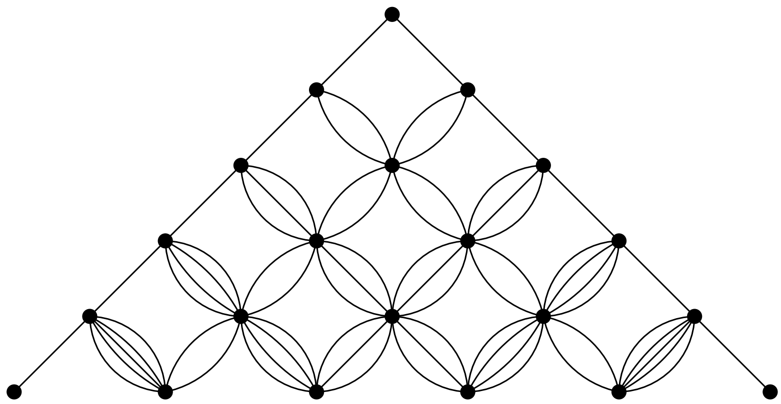

FIGURE 5. The Euler adic

4.4. **The Stirling adic.** When $s = (1, 1), R(0) = (1, 0)$, and $R(1) = (1, 0)$, so that the same edge of G_2 is always reinforced no matter which edge the walker follows, we obtain the Bratteli diagram for which $l(n, k) = n+1$ and $r(n, k) = 1$ for all n, k. This diagram does not have constant diagonals. The dimension of vertex (n, k) in this case is $s_1(n, k) = |s(n, k)|$, where $s(n, k)$ is the Stirling number of the first kind. Thus $\dim(n, k) = s_1(n, k)$ is the number of permutations of $1, \ldots, n$ with exactly k orbits (or cycles). We have $s_1(0, 0) = 1, s_1(n, k) = 0$ for $k \notin \{0, 1, \ldots, n\}$, $s_1(n + 1, k) = s_1(n, k - 1) + ns_1(n, k)$ for all n, k.

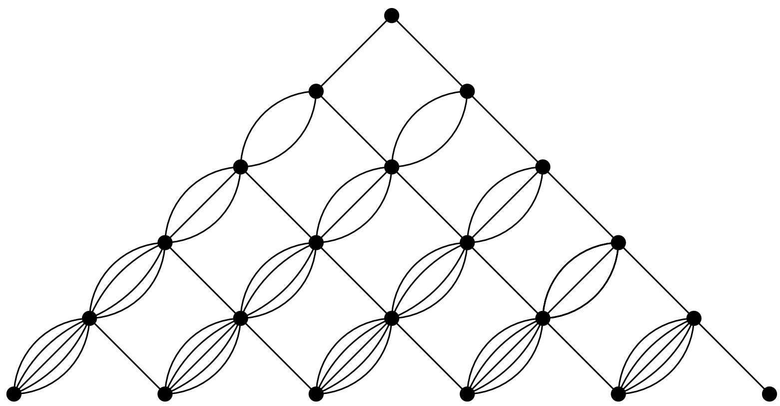

FIGURE 6. The Stirling adic

Theorem 4.1. *For each $\alpha \in [0, 1]$ the measure μ_α determined by the weights $\lambda(n, k) = \alpha/(n\alpha + 1)$ on each edge from vertex (n, k) to vertex $(n + 1, k)$ and $\rho(n, k) = (1 - \alpha)/(n\alpha + 1)$ on each edge from vertex (n, k) to vertex $(n + 1, k + 1)$, for all $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, n$, is invariant and ergodic for the Stirling adic. (This includes the walk measure, which is $\mu_{1/2}$.)*

Proof. Adic-invariance of each measure μ_α defined by weights given as above is a consequence of the Diamond Law (Proposition 3.1). To prove ergodicity, we use the "collision argument" from [17], i.e. the equivalents in our setting of Propositions 1 and 2 and the Theorem of [2]. Since the analogs of Proposition 1 and the proof of the Theorem in [2] hold just as well in our setting, we need only prove the equivalent of Proposition 2 in [2], i.e. that for $(\mu_\alpha \times \mu_\alpha)$-a.e. $(x,y) \in X \times X$, $D_n(x,y) = k_n(x) - k_n(y) = 0$ for infinitely many $n \geq 1$. We start with $D_0(x,y) = 0$. The increments $J_n = D_{n+1} - D_n$ are independent and $\sum(\mu_\alpha \times \mu_\alpha)\{J_n \neq 0\} = \infty$, so by Borel-Cantelli a.s. $J_n \neq 0$ for infinitely many n. But along the sequence of random times n for which $J_n \neq 0$, where the two paths do not both go left or both go right, the process (D_n) is a simple, symmetric random walk and thus hits 0 infinitely many times. Off of this set of times (D_n) cannot change, so with $\mu_\alpha \times \mu_\alpha$ probability 1 there are necessarily infinitely many n with $D_n = 0$. □

Remark 4.2. It is proved in [12, 14, 19] that every ergodic measure on the Stirling system is one of the measures μ_α described above.

