Some Sturmian Symbolic Dynamics

Karl Petersen

University of North Carolina at Chapel Hill

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- ideals in C^* algebras.

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- Codings of irrational rotations: There are x and irrational θ such that for all n, $\omega(n) = 1_{[1-\theta,1)}(x + n\theta)$ or for all n, $\omega(n) = 1_{(1-\theta,1]}(x + n\theta)$. (A *Sturmian system* is then the closure of the orbit of ω under the shift. It is minimal, uniquely ergodic, and isomorphic to the irrational translation.)

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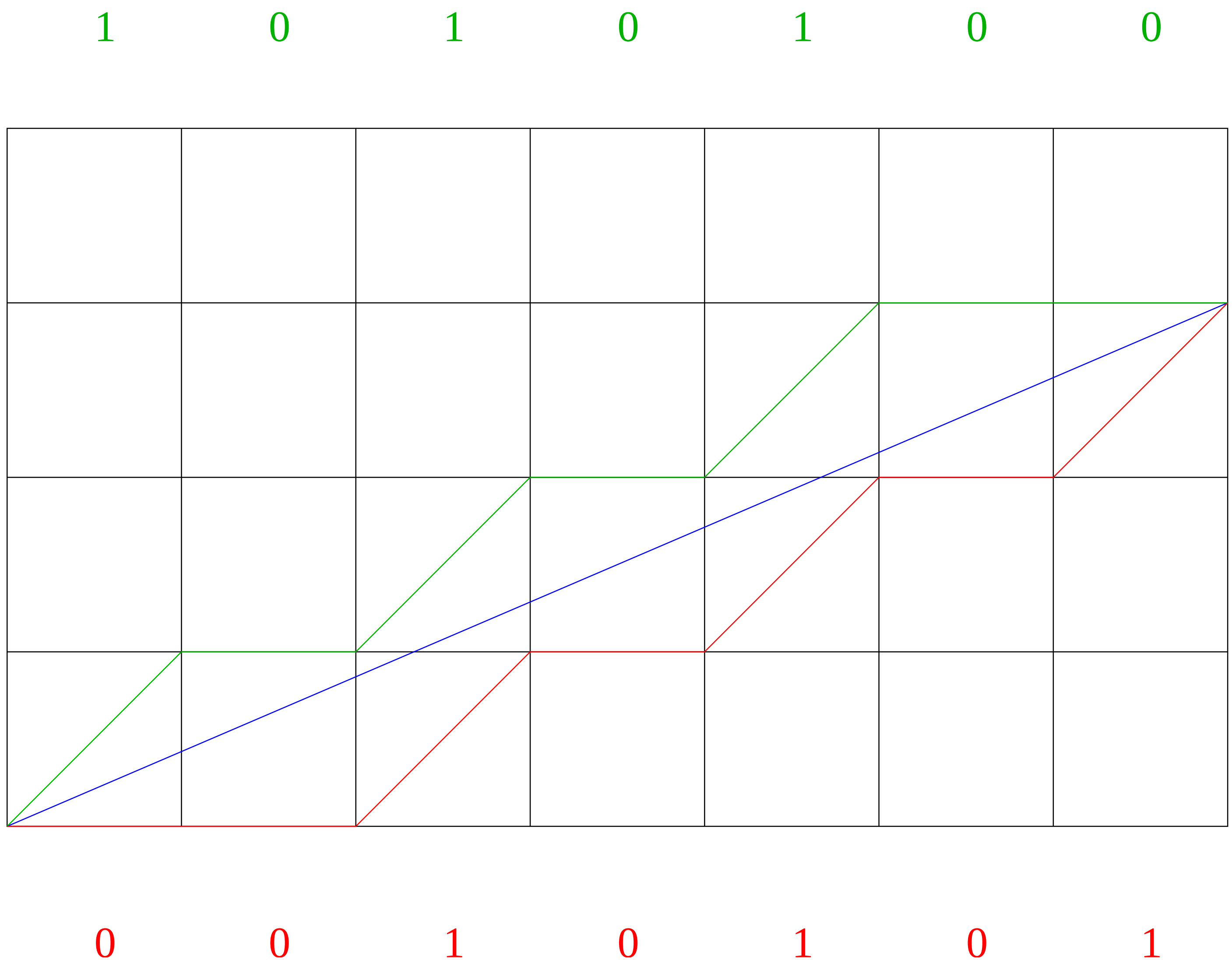
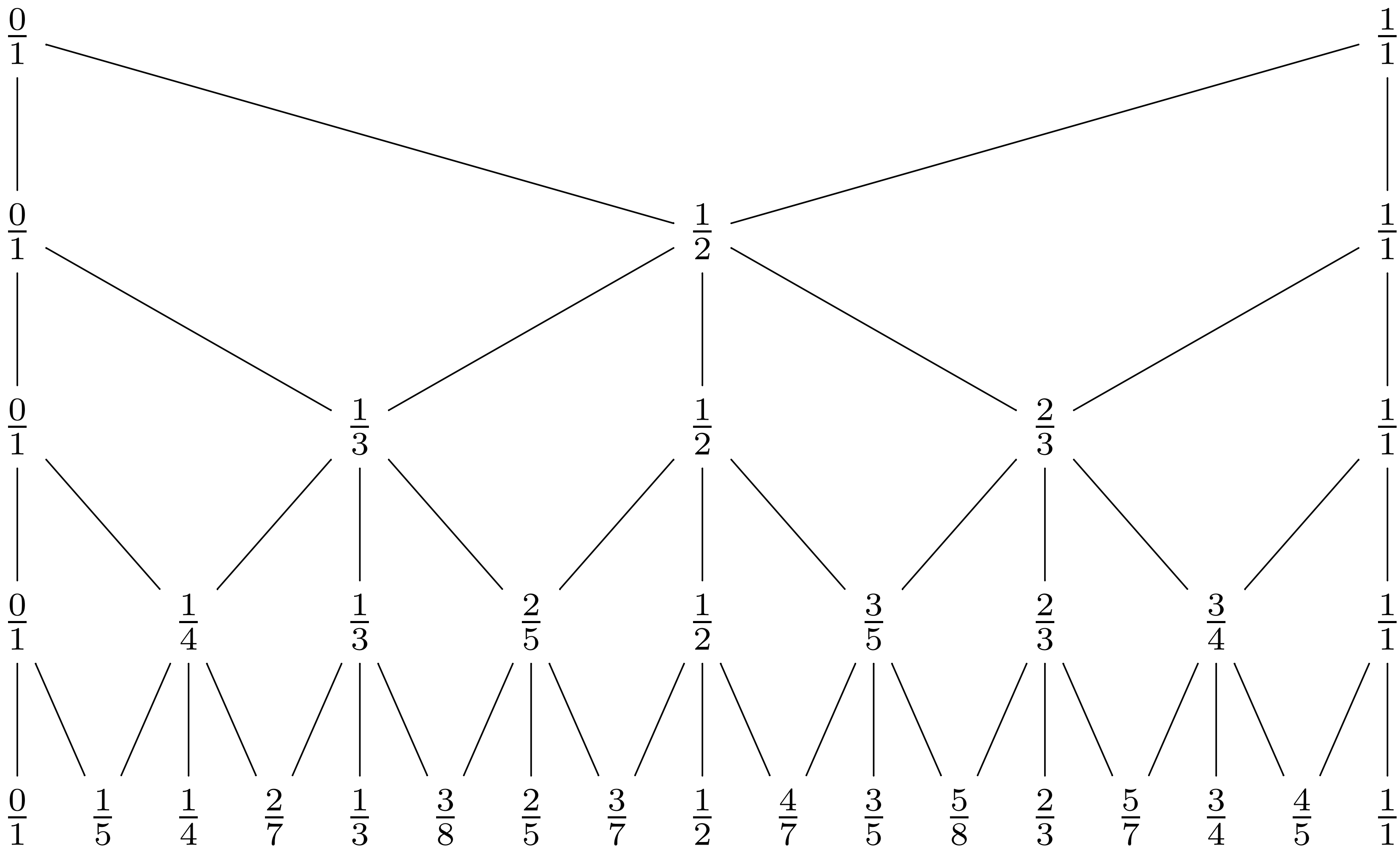
- Staircase coding: There are x and irrational θ such that for all n, $\omega(n) = \lfloor x + (n+1)\theta \rfloor - \lfloor x + n\theta \rfloor$ or for all n, $\omega(n) = \lceil x + (n+1)\theta \rceil - \lceil x + n\theta \rceil$. (Look at jumps between lattice points above or below line through origin of slope θ. Get jump (of floor) when $n\theta$ is in $[1 - \theta, 1)$.)

Upper and lower staircase codings, by jumps



Farey, Stern-Brocot, or *C. Haros* Diagram



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- I learned about the Farey shift from papers of Jeff Lagarias and about this “Farey diagram with memory" from Oliver Jenkinson and Florin Boca.

Ordinary and intermediate continued fractions

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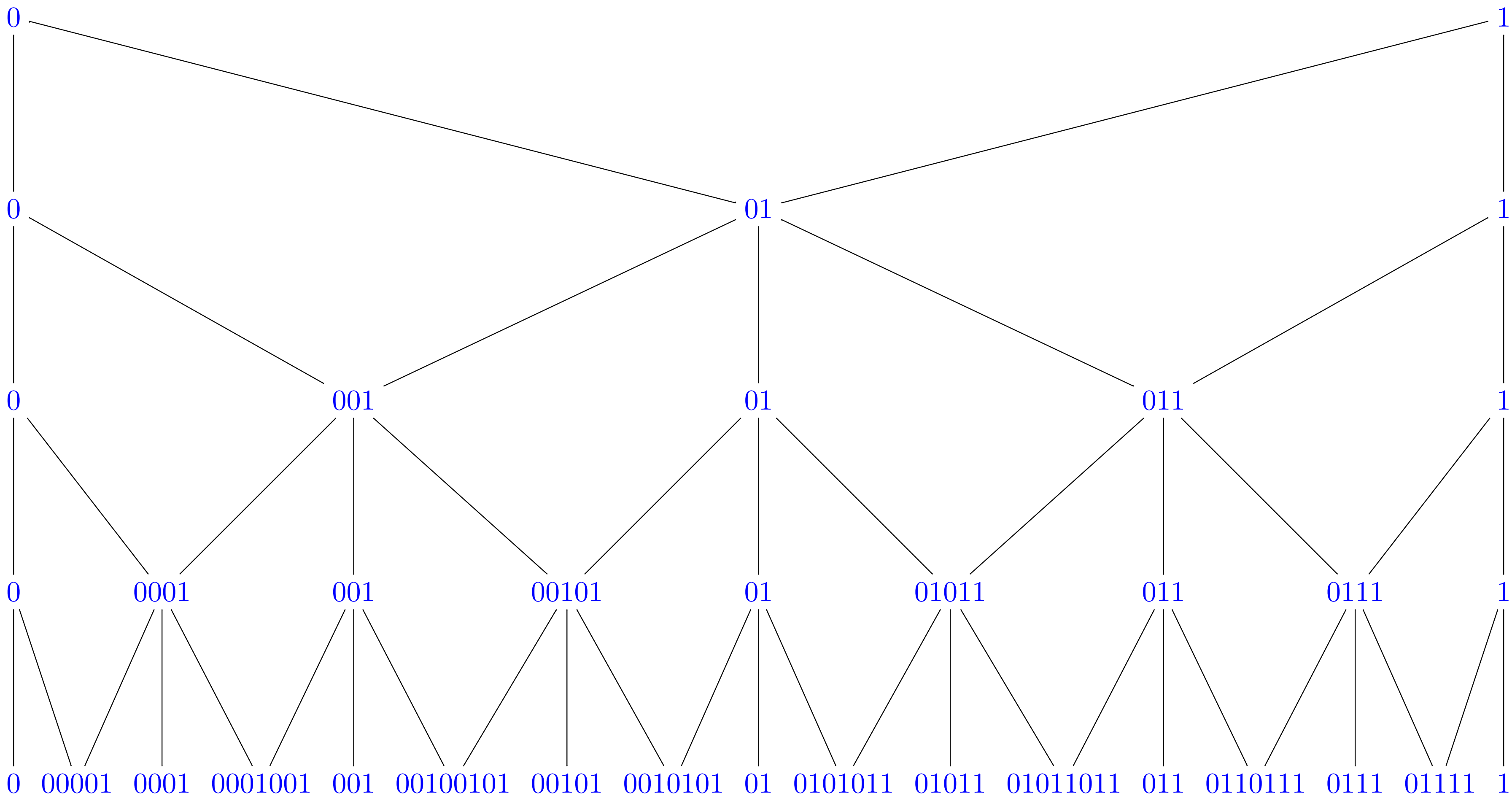
$$x = [2, 3, 2, 4, \ldots] \approx 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{7}{16}, \frac{10}{23}, \frac{17}{39}, \frac{24}{55}, \frac{31}{71}, \ldots$$

Farey Diagram of Blocks



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- Every balanced word of length $p+q$ with exactly p 1's is a rotation of the word in the Farey diagram that corresponds to $p/(p+q)$. There are exactly $p+q$ of them.
- Infinite nonperiodic Sturmian sequences are found as “ends" of infinite paths in the Farey diagram.