5. Multi-dimensional adic systems

Definition 5.1. Define the *d-dimensional Pascal graph* to be the Bratteli diagram for which the vertex set $\mathcal{V}_n$ at level n is indexed by $(k_0, \dots, k_{d-1})$ where $k_0 + \cdots + k_{d-1} = n$. From vertex $(k_0, k_1, \dots, k_{d-1})$ in $\mathcal{V}_n$ there is exactly one edge connecting $(k_0, \dots, k_i + 1, k_{i+1}, \dots, k_{d-1})$ in $\mathcal{V}_{n+1}$ for each $i = 0, \dots, d-1$ and no others. Then the number of paths from $(0, \dots, 0)$ to $(k_0, \dots, k_{d-1})$ where $k_0 + \cdots + k_{d-1} = n$ is exactly the coefficient on $x_0^{k_0} x_i^{k_1} \dots x_{d-1}^{k_{d-1}}$ in the polynomial $(x_0 + x_1 + \cdots + x_{d-1})^n$. If the edge E connects vertices $(k_0, \dots, k_{d-1})$ and $(k_0, \dots, k_i + 1, k_{i+1}, \dots, k_{d-1})$ we say that the edge E is *in the direction of* x_i.

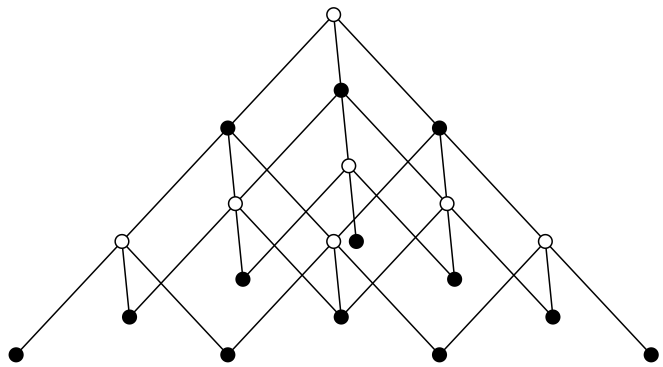

FIGURE 7. The 3-dimensional Pascal, with vertex levels shaded in an alternating fashion.

Let us say that a Bratteli diagram, and the adic system that it defines, is *d-dimensional Pascal based* if it has the same vertex set as that of the Pascal system, and its edge set arises from that of the d-dimensional Pascal just by adding edges in some places where the Pascal system has just one edge: its edge set contains that of the Pascal, and there are no edges between vertices which are not connected by edges in the Pascal system. (Examples of such systems will arise from reinforcement schemes for walks on G_d.) For a d-dimensional Pascal based system, for each vertex $(k_0, \ldots, k_{d-1})$ let $r_i(k_0, \ldots, k_{d-1})$ be the number of edges connecting $(k_0, \ldots, k_{d-1})$ to $(k_0, \ldots, k_{i-1}, k_i+1, k_{i+1}, \ldots, k_{d-1})$, and let $t(k_0, \ldots, k_{d-1})$ be the total number of edges leaving vertex $(k_0, \ldots, k_{d-1})$. Again Borel probability measures on the space X of infinite paths in the diagram beginning at the root can be defined by specifying nonnegative weights $\lambda_i(k_0, \ldots, k_{d-1})$ on the edges connecting vertex $(k_0, \ldots, k_{d-1})$ and $(k_0, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_{d-1})$, subject to the conditions (1) and (2) in section 3.

The proofs of the following propositions, theorems, and corollaries are similar to those for the two-dimensional case appearing in Section 3. We omit the proofs in this section in the interest of brevity.

Proposition 5.2 ((Diamond Law))**.** *Let $(\mathcal{V}, \mathcal{E})$ be a d-dimensional Pascal based Bratteli-Vershik system with adic transformation T, and let μ be a measure on the path space X of the system defined by the weights $\lambda_i(k_0, \ldots, k_{d-1})$. Then μ is T-invariant if and only if for each $0 \leq i, j \leq d-1$,*

$$\begin{aligned}&\lambda_i(k_0, \ldots, k_{d-1})\lambda_j(k_0, \ldots, k_i + 1, k_{i+1}, \ldots, k_{d-1})\\&= \lambda_j(k_0, \ldots, k_{d-1})\lambda_i(k_0, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_{d-1}).\end{aligned}$$

Theorem 5.3. *The ergodic, adic-invariant probability measures on the d-dimensional Pascal graph form a $(d-1)$-parameter family of measures, with weights*

$$\lambda_i(k_0, \ldots, k_{d-1}) = \alpha_i, \quad \text{where } \alpha_0 + \cdots + \alpha_{d-1} = 1.$$

Proof. The proof in [20] for the 2-dimensional Pascal is easily extended to the d-dimensional case. □

Proposition 5.4. *The walk measure on a d-dimensional Pascal based Bratteli-Vershik system is adic-invariant if and only if for each vertex $(k_0, \ldots, k_{d-1})$ the total number $t(k_0, \ldots, k_{d-1})$ of edges downward from $(k_0, \ldots, k_{d-1})$ depends only on the level, $k_0 + \cdots + k_{d-1}$.*

Corollary 5.5. *The walk measure on a ramified adic $\mathcal{R}(G_d)$ with $s = (s_0, \ldots, s_{d-1})$ is adic-invariant if and only if for each $j = 0, 1, \ldots, d-1$, $\sum_{i=0}^{d-1} [R(j)]_i$ is constant.*

Definition 5.6. We say that a d-dimensional Pascal based adic system has *constant diagonals* if for each $i = 0, 1, \ldots, d-1$ there is a sequence $(a_n^{(i)})$ in $\mathbb{N}$ such that $r_i(k_0, \ldots, k_{d-1}) = a_{k_i}^{(i)}$.

Proposition 5.7. *A ramified adic $\mathcal{R}(G_d)$ with $s = (1, \ldots, 1)$ has constant diagonals if and only if for each $i = 0, 1, \ldots, d-1$ there is $a_i \in \mathbb{Z}_+$ such that $R(i) = a_i u_i$, where u_i is the standard i'th basis vector of $\mathbb{Z}^d$.*

Corollary 5.8. *The ramified adic $\mathcal{R}(G_d)$ with $s = (1, 1, \ldots, 1)$ has constant diagonals and adic-invariant walk measure if and only if there is $a \in \mathbb{Z}_+$ such that for each $i = 0, 1, \ldots, d-1$, $R(i) = au_i$, where u_i is the standard i'th basis vector of Z^d.*

Theorem 5.9. *Let $(\mathcal{V}, \mathcal{E})$ be a d-dimensional Pascal based adic system with constant diagonals. Then (as in Theorem 3.7), the set of adic-invariant ergodic measures on the path space X of the system consists of exactly those measures μ for which there are $\alpha_0, \alpha_1, \dots, \alpha_{d-1}$ in $[0,1]$ with $\alpha_0 + \alpha_1 + \dots + \alpha_{d-1} = 1$ such that $\mu = \mu_{\vec{\alpha}}$ is determined by the weights $\lambda_i(k_0, \dots, k_{d-1}) = \alpha_i/(r_i(k_0, \dots, k_{d-1})$.*

5.1. **The d-dimensional reverse Euler adic.** Simple positive reinforcement for a walk on the d-loop graph G_d starting with the number 1 on each edge is modeled by a reinforcement scheme as above with $s = (1, \dots, 1)$, $R(i) = u_i$ where u_i is the standard basis vector of $\mathbb{Z}^d$. Thus each edge of G_d has 1 added to its number each time that the walker uses that edge. The resulting ramified adic again has constant diagonals: $a_n^{(i)} = n + 1$ for all $i = 0, 1, \dots, d-1$ and $n = 0, 1, \dots$. The dimension of vertex $(k_0, k_1, \dots, k_{d-1})$ appearing at level $n = \sum_{i=0}^{d-1} k_i$ is $n!$. By Theorem 5.9, the ergodic invariant measures are again a $(d-1)$-parameter family indexed by $\alpha_i \in [0,1]$ with $\alpha_0 + \dots + \alpha_{d-1} = 1$. For each nonatomic invariant measure, the system has all roots of unity as eigenvalues. The walk measure is *not* ergodic. Its ergodic decomposition is discussed below.

5.2. **The d-dimensional Euler adic.** The d-dimensional Euler adic is the result of opposite reinforcement for a walker on G_d: $s = (1, \dots, 1)$ and $R(i) = (1, \dots, 1, 0, 1, \dots 1)$, where the 0 appears in the i'th place, so that at each step the numbers on all the edges of G_d *not* taken are increased by 1. Extensions of the results of [2, 8, 9] to these systems will be discussed in forthcoming papers.

6. EDGE TRAVERSAL FREQUENCIES AND ERGODIC DECOMPOSITIONS

In this section we investigate the relationship between the ergodic decomposition of an adic-invariant walk measure and its vector of edge traversal frequencies. Let η be the walk measure on the path space X of a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ as at the beginning of Section 3 arising from nonnegative weights $w(E), E \in \mathcal{E}$, determined by the numbers of succeeding edges, as in Formula 3.2. We assume that the system is the reinforced adic system $\mathcal{R}(G)$ arising from a reinforcement scheme (s, R) for the numbers on the edges $e \in \mathcal{E}(G) = A$ of a directed finite graph G. Recall that infinite paths $x \in X$ in the Bratteli diagram $\mathcal{R}(G)$ starting at the root are coordinatized as sequences of edges, $x = x_1 x_2 \dots$, with $x_k \in \mathcal{E}_k$ for all k, and that for each edge $E \in \mathcal{C}(G)$ (or $\mathcal{R}(G)$) there is a unique corresponding edge $e(E) \in A = \mathcal{E}(G)$ (see Section 2). This map extends to a map $e : X \to A^{\mathbb{N}}$ by $e(x) = e(x_1)e(x_2)\dots$ for each $x \in X$.

The walk measure η defines a measure η_G on anchored cylinder sets $[e_1 \dots e_n] = \{\omega \in A^{\mathbb{N}} : \omega_i = e_i, i = 1, \dots, n\} \subset A^{\mathbb{N}}$ by

$$\eta_G[e_1 \dots e_n] = \eta\{x \in X : e(x_i) = e_i, i = 1, \dots, n\} \tag{6.1}$$

and hence a stochastic process, that is to say, a measure on $A^{\mathbb{N}}$, taking values in the set A of edges of G. This measure may not be Markov, but for each $j \in A$, the probability that $\omega_{n+1} = j$ depends only on $\omega_1, \dots, \omega_n$, and indeed only on the symbol count $C(\omega_1 \dots \omega_n)$ and the final vertex $t(\omega_n)$. In terms of the map $e : X \to A^{\mathbb{N}}$ just defined, we have $\eta_G = \eta e^{-1}$.

Because of this last observation, the measure η_G is *exchangeable, or* symmetric: two anchored cylinder sets determined by strings that are permutations of one another have the same measure. Consequently η_G is also *partially exchangeable*, in the sense of Diaconis and Freedman [6, 7]: Any two path segments that start at the same symbol (edge of G) and have the same *transition counts* have the same probability. (Any two such path segments must have the same symbol counts and same length and must end at the same symbol, since just one is entered more times than it is left.)

Proposition 6.1. *The vector of edge traversal frequencies $\bar{v}(\omega) \in \mathbb{Z}_+^d$ with the e'th coordinate ($e \in A$) equal to*

$$\bar{v}(\omega)_e = \lim_{n\to\infty} \frac{1}{n} |\{j \in [1, n] : \omega_j = e\}| \tag{6.2}$$

exists for η_G-almost all $\omega \in A^{\mathbb{N}}$, and thus $\bar{v}^X(x) = \bar{v}(e(x))$ exists for η-almost all $x \in X$.