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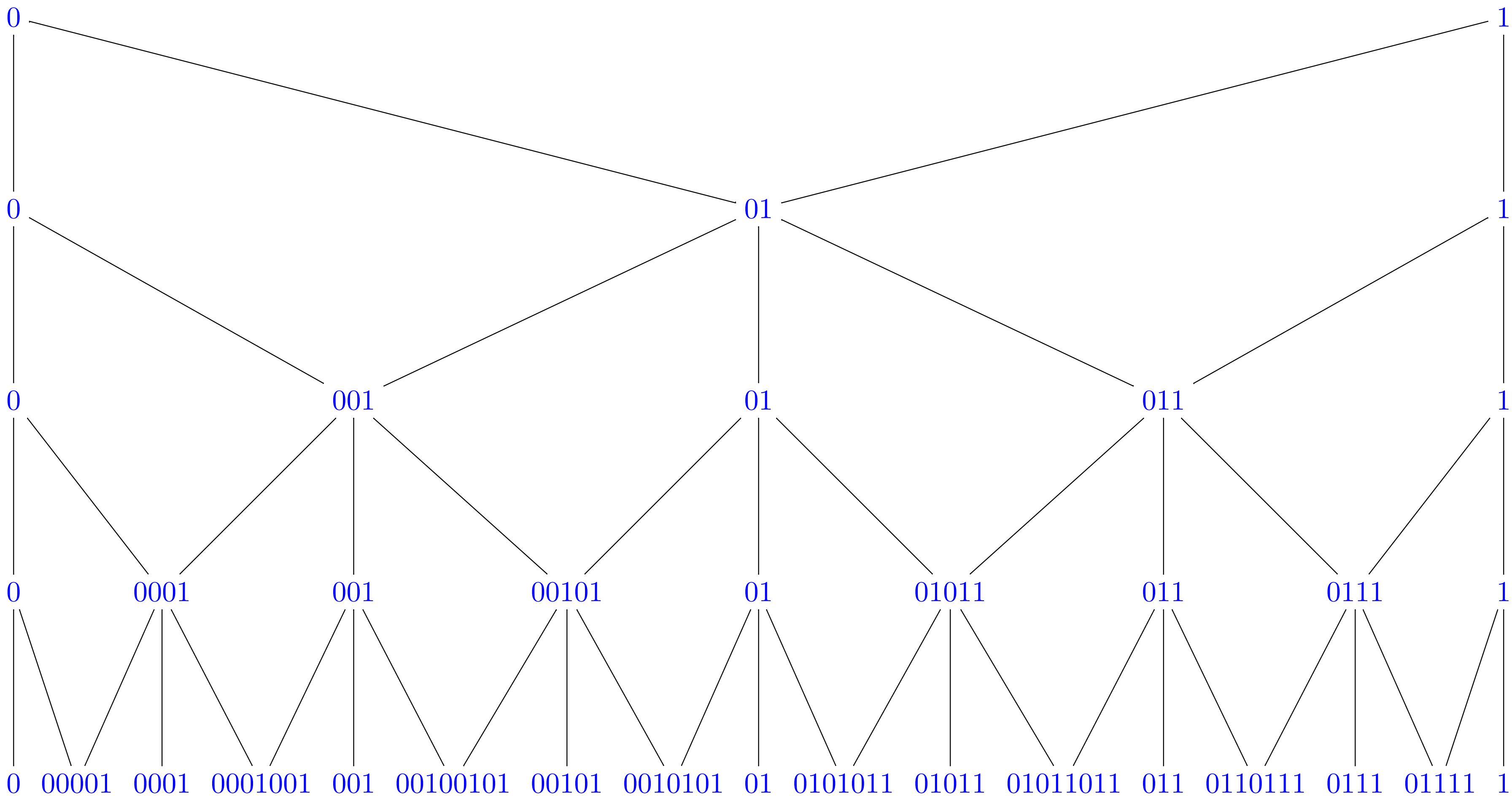

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- Besides Coven-Hedlund (1973) and Hedlund-Morse (1940), we should also mention Jenkinson-Zamboni (2004), Arnoux (2002—in Pytheas Fogg), Berstel-Séébold (2002—in Lothaire), Jenkinson (1996–), Bullett-Sentenac (1994), Borel-Laubie (1993), Rauzy (1985), Gambaudo-Lanford-Tresser (1984), Hedlund (1944), Christoffel (1875), J. Bernoulli (1772), and probably others.

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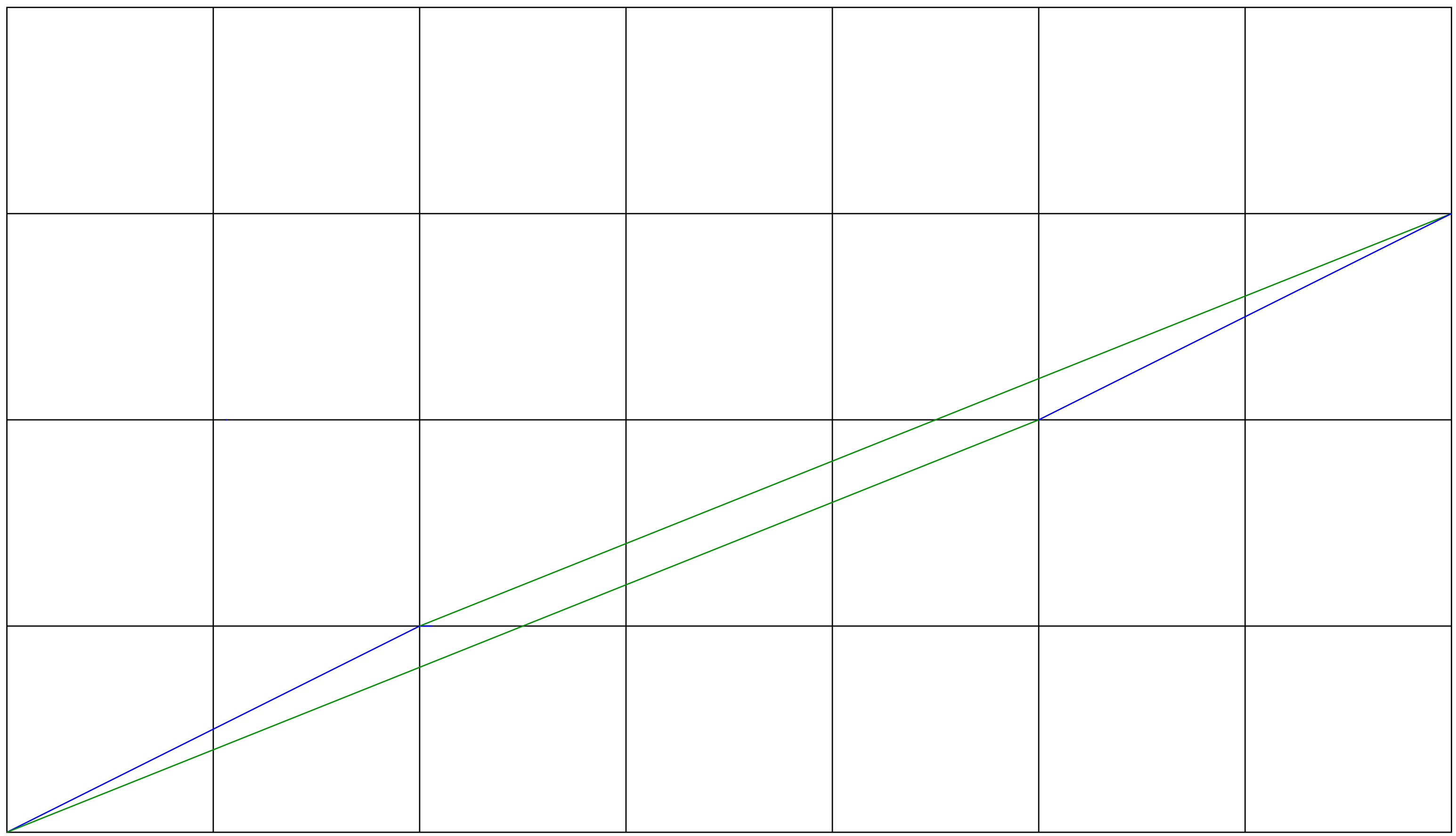
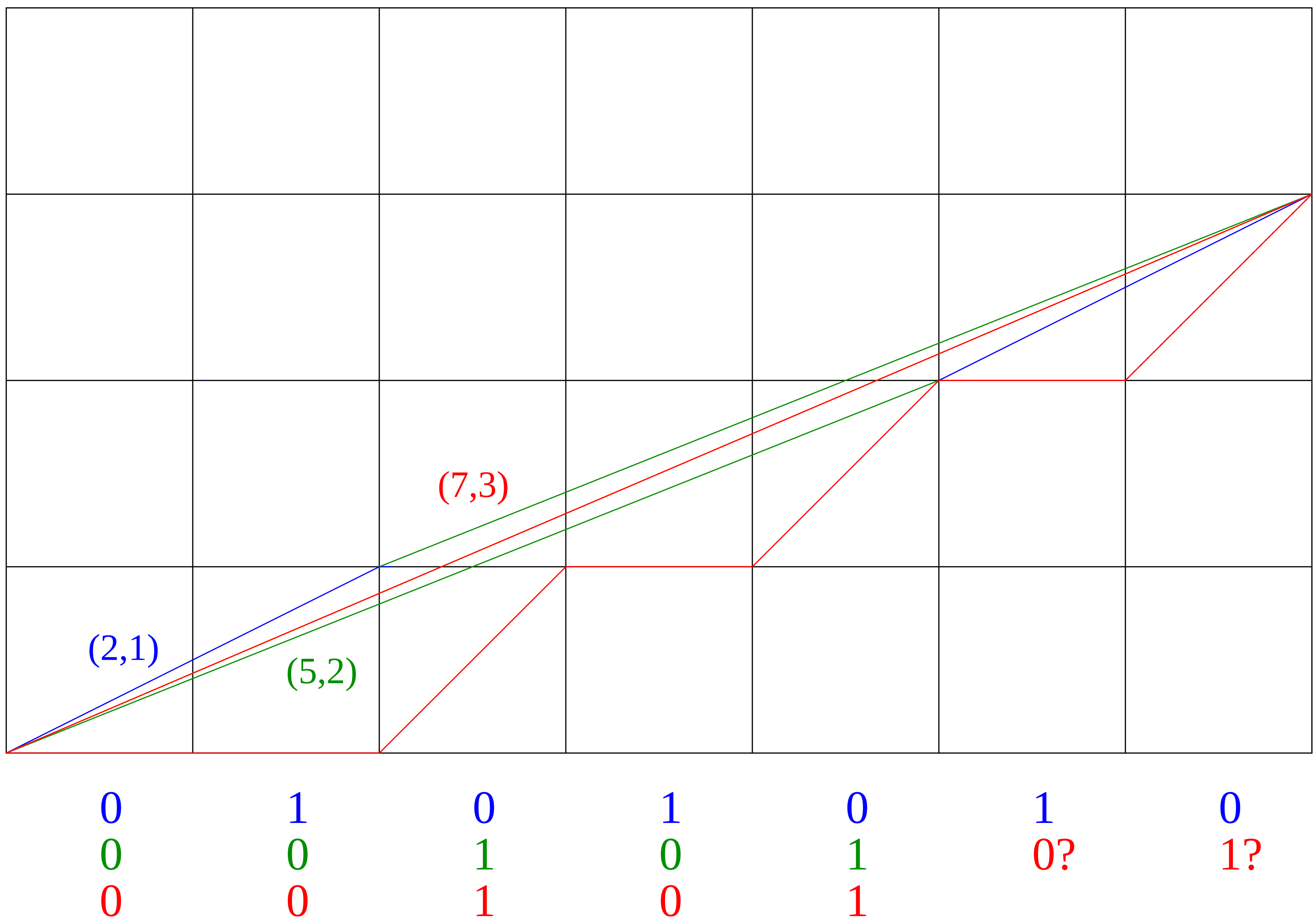
Parallelogram containing no interior lattice points

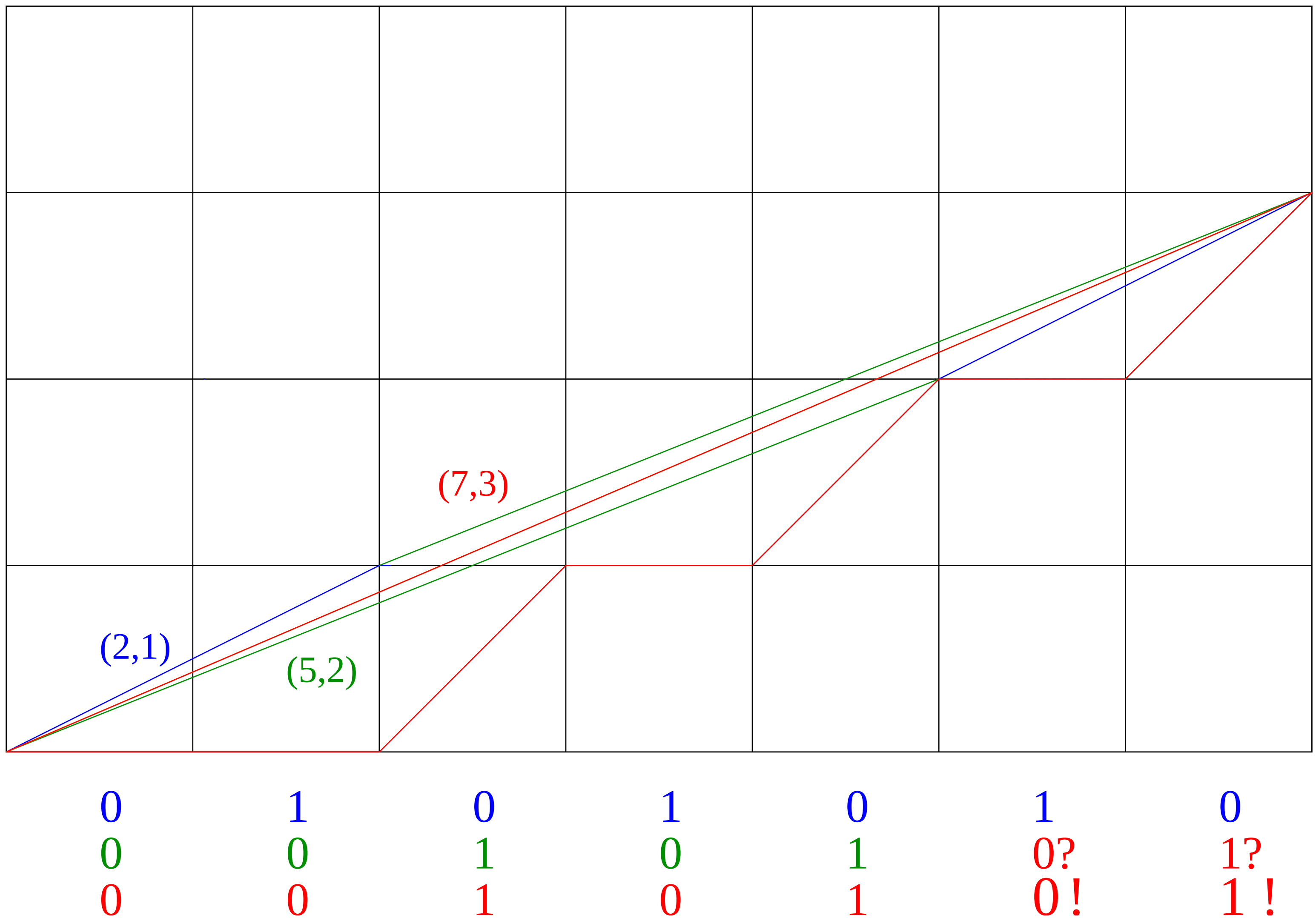


(2,1) (5,2)

First part of coding of (7,3) follows (5,2)

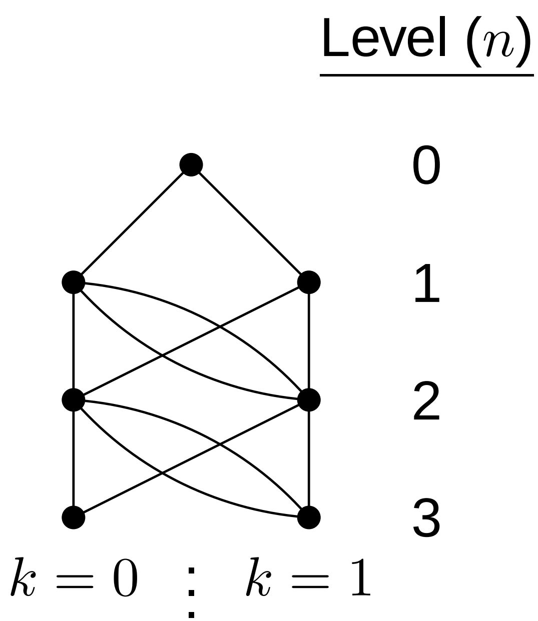


Last part of coding of (7,3) follows translate of (2,1)



Bratteli Diagrams

- Infinite downward directed graphs



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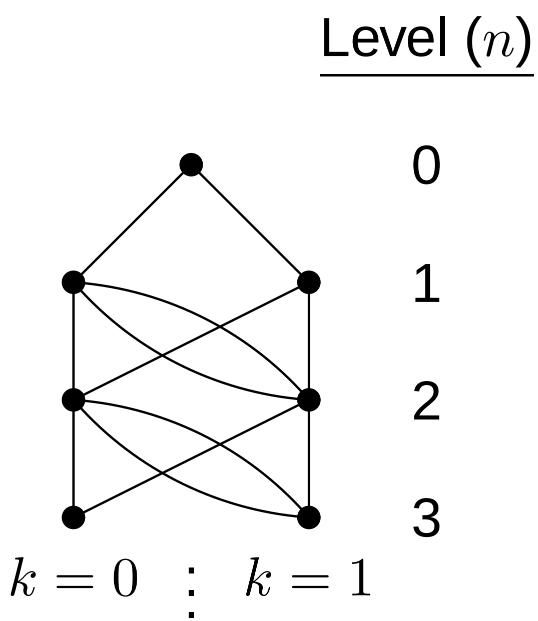

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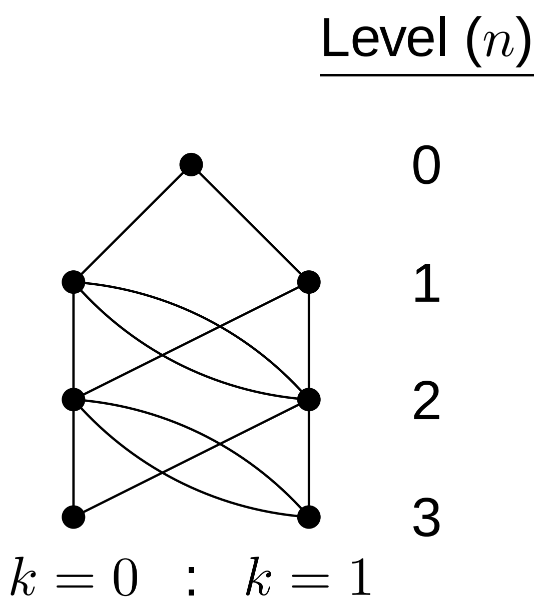

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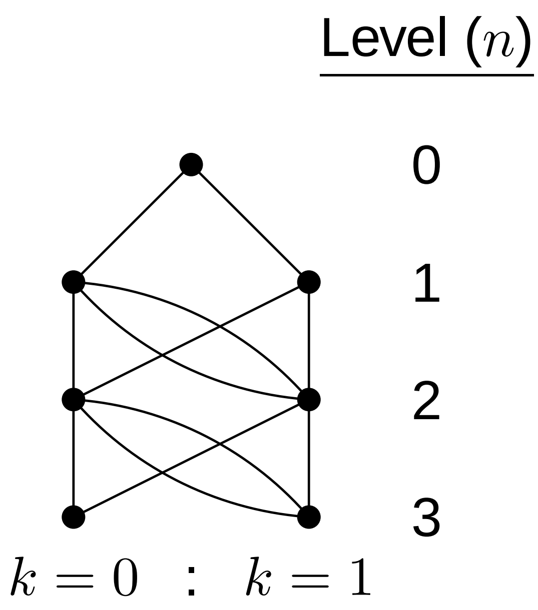
- Incidence matrices describe the number of edges connecting levels n and $n+1$



$$A_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

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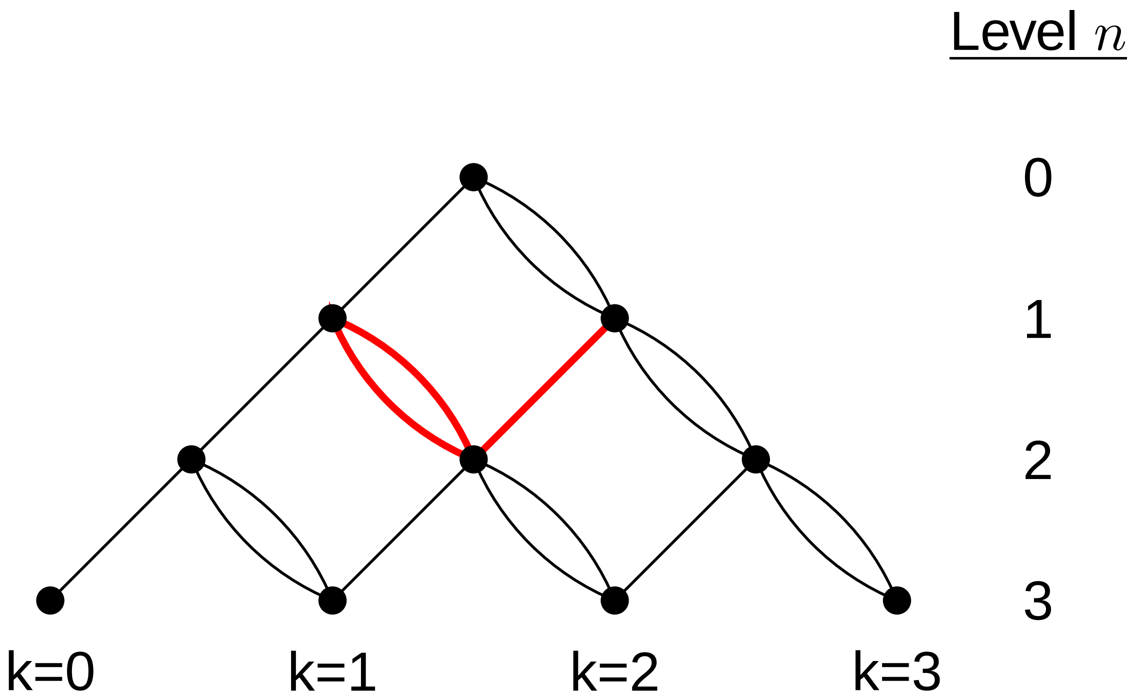
- For $x = x_0x_1x_2 \cdots \in X$ denote by x_i the i'th edge of x, which connects a vertex in level i to a vertex in level $i+1$.
- X is a compact metric space with metric given by:
  For $x, y \in X$, $d(x,y) = 2^{-i}$ where $i = \inf\{j | x_j \neq y_j\}$.

Edge ordering yields a partial order on the set of paths



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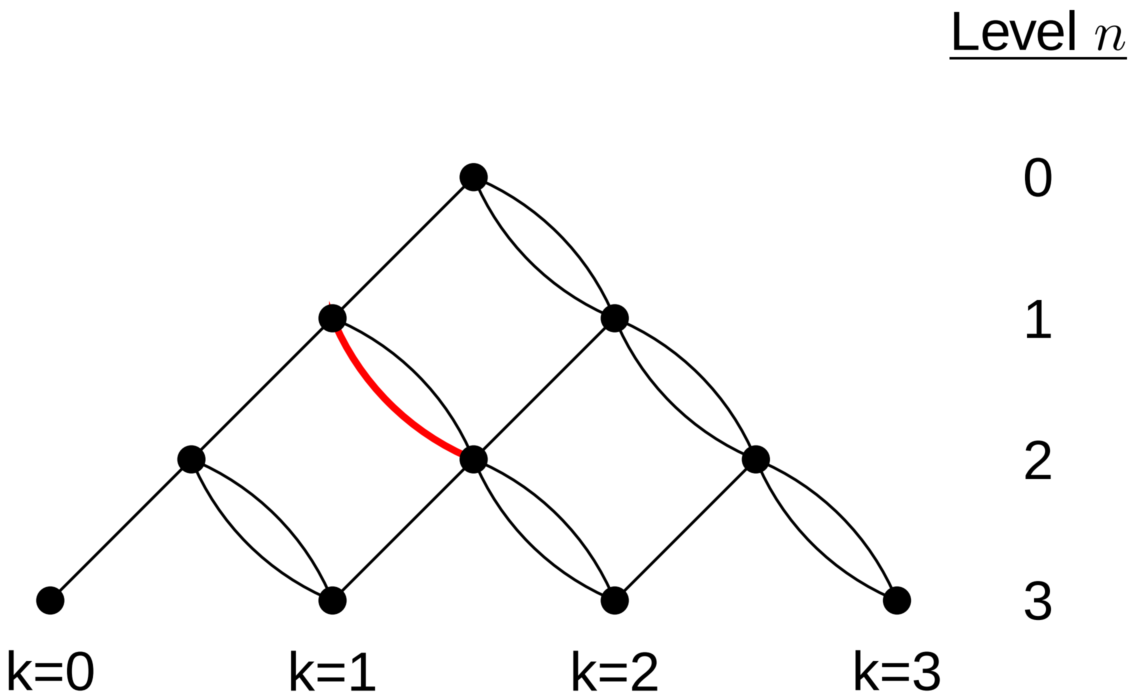

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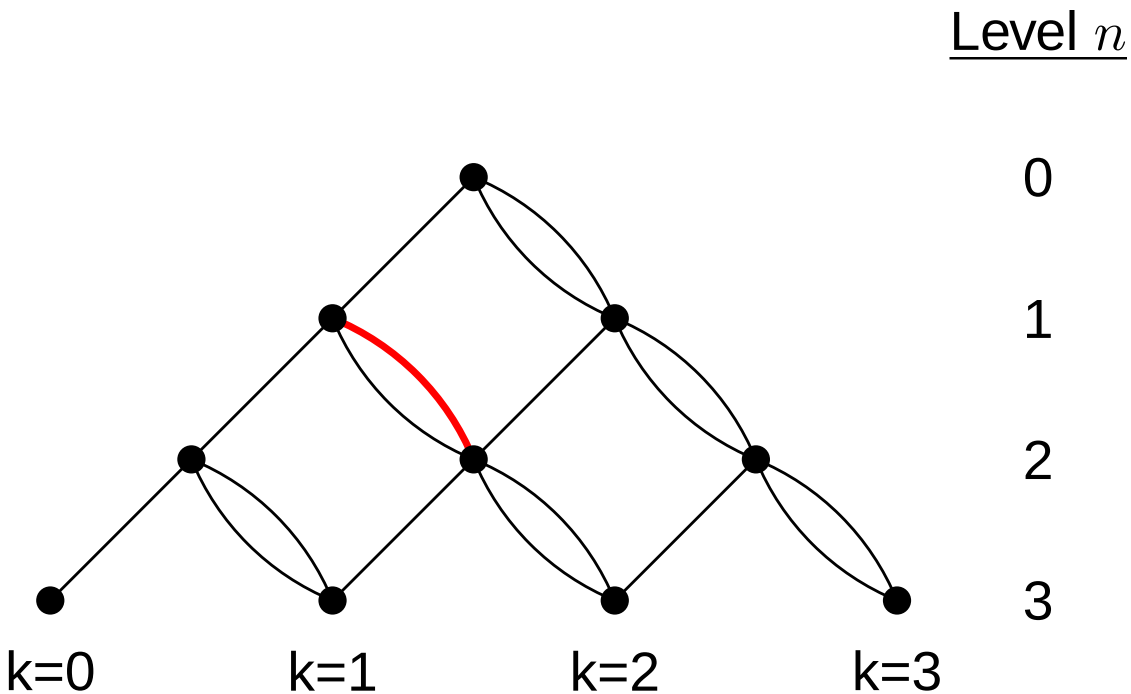

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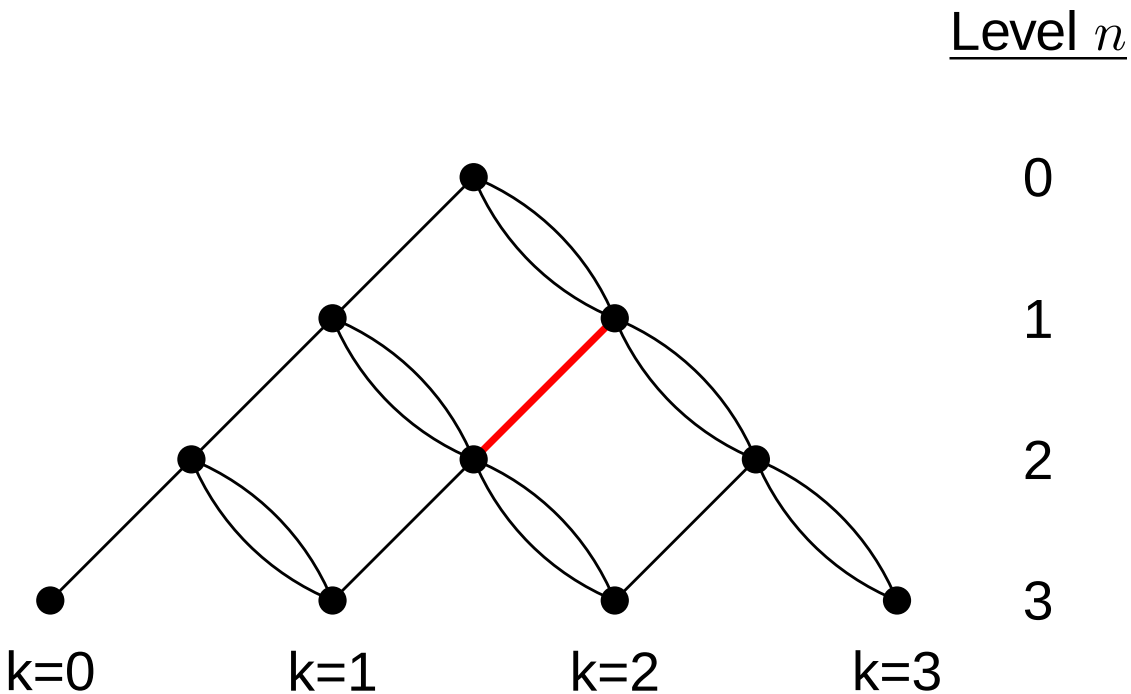