Proof. According to [7, p. 121], for each extreme point measure π_θ of the set of partially exchangeable probabilities on a finite state space I, the process runs through finitely many transient states (perhaps none) before settling in a recurrent class described by an irreducible stochastic matrix. Therefore the vector of state occupation frequencies exists a.s $d\pi_\theta$. Further, according to Theorem (22) of [7], any exchangeable probability (such as η or any of its ergodic components) is a mixture of such extreme points π_θ, and hence the vector of state occupation frequencies exists a.s $d\eta$. (There is a measure μ_η on the set U of extreme points such that, denoting by B the event that the sequence of occupation vectors does not converge, $\eta(B) = \int_U \pi_\theta(B)\, d\mu_\eta(\theta) = 0$. For a stationary Markov chain the state occupation frequencies exist by the Ergodic Theorem, and using the same transition matrix and different positive initial vector defines an equivalent measure.) □

For the case of a simply reinforced random walk ($s_j = 1, R(j) = u_j = j$'th standard basis vector of $\mathbb{Z}^d$ for all $j \in A$), Coppersmith and Diaconis [5] stated that the distribution of the limiting edge traversal frequency vector $\bar{v}(x)$ is absolutely continuous with respect to Lebesgue measure m_{d-1} on the $d-1$-dimensional simplex $\triangle_{d-1}$ in $\mathbb{R}^d$, so that there is a nonnegative integrable function f on $\triangle_{d-1}$ such that

$$\eta_G\{\omega : \bar{v}(\omega) \in B\} = \int_B f\, dm_{d-1} \tag{6.3}$$

for all Borel sets $B \subset \triangle_{d-1}$. They also gave a formula for the density. Keane and Rolles [18] gave a proof of this assertion and formula.

Here we examine the relationship of such a statement to the ergodic decomposition of the walk measure η. Assume henceforth that η is adic-invariant. Then it has an ergodic decomposition: there is a measure ν_η on the compact metric space

Ξ of ergodic adic-invariant measures on X such that

$$\eta = \int_{\Xi} \xi \, d\nu_\eta(\xi) \tag{6.4}$$

(see [22] and its references, as well as [25, 26]).

Proposition 6.2. *If the walk measure η for a reinforced-walk adic system $\mathcal{R}(G)$ is adic-invariant and ξ is one of its ergodic components, then the vector $\bar{v}^X(x)$ of limiting edge traversal frequencies is ξ-a.e. constant.*

Proof. Since any ergodic component ξ of η is by definition adic-invariant, it is exchangeable, hence partially exchangeable, and so by Proposition 6.1 $\bar{v}^X(x)$ exists a.e. $d\xi$. Since $\bar{v}^X$ is invariant under the adic transformation T (it depends only on the tail of x) and ξ is ergodic for T, $\bar{v}^X$ must be constant a.e. $d\xi$. □

Denote by $\pi : \Xi \to \triangle_{d-1}$ the map that sends an ergodic measure ξ to its (a.e. constant) vector $\bar{v}_\xi$ of edge traversal frequencies. The ergodic decomposition measure ν_η on Ξ has a *disintegration* with respect to π: there are measures τ_p ($p \in \triangle_{d-1}$) supported on $\pi^{-1}\{p\}$ such that if $\lambda_\eta = \nu_\eta \circ \pi^{-1}$, then

$$\nu_\eta = \int_{\triangle_{d-1}} \tau_p \, d\lambda_\eta(p). \tag{6.5}$$

(See, for example, [10, pp. 108–109].) The key properties are:

$$\begin{aligned} &\text{If } \phi \in L^1(\Xi, \nu_\eta), \text{ then } \phi \in L^1(\Xi, \tau_p) \\ &\text{and } \mathbb{E}(\phi|\triangle_{d-1})(p) = \int_{\Xi} \phi \, d\tau_p \text{ for } \lambda_\eta\text{- a.e. } p \in \triangle_{d-1}; \end{aligned} \tag{6.6}$$

$$\text{if } \phi \in L^1(\Xi, \nu_\eta), \text{ then } \int_{\Xi} \phi \, d\nu_\eta = \int_{\triangle_{d-1}} \left(\int_{\Xi} \phi \, d\tau_p \right) d\lambda_\eta(p). \tag{6.7}$$

(Here $\mathbb{E}(\phi|\triangle_{d-1})$ is the conditional expectation of ϕ with respect to $\pi^{-1}(\mathcal{B}(\triangle_{d-1}))$, where $\mathcal{B}(\triangle_{d-1})$ is the Borel σ-algebra of $\triangle_{d-1}$.) This setup is summarized in the following theorem.

Theorem 6.3. *When the walk measure η on a reinforced adic system $\mathcal{R}(G)$ is adic-invariant, ν_η is its ergodic decomposition measure on the set Ξ of ergodic measures, and λ_η is the image of ν_η under the map π that sends an ergodic measure ξ to its (a.e. constant) vector $\bar{v}^X_\xi$ of limiting edge traversal frequencies, then for all Borel $B \subset \triangle_{d-1}$, the hitting measure*

$$\rho(B) = \eta\{x : \bar{v}^X(x) \in B\} = \lambda_\eta(B). \tag{6.8}$$

Proof. Note that for each ergodic measure ξ and Borel set $B \subset \triangle_{d-1}$ we have

$$\xi\{x : \bar{v}^X(x) \in B\} = 1_B(\pi\xi). \tag{6.9}$$

For each Borel set $B \subset \triangle_{d-1}$, by the ergodic decomposition

$$
\begin{aligned}
\eta\{x : \bar{v}^X(x) \in B\} &= \int_\Xi \xi\{x : \bar{v}^X(x) \in B\}\, d\nu_\eta(\xi) = \int_\Xi 1_B(\pi\xi)\, d\nu_\eta(\xi) \\
&= \int_{\triangle_{d-1}} \left(\int_\Xi 1_B(\pi\xi)\, d\tau_p(\xi) \right) d\lambda_\eta(p) \\
&= \int_{\triangle_{d-1}} \tau_p(\pi^{-1}(B)\, d\lambda_\eta(p) \\
&= \int_{\triangle_{d-1}} 1_B(p)\, d\lambda_\eta(p) = \lambda_\eta(B).
\end{aligned} \tag{6.10}
$$