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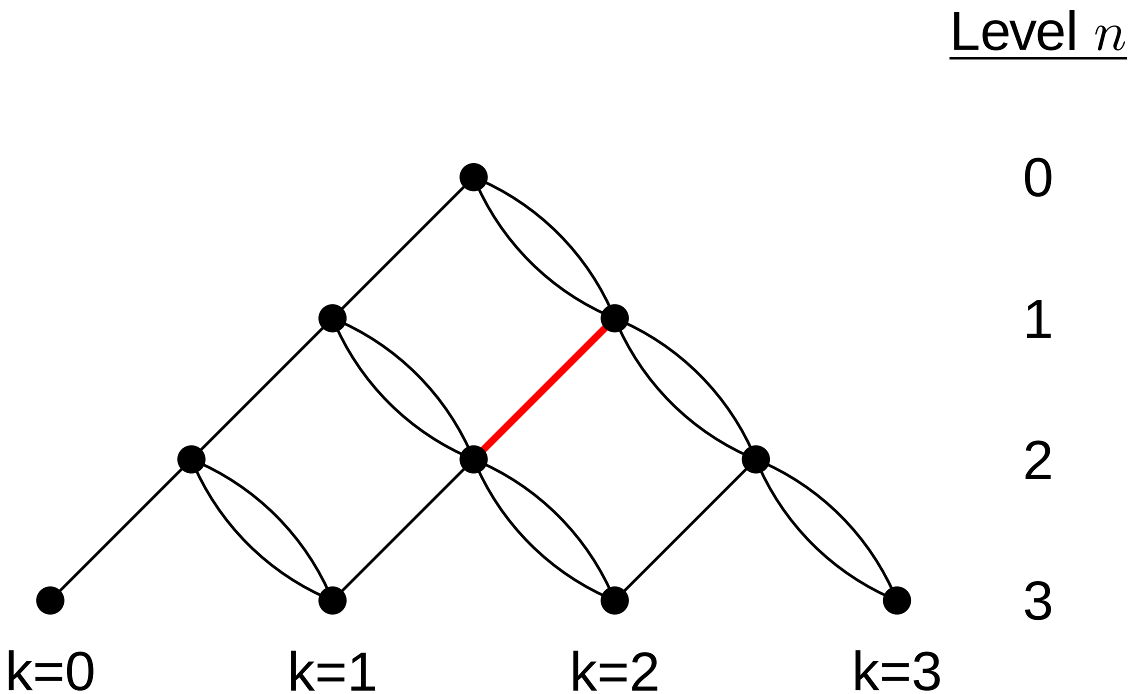
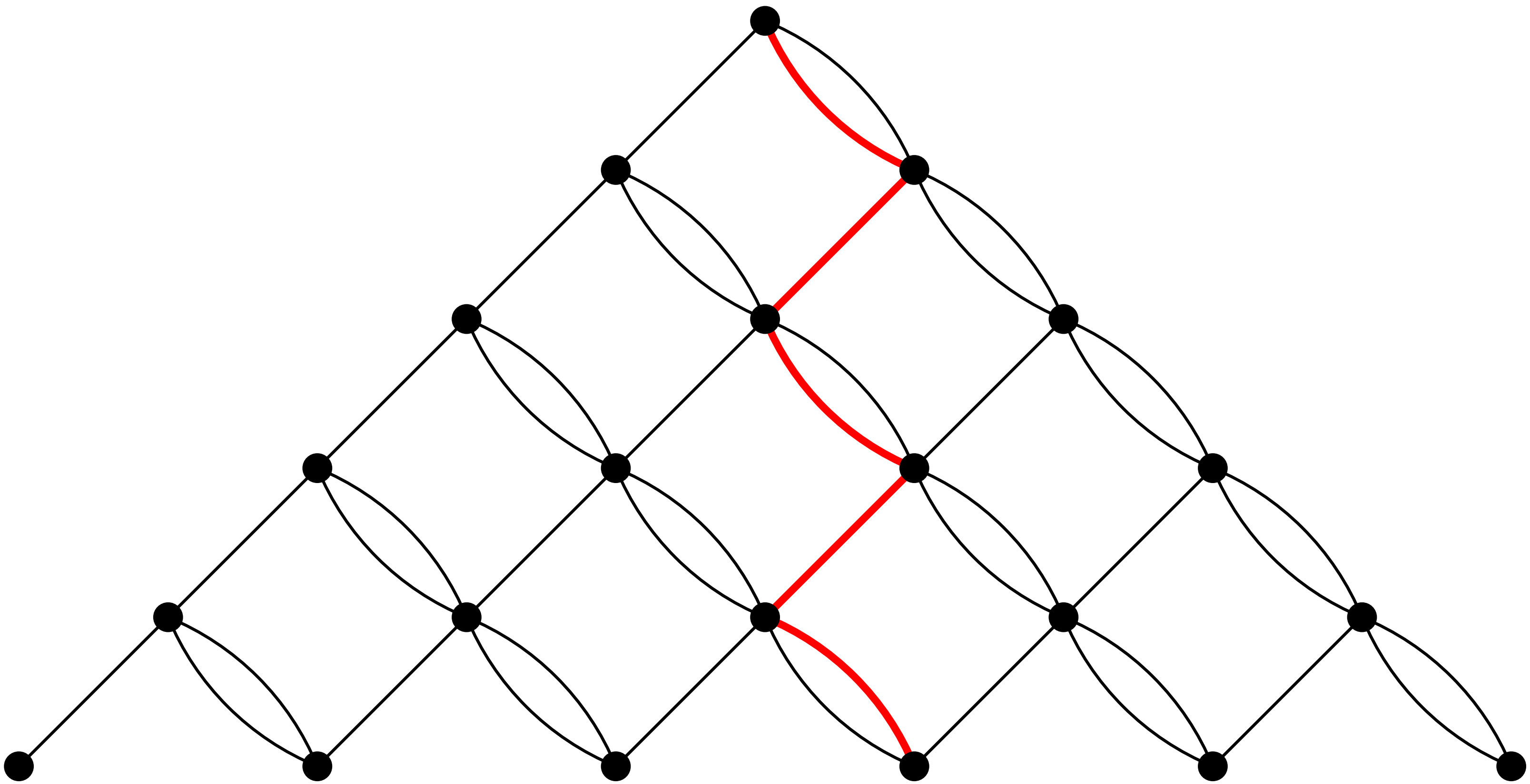


Define $y > x$ if $y_n > x_n$ the last time they differ.

The adic transformation

- $T : X \to X$, $Tx =$ smallest $y > x$ if there is one.



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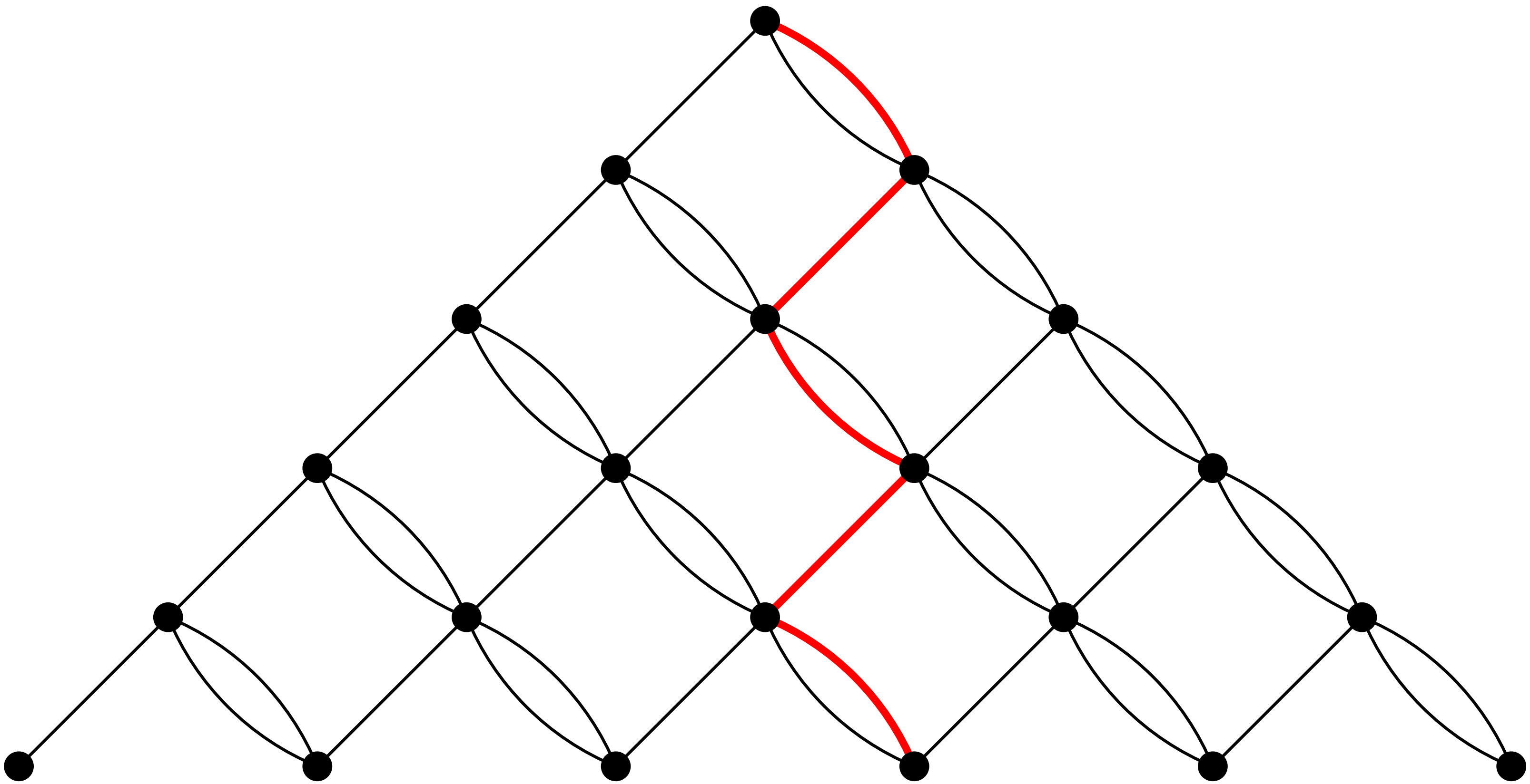

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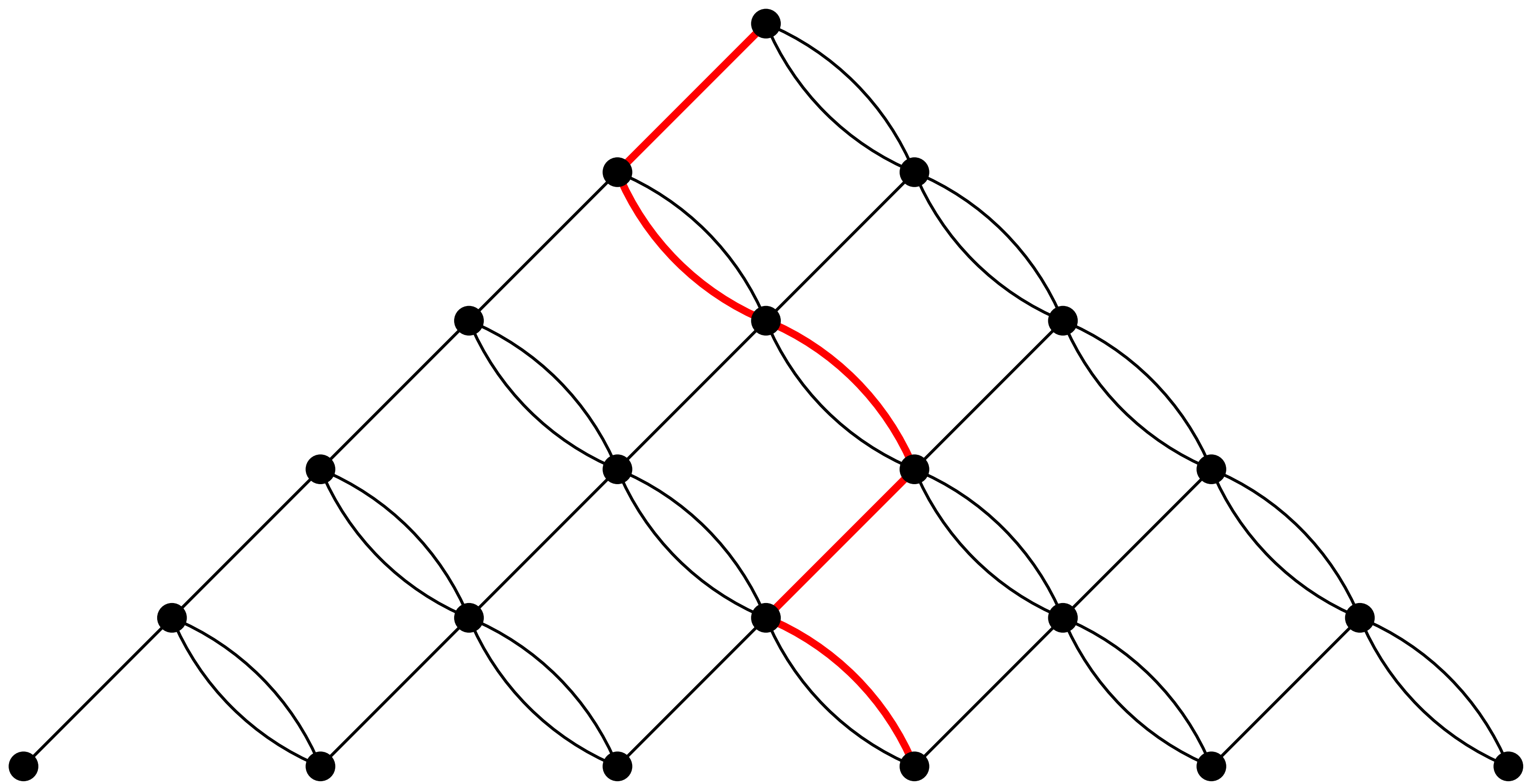

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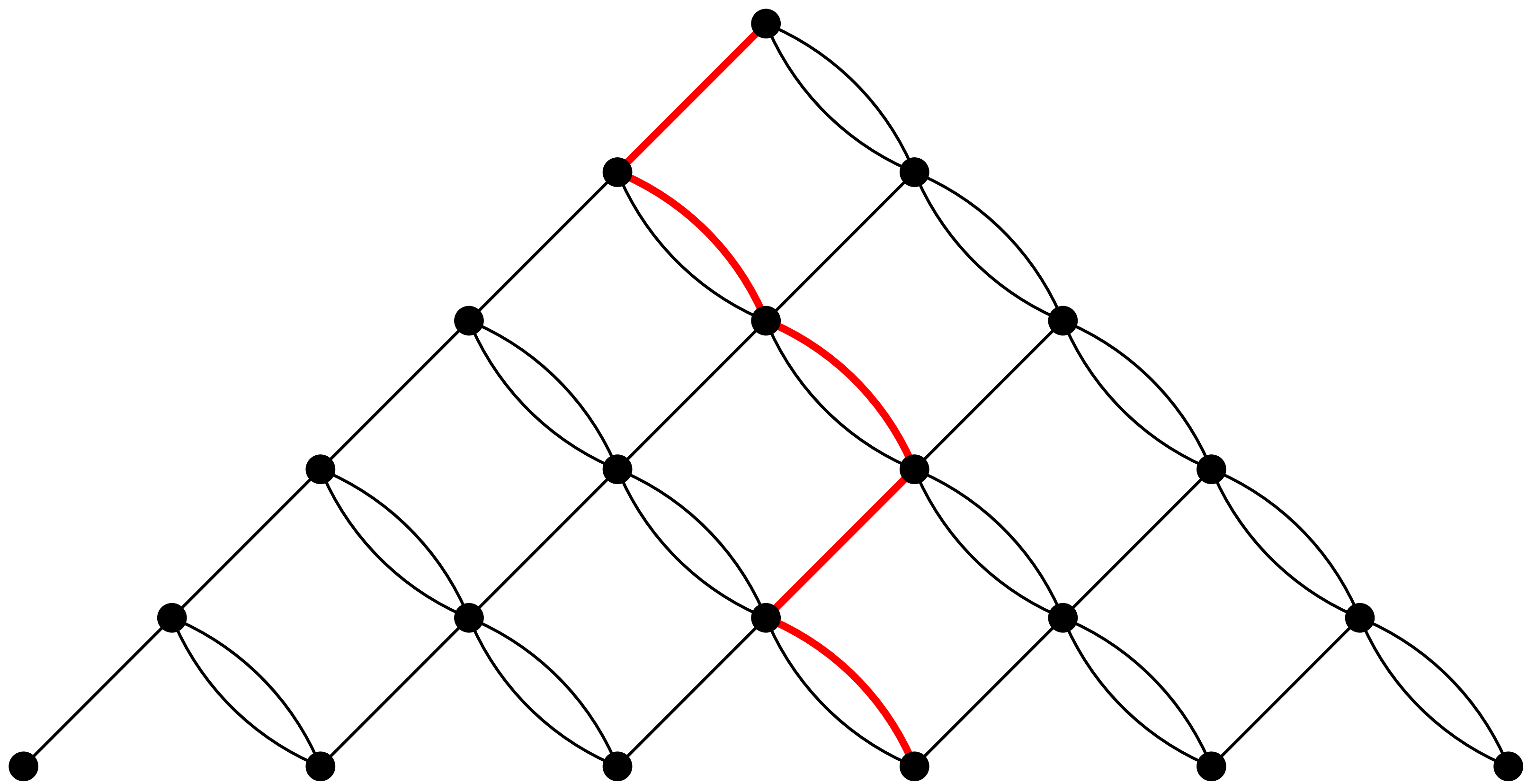

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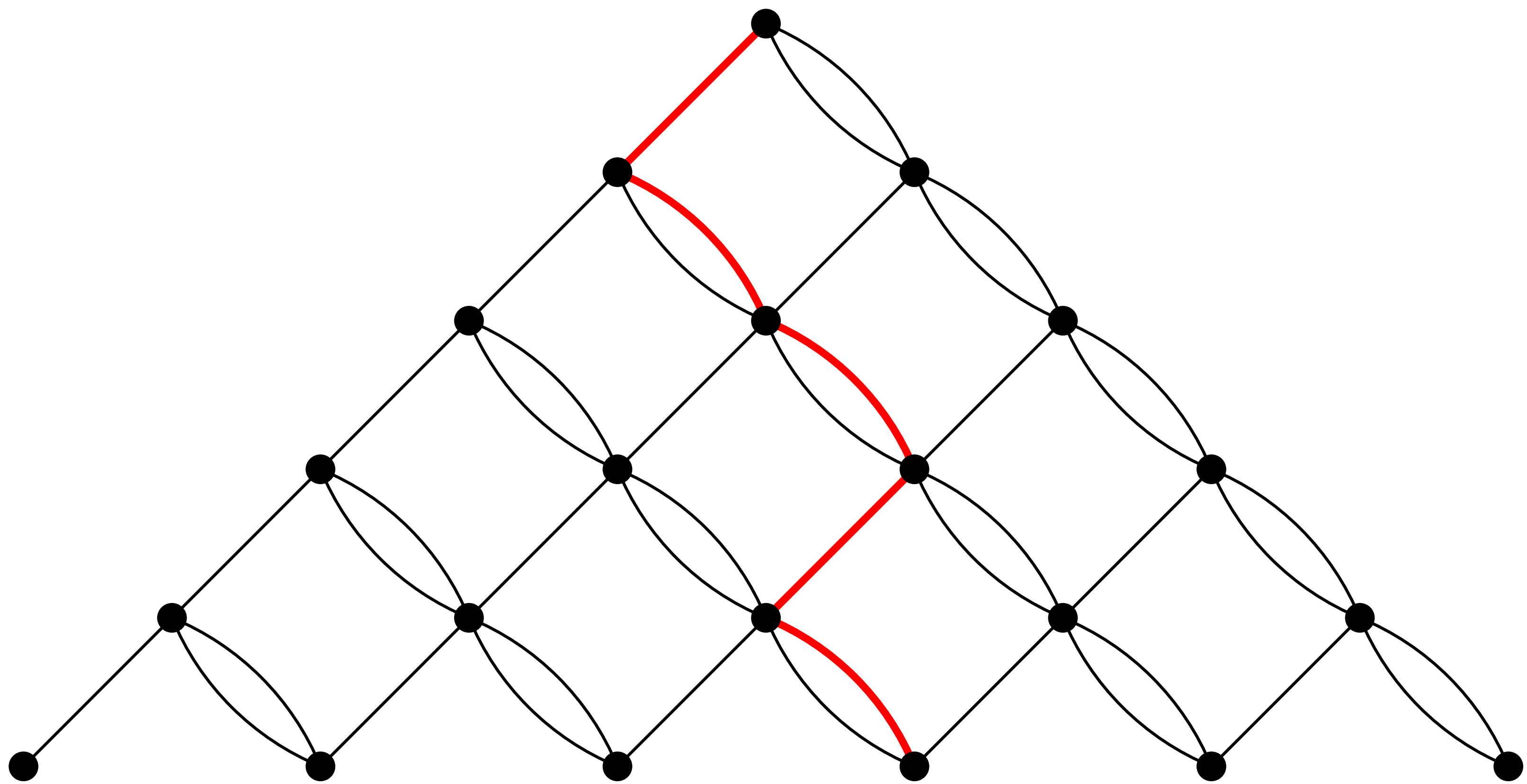

Thanks to Sarah Bailey Frick for this animated introduction.

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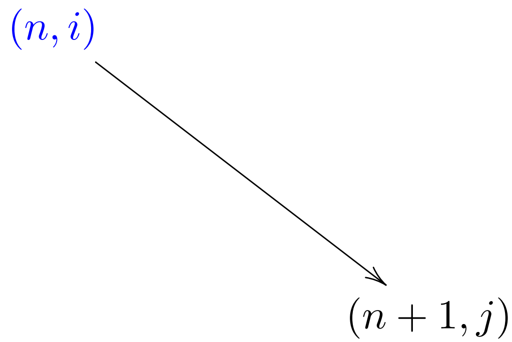

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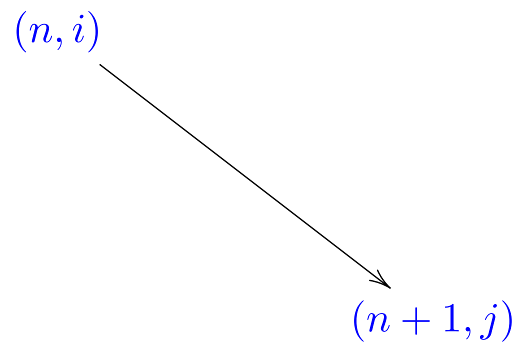

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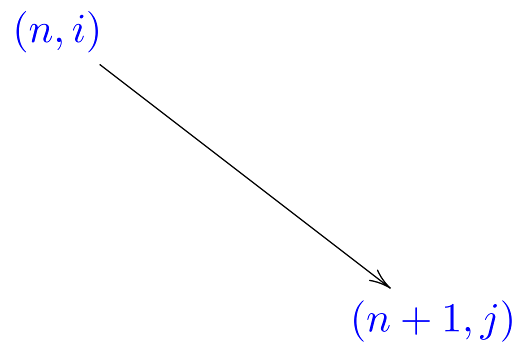

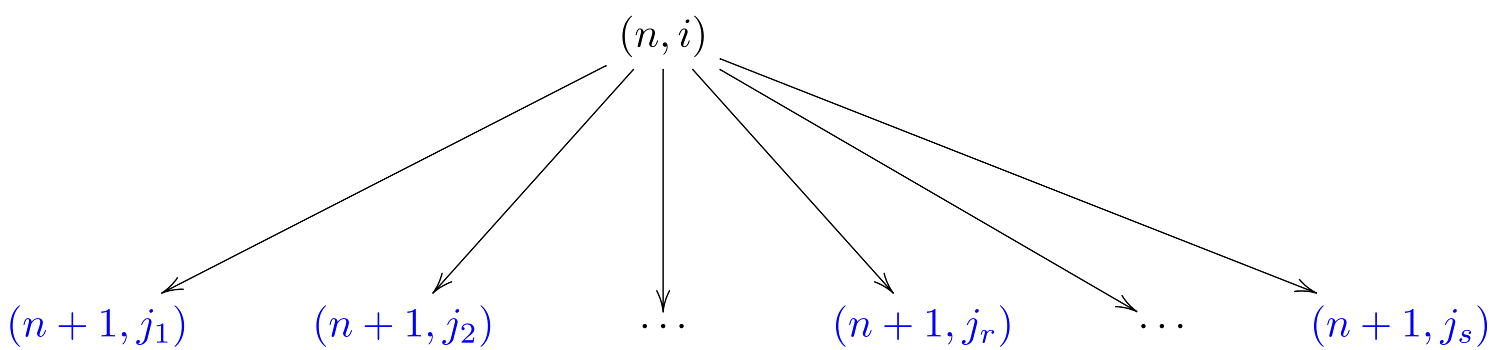

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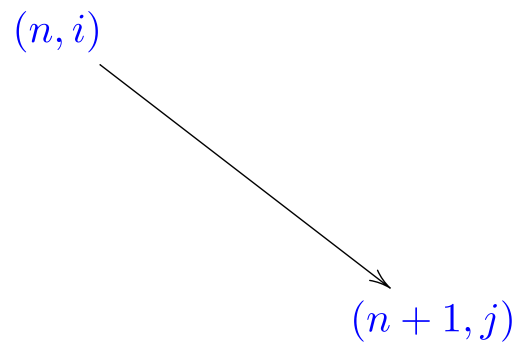

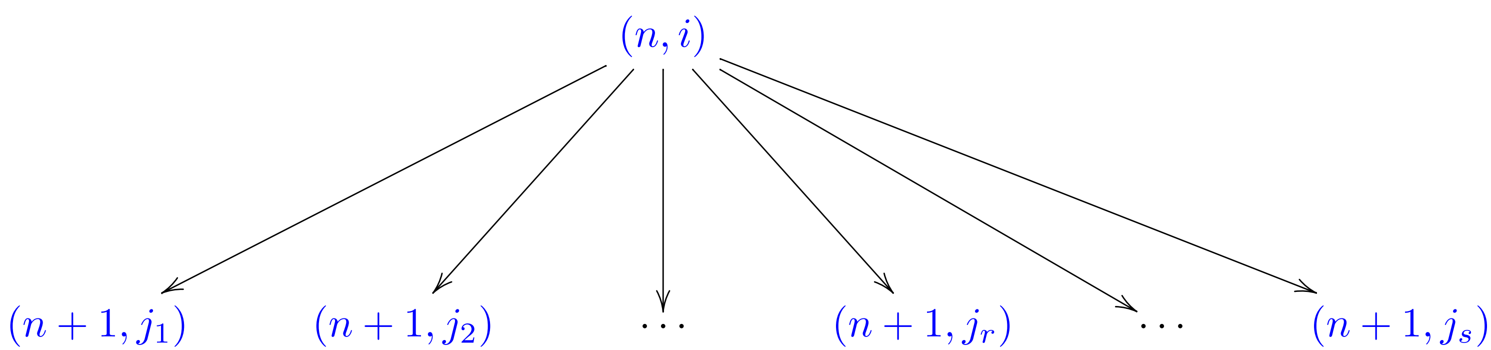

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In terms of the diagram Λ determining I, this means that if $(n,i),(m,j) \notin \Lambda$, then there are $p \geq n,m$ and $(p,k) \notin \Lambda$ such that $(n,i) \searrow (p,k)$ and $(m,j) \searrow (p,k)$.

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In terms of the diagram Λ determining I, this means that if $(n,i),(m,j) \notin \Lambda$, then there are $p \geq n,m$ and $(p,k) \notin \Lambda$ such that $(n,i) \searrow (p,k)$ and $(m,j) \searrow (p,k)$.