□

Corollary 6.4. *If the hitting measure ρ is absolutely continuous with respect to Lebesgue measure m_{d-1} on $\triangle_{d-1}$, then its density function f (the density function of the vector of limiting edge traversal frequencies) is the Radon-Nikodym derivative $d\lambda_\eta / dm_{d-1}$.*

7. Computation of the density for some multiloop systems

We can apply the general discussion in the preceding section to get explicit formulas for the density function f for some special random walks, not necessarily just with simple positive reinforcement.

Example 7.1. For the reverse Euler system (simple positive reinforcement on G_2, starting with $s = (1, 1)$), according to Theorem 3.7 we find that the set of ergodic measures is parametrized by $\alpha \in \triangle_1 \approx [0, 1]$, $\lambda_\eta = m_1$, and $f \equiv 1$. This coincides with the formula of Keane and Rolles for this example.

Example 7.2. We again consider reinforced random walk on the two-loop graph $G = G_2$ with initial number 1 on both edges ($s = (1, 1)$) but now with *double* rather than single reinforcement:

$$
R(0) = (2, 0), R(1) = (0, 2). \tag{7.1}
$$

We can do a computation and find that again the ergodic measures are parametrized by $\alpha \in \triangle_1 = [0, 1]$, $\lambda_\eta = m_1$, but this time

$$
f(\alpha) = \frac{1}{\pi \sqrt{\alpha\,(1-\alpha)}} \quad \text{for all } \alpha \in (0, 1). \tag{7.2}
$$

Rather than present the computations for the preceding examples, we discuss next how to accomplish it in the general case of the graph G_d consisting of d loops at a single vertex, an arbitrary initial positive integer vector s of numbers on the edges of G_d, and a reinforcement scheme

$$
R(j) = a u_j, j = 0, 1, \ldots, d-1, \tag{7.3}
$$

where $a \in \mathbb{N}$ is fixed and u_j denotes, as before, the j'th standard basis vector of $\mathbb{Z}_+^d$. The following computations work also for positive real a and s, not just integers.

Lemma 7.3. *For $a, s \in \mathbb{N}$ let $g_{a,s}(n) = (a+s)(2a+s)\dots((n-1)a+s)$. Then for large $n \in N$,*

$$g_{a,s}(n) \sim \left(\frac{an+s}{e}\right)^n \left(\frac{an+s}{s}\right)^{s/a} \frac{e^{c_{a,s}-\alpha/(12s)}}{\sqrt{s(na+s)}},$$

where

$$c_{a,s} = \int_0^\infty \frac{a^2[(t-\lfloor t\rfloor)^2 - (t-\lfloor t\rfloor) + \frac{1}{6}]}{(at+s)^2}\,dt, \tag{7.4}$$

$\lfloor t\rfloor$ stands for the greatest integer function, and $\sim$ means that the quotient tends to 1 as $n \to \infty$.

Proof. From the Euler-MacLaurin Summation Formula applied to the function $h(t) = \ln(at+s)$, we have

$$\begin{aligned}\int_0^n h(t)\,dt &= \frac{\ln(s)}{2} + \ln(a+s) + \ln(2a+s) + \cdots + \ln((n-1)a+s) + \frac{\ln(na+s)}{2}\\ &-\sum_{k=1}^{p} \frac{b_{k+1}}{(k+1)!}(h^{(k)}(n) - h^{(k)}(0))\\ &-(-1)^p \int_0^n h^{(p+1)}(x)\frac{B_{p+1}(x-\lfloor x\rfloor)}{(p+1)!}\,dx,\end{aligned}$$

where b_{k+1} is the $(k+1)$'st Bernoulli number, B_{p+1} is the $(p+1)$'st Bernoulli polynomial, and $\lfloor x\rfloor$ denotes the greatest integer function of x. Choosing $p = 1$, the above simplifies to

$$\begin{aligned}\int_0^n \ln(at+s)\,dt &= \frac{\ln(s)}{2} + \ln(a+s) + \cdots + \ln((n-1)a+s) + \frac{\ln(na+s)}{2}\\ &-\frac{1}{12}\left(\frac{a}{an+s} - \frac{a}{s}\right) - \int_0^n \frac{a^2[(t-\lfloor t\rfloor)^2 - (t-\lfloor t\rfloor) + \frac{1}{6}]}{2(at+s)^2}\,dt.\end{aligned} \tag{7.5}$$

We note that

$$c_{a,s}(n) = \int_0^n \frac{a^2[(t-\lfloor t\rfloor)^2 - (t-\lfloor t\rfloor) + \frac{1}{6}]}{(at+s)^2}\,dt \tag{7.6}$$

converges as $n \to \infty$ to a finite constant $c_{a,s}$ (since the numerator is bounded by $a^2/6$). Rearranging the terms in Equation 7.5, we see that

$$\ln(g_{a,s}(n)) = \int_0^n \ln(at+s)\,dt - \frac{\ln(s) + \ln(an+s)}{2} - \frac{a^2 n}{12s(an+s)} + c_{a,s}(n). \tag{7.7}$$