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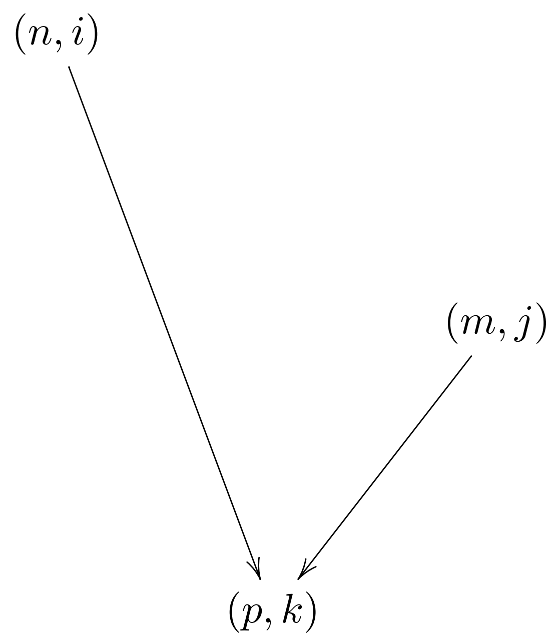

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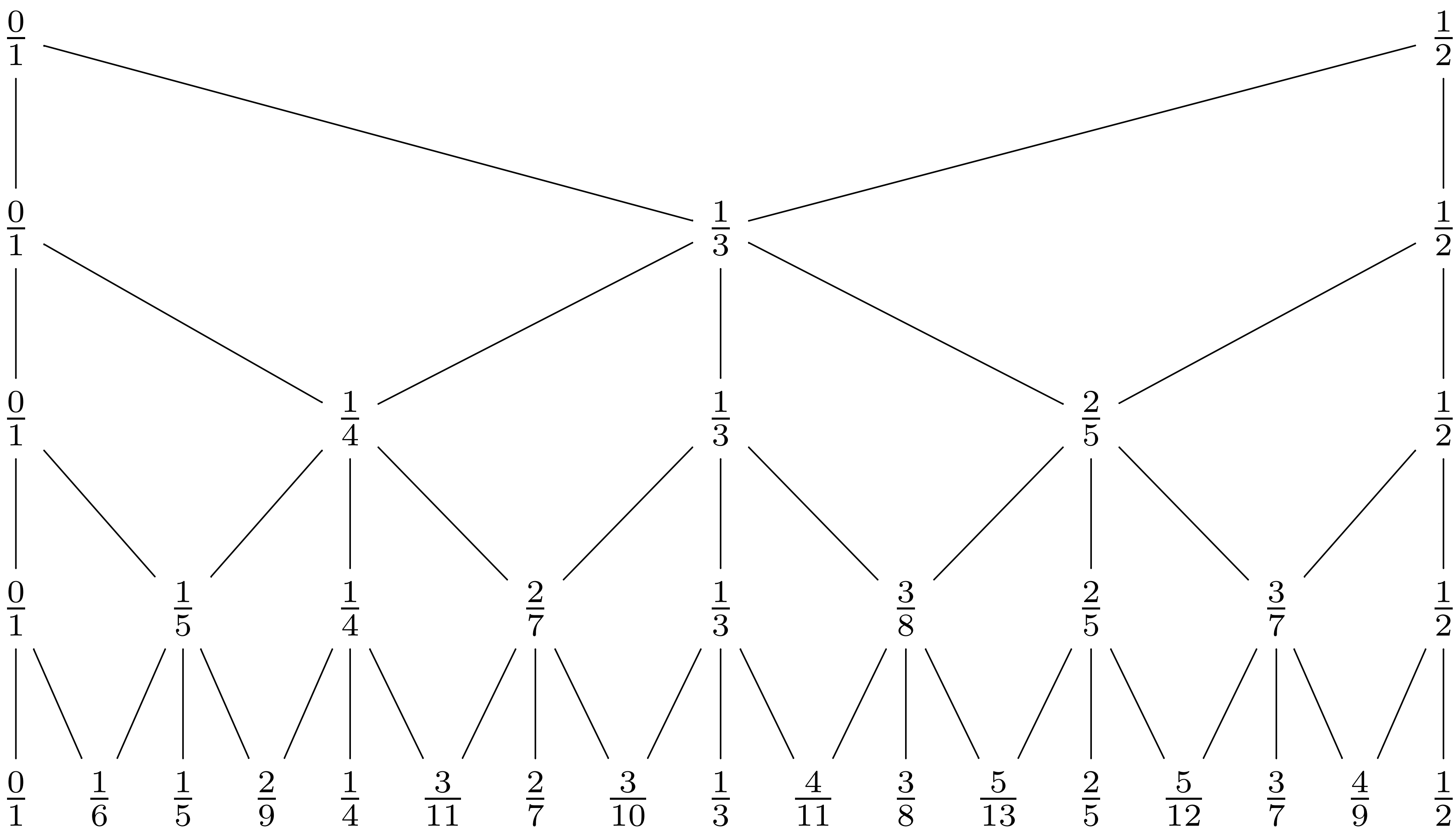
Half of Farey diagram



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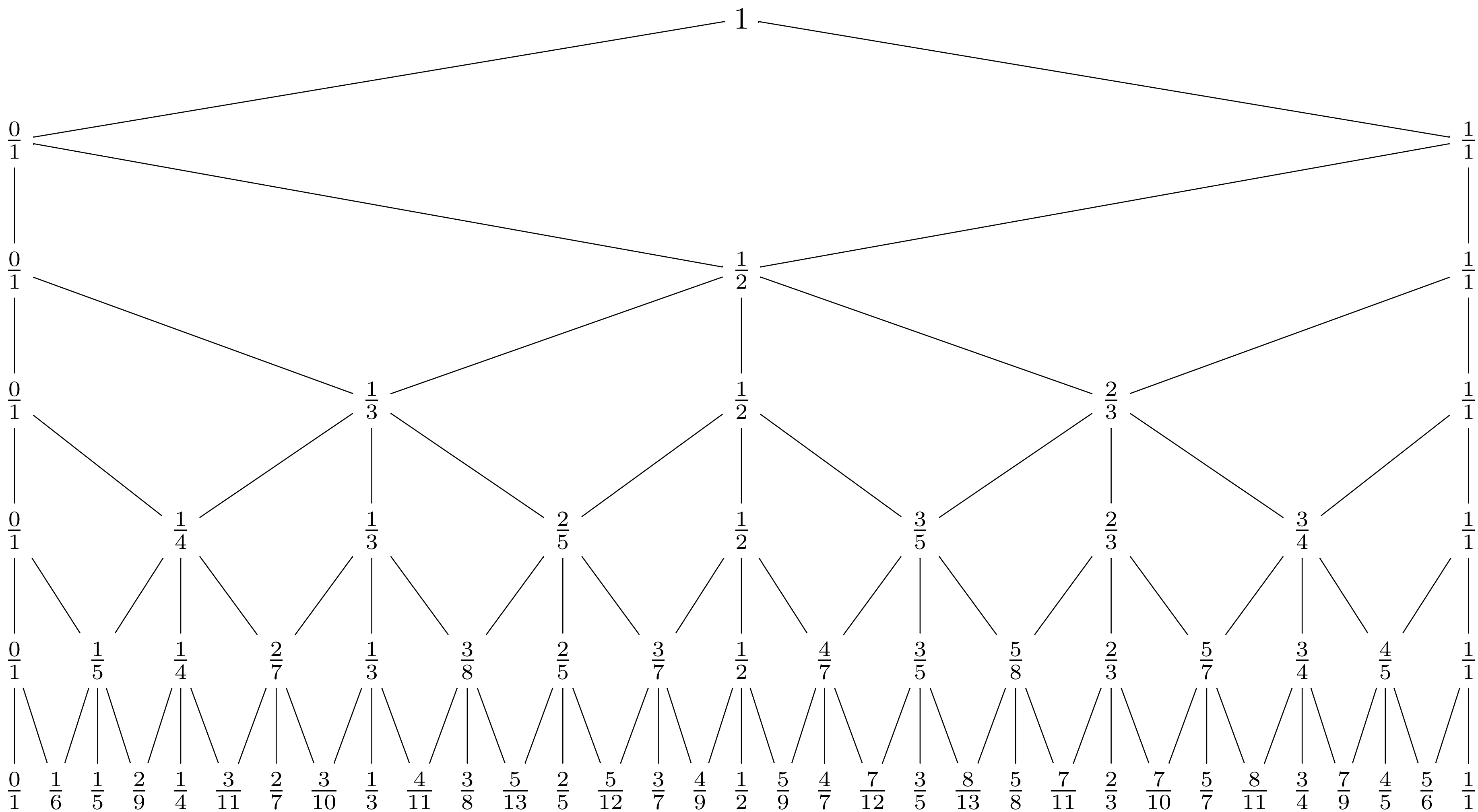
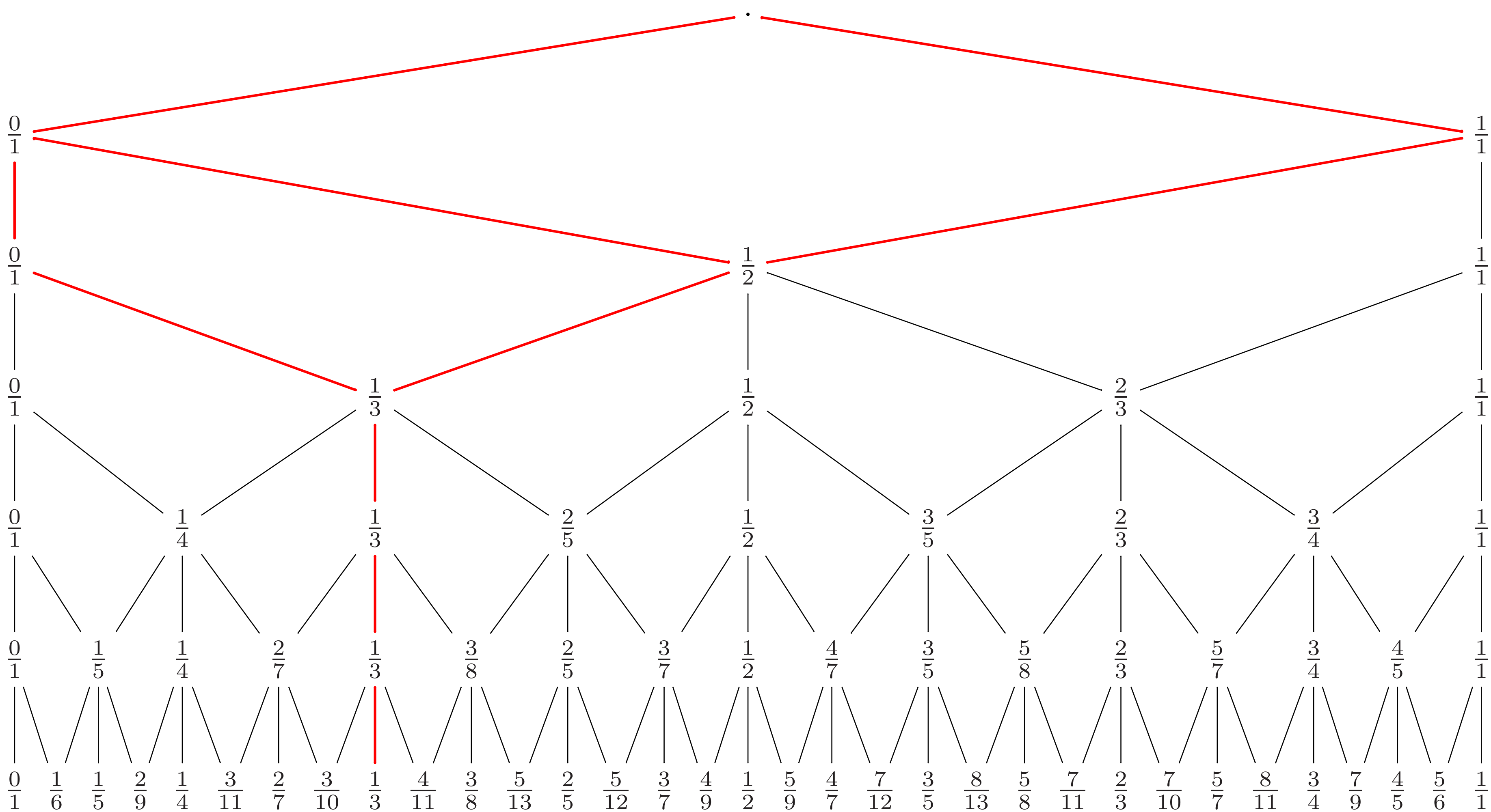
These closed invariant subsets correspond to primitive ideals of the approximately finite C^* algebra determined by the Farey Bratteli diagram.