Therefore

$$g_{a,s}(n) = \exp\left[\int_0^n \ln(at+s)\,dt - \frac{\ln(s(an+s))}{2} - \frac{a^2 n}{12s(an+s)} + c_{a,s}(n)\right]. \tag{7.8}$$

Since $\int_0^n \ln(at+s)\,dt = \frac{1}{a}[(an+s)\ln(an+s) - (an+s) - s\ln(s) + s]$,

$$\exp\left(\int_0^n \ln(at+s)\,dt\right) = \left[\frac{(an+s)^{an+s}}{s^s e^{na}}\right]^{1/a} = \left(\frac{an+s}{e}\right)^n \left(\frac{an+s}{s}\right)^{s/a}. \tag{7.9}$$

For large n, $\exp\left(\frac{-a^2n}{12s(an+s)}\right) \sim \exp\left(\frac{-a}{12s}\right)$. Combining these observations with Equation 7.8 concludes the proof. □

Consider a multidimensional system for which the number of edges is positively reinforced by a summand of a, with each entry $s_0, s_1, \dots, s_{d-1}$ of the start vector s representing the number of edges connecting the root vertex $(0, \dots, 0)$ to vertex $(1, 0, \dots, 0)$, $(0, 1, 0 \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ respectively. By Corollary 5.8, the system has constant diagonals and adic-invariant walk measure. Let $\mu_{\alpha_0,\dots,\alpha_{d-1}}$ be the invariant ergodic measure described in Theorem 5.9 for this system which gives each cylinder into vertex $(k_0, k_1, \dots, k_{d-1})$, where $k_0 + \cdots + k_{d-1} = n$, measure

$$\frac{\alpha_0^{k_0}\alpha_1^{k_1}\dots\alpha_{d-1}^{k_{d-1}}}{s_0 s_1 \dots s_{d-1} g_{a,s_0}(k_0) g_{a,s_1}(k_1) \dots g_{a,s_{d-1}}(k_{d-1})}. \tag{7.10}$$

Let η be the walk measure, which gives each cylinder into vertex $(k_0, k_1, \dots, k_{d-1})$ measure

$$\frac{n}{(s_0 + s_1 + \cdots + s_{d-1}) g_{a,s_0+s_1+\cdots+s_{d-1}}}. \tag{7.11}$$

Theorem 6.3 and Corollary 6.4 apply to this system.

Theorem 7.4. *The density function $f_{a,s_0,s_1,\dots,s_{d-1}}(\alpha_0, \dots, \alpha_{d-1})$ corresponding to the ergodic decomposition of η is given by the following formula:*

$$f_{a,s_0,s_1,\dots,s_{d-1}}(\alpha_0, \dots, \alpha_{d-1}) = \frac{\Gamma\left(\frac{s_0 + \cdots + s_{d-1}}{a}\right)}{\Gamma\left(\frac{s_0}{a}\right) \dots \Gamma\left(\frac{s_{d-1}}{a}\right)} \left(\prod_{i=0}^{d-1} \alpha_i^{(s_i/a)-1}\right). \tag{7.12}$$

Proof. Abbreviate $f_{a,s_0,s_1,\dots,s_{d-1}}$ by f. Note first that the functions $x_0^{k_0} x_1^{k_1} \dots x_{d-1}^{k_{d-1}}$ on the d-simplex, when normalized, form an approximate identity peaking at $(\alpha_0, \dots, \alpha_{d-1})$ when $k_0 + \cdots + k_{d-1} = n \to \infty$ and each $k_i/n \to \alpha_i$. Therefore $f(\alpha_0, \dots, \alpha_{d-1})$ can be approximated by

$$\frac{1}{N} \int \dots \int f(x_0, \dots, x_{d-1}) x_0^{k_0} x_1^{k_1} \dots x_{d-1}^{k_{d-1}}\, dx_0\, dx_1 \dots dx_{d-1}, \tag{7.13}$$

which runs over all positive $x_0, x_1, \dots, x_{d-1}$ for which $x_0 + x_1 + \cdots + x_{d-1} = 1$, with

$$N = \int \dots \int x_0^{k_0} x_1^{k_1} \dots x_{d-2}^{k_{d-2}} (1 - (x_0 + \cdots + x_{d-2}))^{k_{d-1}}\, dx_0\, dx_1 \dots dx_{d-2}$$

running over the all positive $x_0, x_1, \dots, x_{d-2}$ for which $x_0 + x_1 + \cdots + x_{d-2} < 1$. This is precisely *Dirichlet's integral* (see [30, p. 258]) and is known to have the value

$$\frac{\Gamma(k_0 + 1)\Gamma(k_1 + 1) \dots \Gamma(k_{d-2} + 1)}{\Gamma(k_0 + k_1 + \dots k_{d-2} + (d - 1))} \int_0^1 (1 - \tau)^{k_{d-1}} \tau^{k_0+\cdots+k_{d-2}+d-2}\, d\tau. \tag{7.14}$$

The remaining integral in (7.14) is exactly the *Eulerian integral of the first kind* (see [30, pp. 253-255]), which is known to have value

$$\frac{\Gamma(k_{d-1} + 1)\Gamma(k_0 + \cdots + k_{d-2} + d - 1)}{\Gamma(k_0 + \cdots + k_{d-1} + d)}.$$

Therefore (7.14) simplifies to

$$\frac{\Gamma(k_0+1)\Gamma(k_1+1)\ldots\Gamma(k_{d-1}+1)}{\Gamma(k_0+\cdots+k_{d-1}+d-1)}=\frac{k_0!\ldots k_{d-1}!}{n!(n+d-1)(n+d-2)\ldots(n+1)}.$$