Farey diagram again



The orbit of $1/3 \sim 001001001001\cdots = 1/7$



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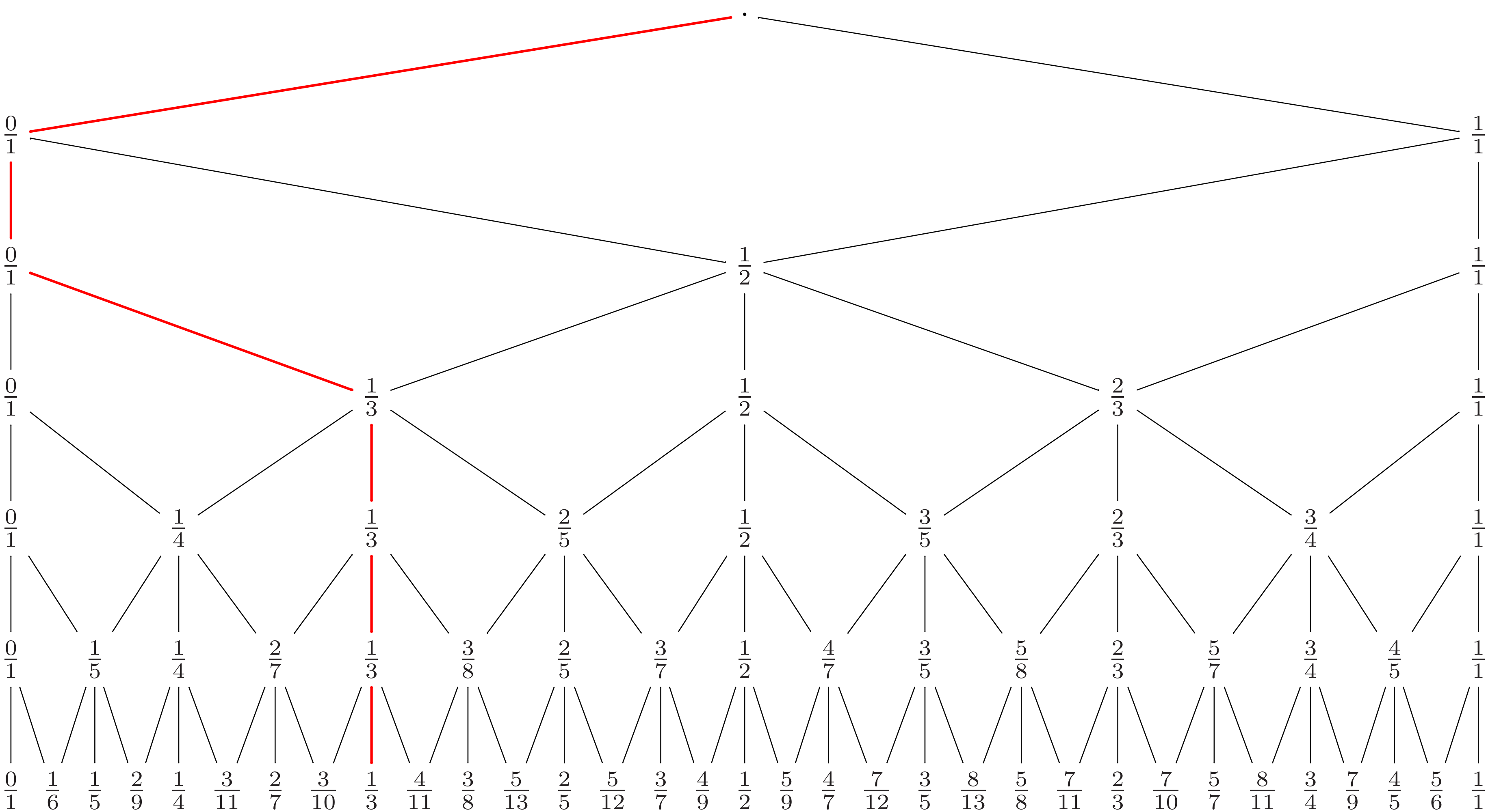

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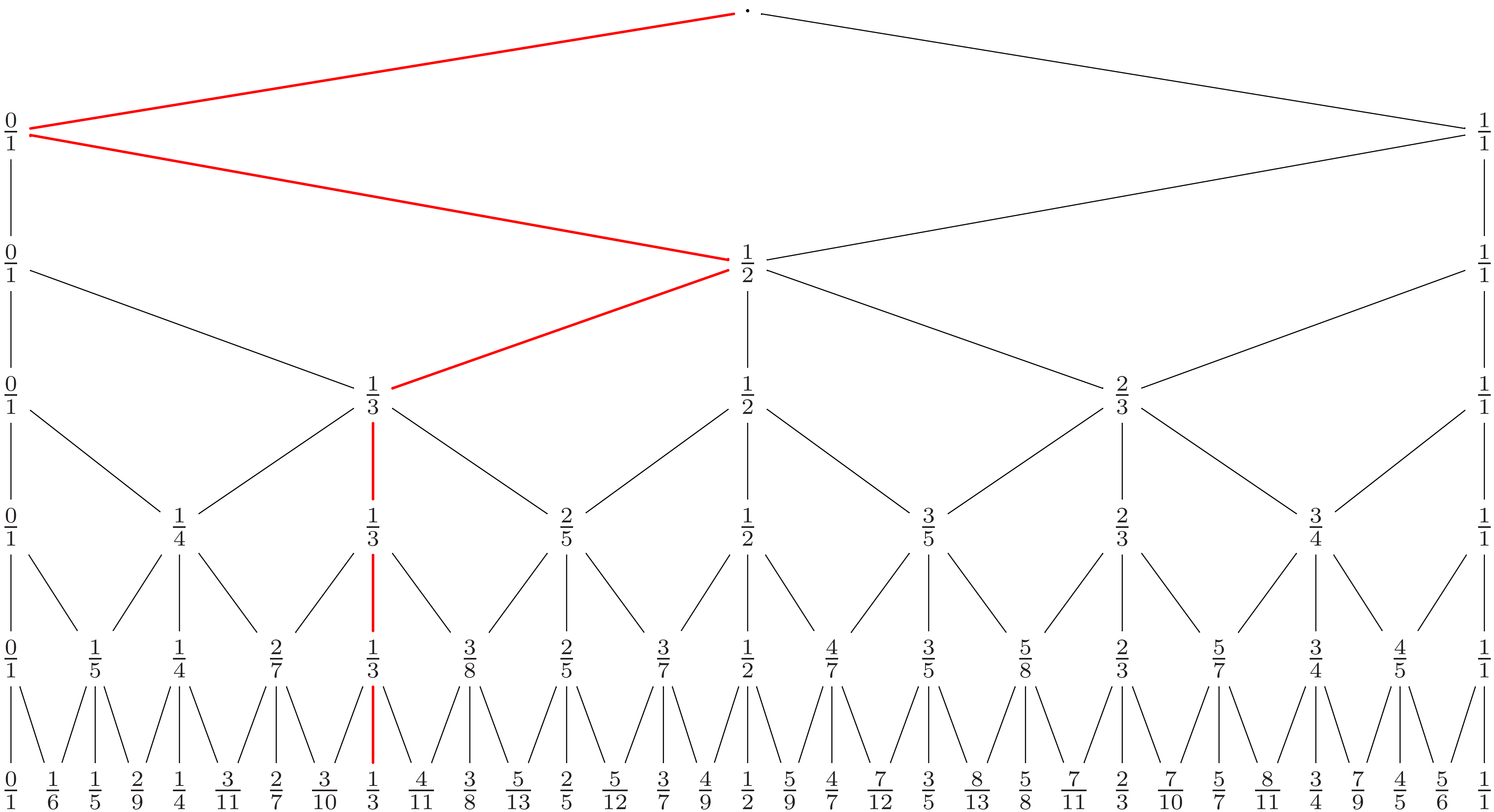

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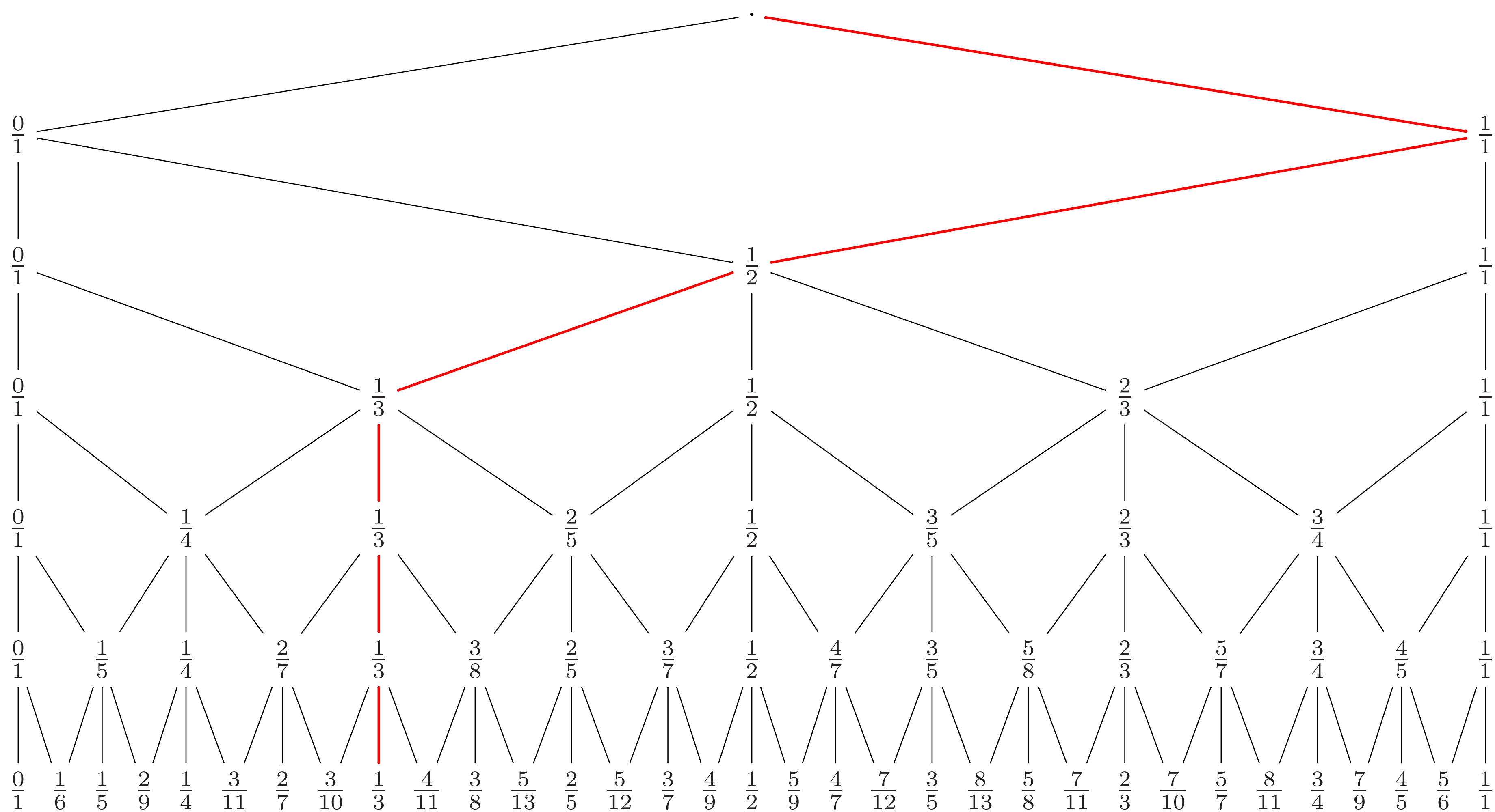

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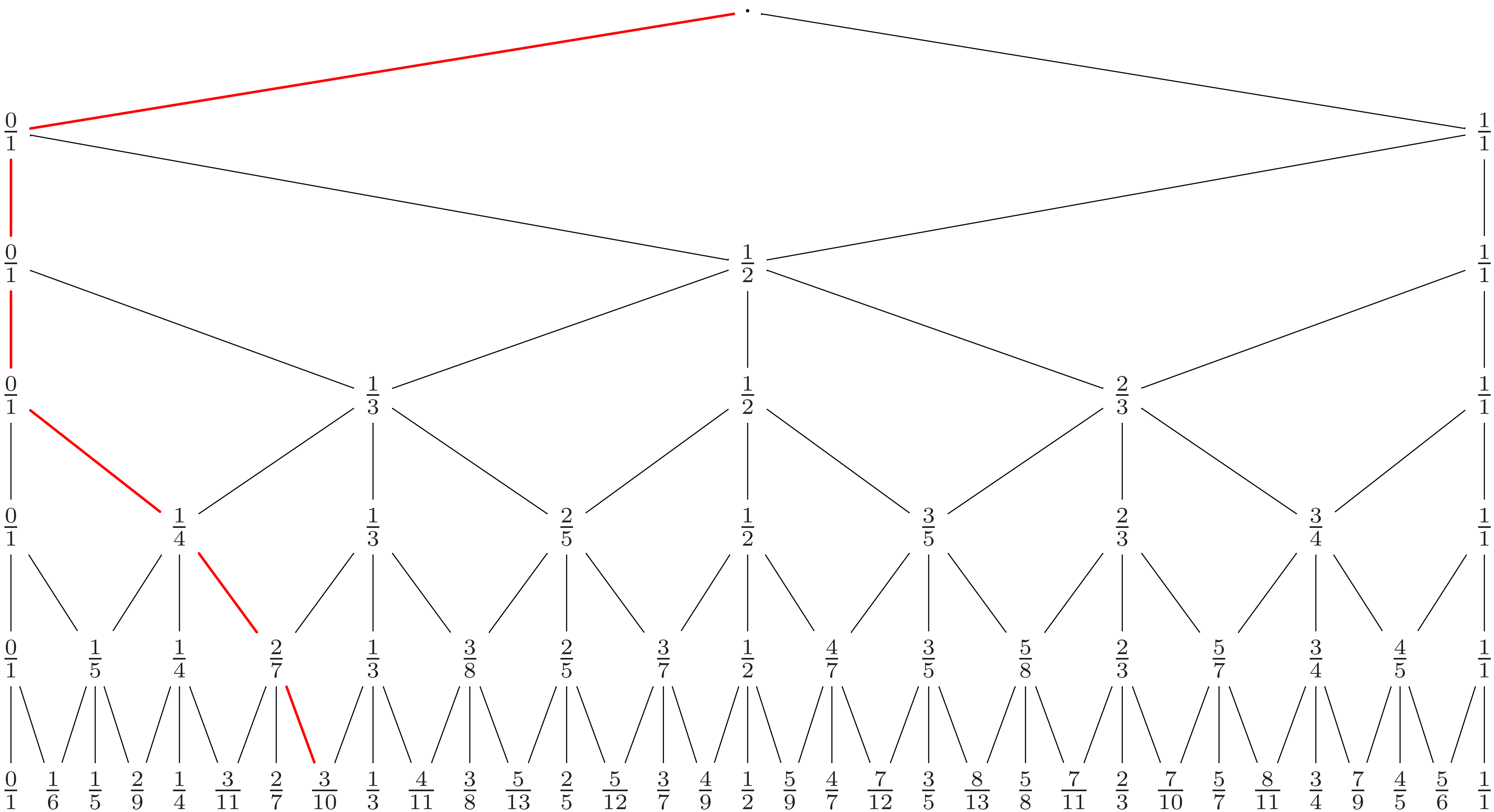