Let W be any cylinder set determined by a path from the root vertex to the vertex $k=(k_0,\ldots,k_{d-1})$, where $k_0+\cdots+k_{d-1}=n$. Referring to Corollary 6.4 and the discussion preceding it as well as the paragraph preceding Theorem 7.4, because $\pi:\Xi\to\Delta_{d-1}$ is one-to-one in this case, the set Ξ of ergodic measures is identified with $\triangle_{d-1}$, ξ corresponding to $\tau_{\pi\xi}$, and $d\nu_\eta$ is identified with $d\lambda_\eta=f\,dm_{d-1}$. (See the discussion near Formula (6.5).) Therefore,

(7.15)

$$\eta(W)=\int_\Xi \xi(W)\,d\nu_\eta(\xi)=\int_{\Delta_{d-1}}\mu_{\alpha_0,\ldots,\alpha_{d-1}}(W)f(\alpha_0,\ldots,\alpha_{d-1})\,d\alpha_0\ldots d\alpha_{d-1},$$

and (see 7.10)

$$\mu_{\alpha_0,\ldots,\alpha_{d-1}}(W)=\frac{\alpha_0^{k_0}\alpha_1^{k_1}\ldots\alpha_{d-1}^{k_{d-1}}}{s_0s_1\ldots s_{d-1}g_{a,s_0}(k_0)g_{a,s_1}(k_1)\ldots g_{a,s_{d-1}}(k_{d-1})}.$$

Thus the expression in (7.13) is equal to

$$\frac{1}{N}s_0s_1\ldots s_{d-1}g_{a,s_0}(k_0)g_{a,s_1}(k_1)\ldots g_{a,s_{d-1}}(k_{d-1})\eta(W),$$

and by (7.11) this equals

$$\frac{ns_0s_1\ldots s_{d-1}g_{a,s_0}(k_0)g_{a,s_1}(k_1)\ldots g_{a,s_{d-1}}(k_{d-1})}{N(s_0+\cdots+s_{d-1})g_{a,s_0+\ldots s_{d-1}}(n)}. \tag{7.16}$$

Making use of Lemma 7.3, we approximate

$$\frac{g_{a,s_0}(k_0)\ldots g_{a,s_{d-1}}(k_{d-1})}{g_{a,s_0+\cdots+s_{d-1}}(n)}$$

$$\approx\frac{\prod_{i=0}^{d-1}\left(\frac{k_ia+s_i}{e}\right)^{k_i}\left(\frac{k_ia+s_i}{s_i}\right)^{s_i/a}\frac{\exp(c_{a,s_i}-a/(12s_i))}{\sqrt{s_i(k_ia+s_i)}}}{\left(\frac{na+s_0+\cdots+s_{d-1}}{e}\right)^n\left(\frac{na+s_0+\cdots+s_{d-1}}{s_0+\cdots+s_{d-1}}\right)^{(s_0+\cdots+s_{d-1})/a}} \tag{7.17}$$

$$\times\frac{\sqrt{(s_0+\cdots+s_{d-1})(na+s_0+\cdots+s_{d-1})}}{\exp(c_{a,s_0+\cdots+s_{d-1}}-a/(12(s_0+\cdots+s_{d-1})))}.$$

Since $k_0+\cdots+k_{d-1}=n$, we split

$$\left(\frac{na+s_0+\cdots+s_{d-1}}{e}\right)^n=\prod_{i=0}^{d-1}\left(\frac{na+s_0+\cdots+s_{d-1}}{e}\right)^{k_i}$$

and combine each factor with the corresponding $((k_ia+s_i)/e)^{k_i}$. Likewise, split

$$\left(\frac{na+s_0+\cdots+s_{d-1}}{s_0+\cdots+s_{d-1}}\right)^{(s_0+\cdots+s_{d-1})/a}=\prod_{i=0}^{d-1}\left(\frac{na+s_0+\cdots+s_{d-1}}{s_0+\cdots+s_{d-1}}\right)^{s_i/a}$$

and combine each factor with the corresponding $((k_i a+s_i)/s_i)^{s_i/a}$. Then (7.17) simplifies to

$$\prod_{i=0}^{d-1}\left(\frac{k_i a+s_i}{na+s_0+\cdots+s_{d-1}}\right)^{k_i}\left(\frac{(s_0+\cdots+s_{d-1})(k_i a+s_i)}{s_i(na+s_0+\cdots+s_{d-1})}\right)^{s_i/a}$$

$$\times\exp\left(\left(\sum_{i=1}^{d}c_{a,s_i}-\frac{a}{12s_i}\right)-\left(c_{a,s_0+\cdots+s_{d-1}}-\frac{a}{12(s_0+\cdots+s_{d-1})}\right)\right) \tag{7.18}$$

$$\times\frac{\sqrt{(s_0+\cdots+s_{d-1})(na+s_0+\cdots+s_{d-1})}}{\sqrt{s_0s_1\ldots s_{d-1}(k_0a+s_0)(k_1a+s_1)\ldots(k_{d-1}a+s_{d-1})}}.$$

As we will not simplify the exponential any more, for readability we will denote it by e^C until the end of the calculation.

Note that for each i, large n, and $\mu_{\alpha_0,\ldots,\alpha_{d-1}}$-almost every path in the multidimensional reinforced Pascal adic we have $k_i/n\to\alpha_i$. We introduce factors of n and $1/n$ so that we may simplify further. Now (7.18) becomes

(7.19)
$$\left(\prod_{i=0}^{d-1}\left(\frac{(k_i/n)a+(s_i/n)}{a+(s_0+\cdots+s_{d-1})/n}\right)^{ki}\left(\frac{(s_0+\cdots+s_{d-1})(k_ia/n+s_i/n)}{s_i(a+(s_0+\cdots+s_{d-1})/n)}\right)^{s_i/a}\right)e^C$$

$$\times\sqrt{\frac{n(s_0+\cdots+s_{d-1})\left(a+\dfrac{s_0+\cdots+s_{d-1}}{n}\right)}{s_0s_1\ldots s_{d-1}n^d\left(\dfrac{k_0}{n}a+\dfrac{s_0}{a}\right)\left(\dfrac{k_1}{n}a+\dfrac{s_1}{a}\right)\cdots\left(\dfrac{k_{d-1}}{n}a+\dfrac{s_{d-1}}{a}\right)}}$$

$$\approx\left(\prod_{i=0}^{d-1}\alpha_i^{k_i}\alpha_i^{s_i/a}\left(\frac{s_0+\cdots+s_{d-1}}{s_i}\right)^{s_i/a}\right)e^C n^{(1-d)/2}\sqrt{\frac{s_0+\cdots+s_{d-1}}{a^{d-1}\prod_{i=0}^{d-1}s_i\alpha_i}}.$$

This ends the approximation of (7.17). We now approximate

$$1/N=\frac{n!(n+d-1)(n+d-2)\ldots(n+1)}{k_0!\ldots k_{d-1}!}$$

using Stirling's approximation $n! \approx \sqrt{2\pi n}\,(n/e)^n$:
(7.20)

$$\begin{aligned}
\frac{1}{N} &\approx (n+d-1)(n+d-2)\dots(n+1)\frac{\sqrt{2\pi n}n^n e^n}{e^n \prod_{i=0}^{d-1} k_i^{k_i}\sqrt{2\pi k_i}} \\
&= (n+d-1)(n+d-2)\dots(n+1)\frac{\sqrt{2\pi n}}{\prod_{i=0}^{d-1}(k_i/n)^{k_i}\sqrt{n}\sqrt{2\pi k_i/n}} \\
&\approx (n+d-1)(n+d-2)\dots(n+1)\frac{1}{\left(\sqrt{2\pi n}\right)^{d-1}\prod_{i=0}^{d-1}\alpha_i^{k_i}\sqrt{\alpha_i}} \\
&= (n+d-1)(n+d-2)\dots(n+1)\left(\frac{1}{2\pi}\right)^{(d-1)/2} n^{(1-d)/2}\frac{1}{\prod_{i=0}^{d-1}\alpha_i^{k_i}\sqrt{\alpha_i}}.
\end{aligned}$$

Combining (7.20) and (7.19), we have that (7.16) becomes

$$\frac{(n+d-1)(n+d-2)\dots(n+1)}{n^{d-1}}\left(\frac{1}{2a\pi}\right)^{(d-1)/2}\left(\prod_{i=0}^{d-1}\alpha_i^{(s_i/a)-1}\left(\frac{s_0+\dots+s_{d-1}}{s_i}\right)^{s_i/a}\right) \times e^C\sqrt{\frac{s_0 s_1\dots s_{d-1}}{s_0+\dots+s_{d-1}}}.$$

As we let $n\to\infty$, we have the approximation for $f_{a,s_0,\dots,s_{d-1}}(\alpha_0,\dots,\alpha_{d-1})$ as

$$e^C(2a\pi)^{(1-d)/2}\sqrt{\frac{s_0\dots s_{d-1}}{s_0+\dots+s_{d-1}}}\cdot\left(\prod_{i=0}^{d-1}\alpha_i^{(s_i/a)-1}\left(\frac{s_0+\dots+s_{d-1}}{s_i}\right)^{s_i/a}\right). \tag{7.21}$$

We also know that the integral of the density function over all possible values of the α_i should give us 1. Therefore

$$\begin{aligned}
&e^C(2a\pi)^{(1-d)/2}\sqrt{\frac{s_0\dots s_{d-1}}{s_0+\dots+s_{d-1}}}\left(\prod_{i=0}^{d-1}\left(\frac{s_0+\dots+s_{d-1}}{s_i}\right)^{s_i/a}\right) \\
&= \left(\int\dots\int\prod_{i=0}^{d-1}\alpha_i^{(s_i/a)-1}\,d\alpha_0\dots d\alpha_{d-1}\right)^{-1} \\
&= \left(\int\dots\int\left(1-\sum_{k=0}^{d-2}\alpha_k\right)^{(s_{d-1}/a)-1}\prod_{i=0}^{d-2}(\alpha_i)^{(s_i/a)-1}\,d\alpha_0\dots d\alpha_{d-2}\right) \\
&= \frac{\Gamma\left(\dfrac{s_0+\dots+s_{d-1}}{a}\right)}{\Gamma\left(\dfrac{s_0}{a}\right)\dots\Gamma\left(\dfrac{s_{d-1}}{a}\right)},
\end{aligned}$$

again by Dirichlet's integral.

Therefore

$$f_{a,s_0,s_1,\ldots,s_{d-1}}(\alpha_0,\ldots,\alpha_{d-1}) = \frac{\Gamma\left(\dfrac{s_0+\cdots+s_{d-1}}{a}\right)}{\Gamma\left(\dfrac{s_0}{a}\right)\ldots\Gamma\left(\dfrac{s_{d-1}}{a}\right)} \prod_{i=0}^{d-1} \alpha_i^{(s_i/a)-1}.$$

□

For the appropriate choices of a and s, this formula reduces to the ones given in Examples 7.1 and 7.2.

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