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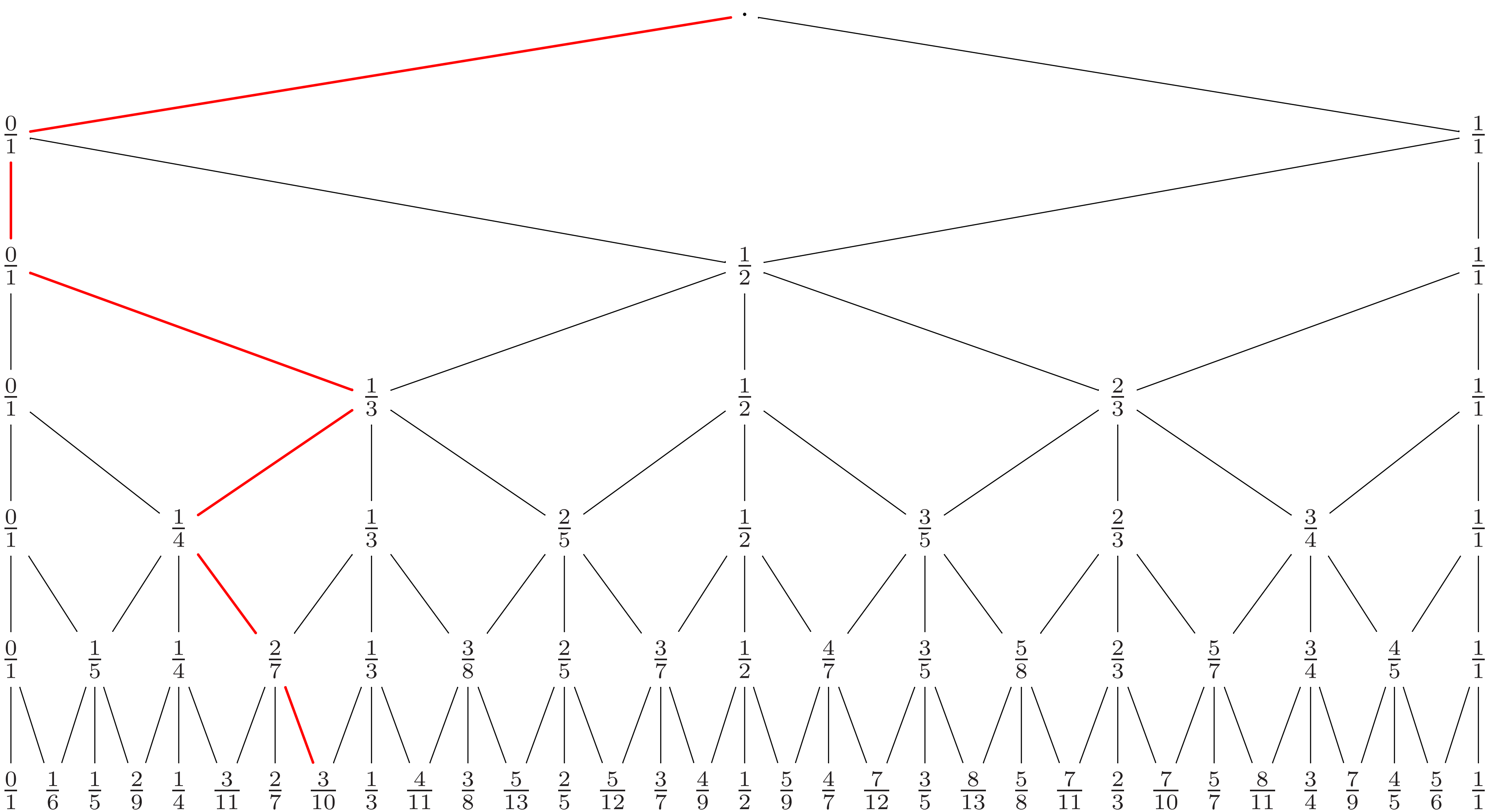

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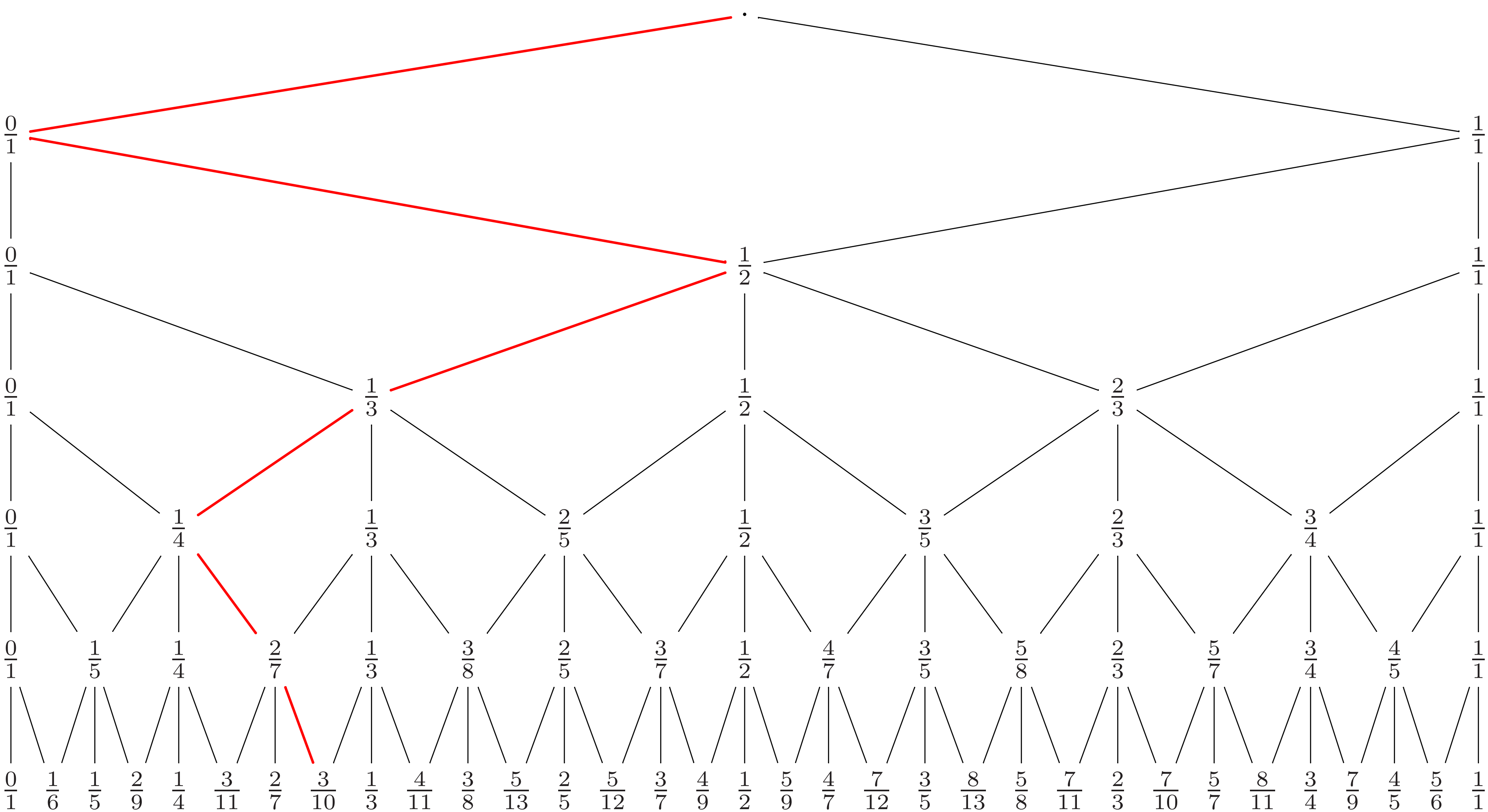

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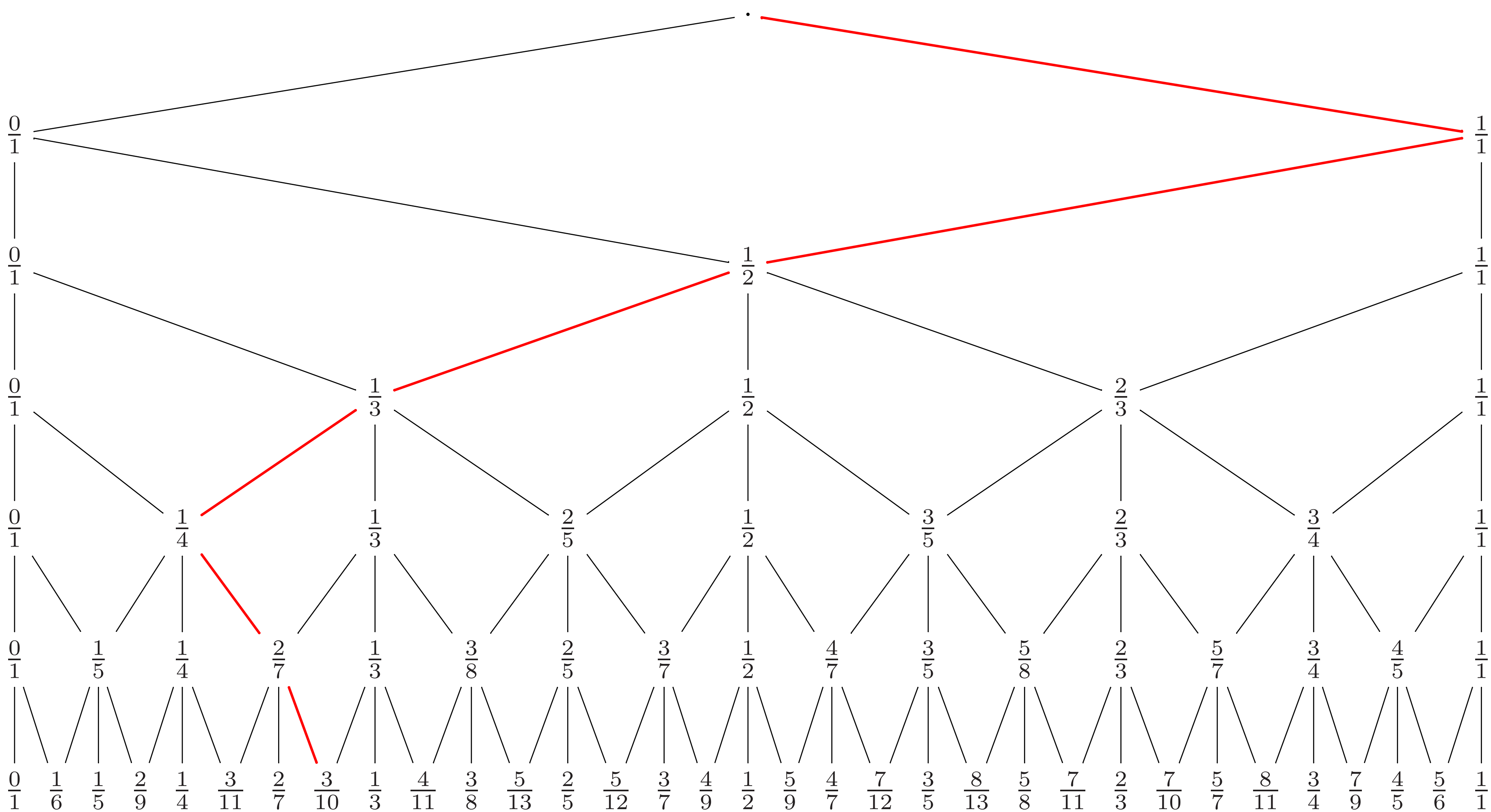

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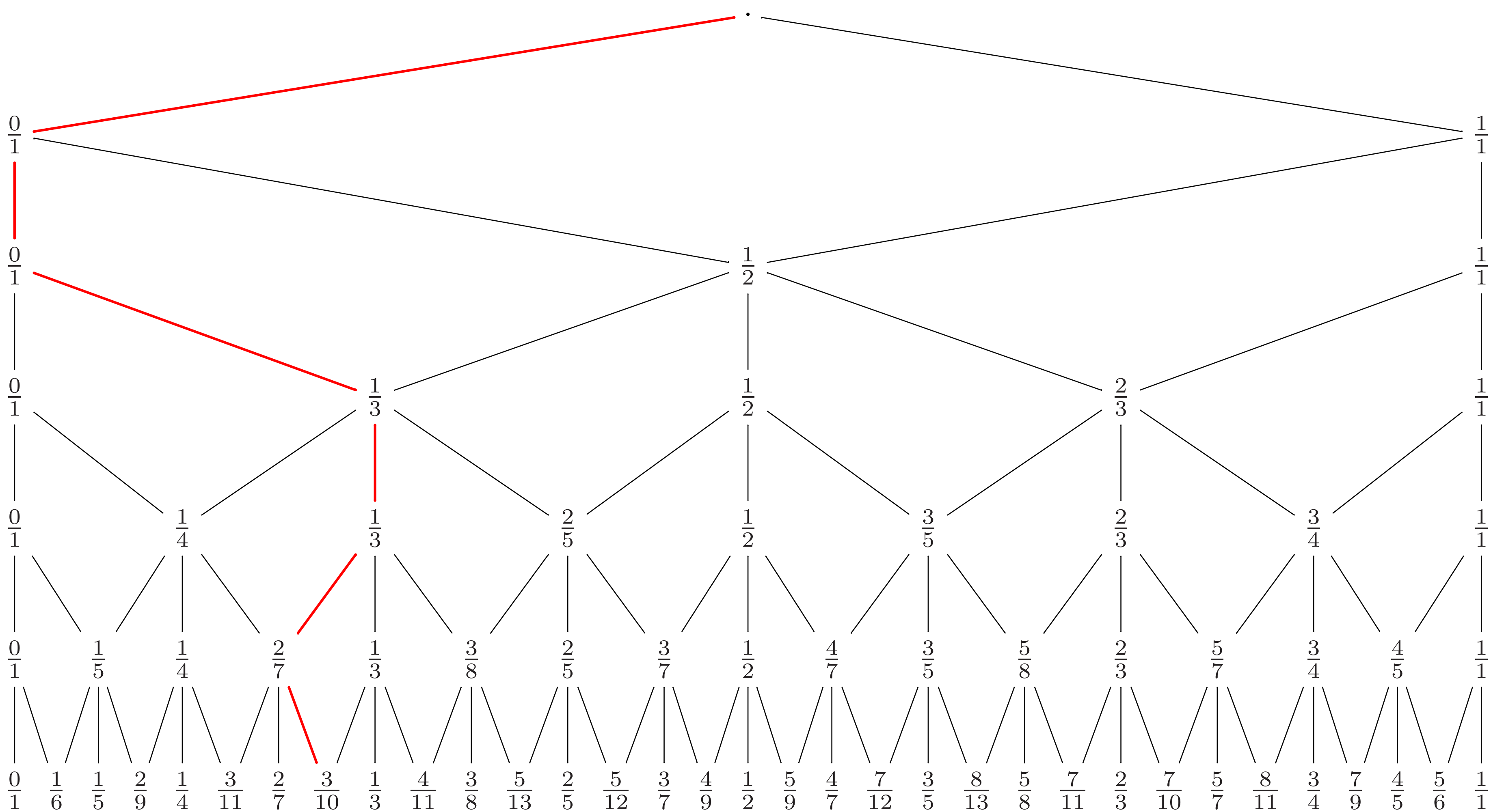

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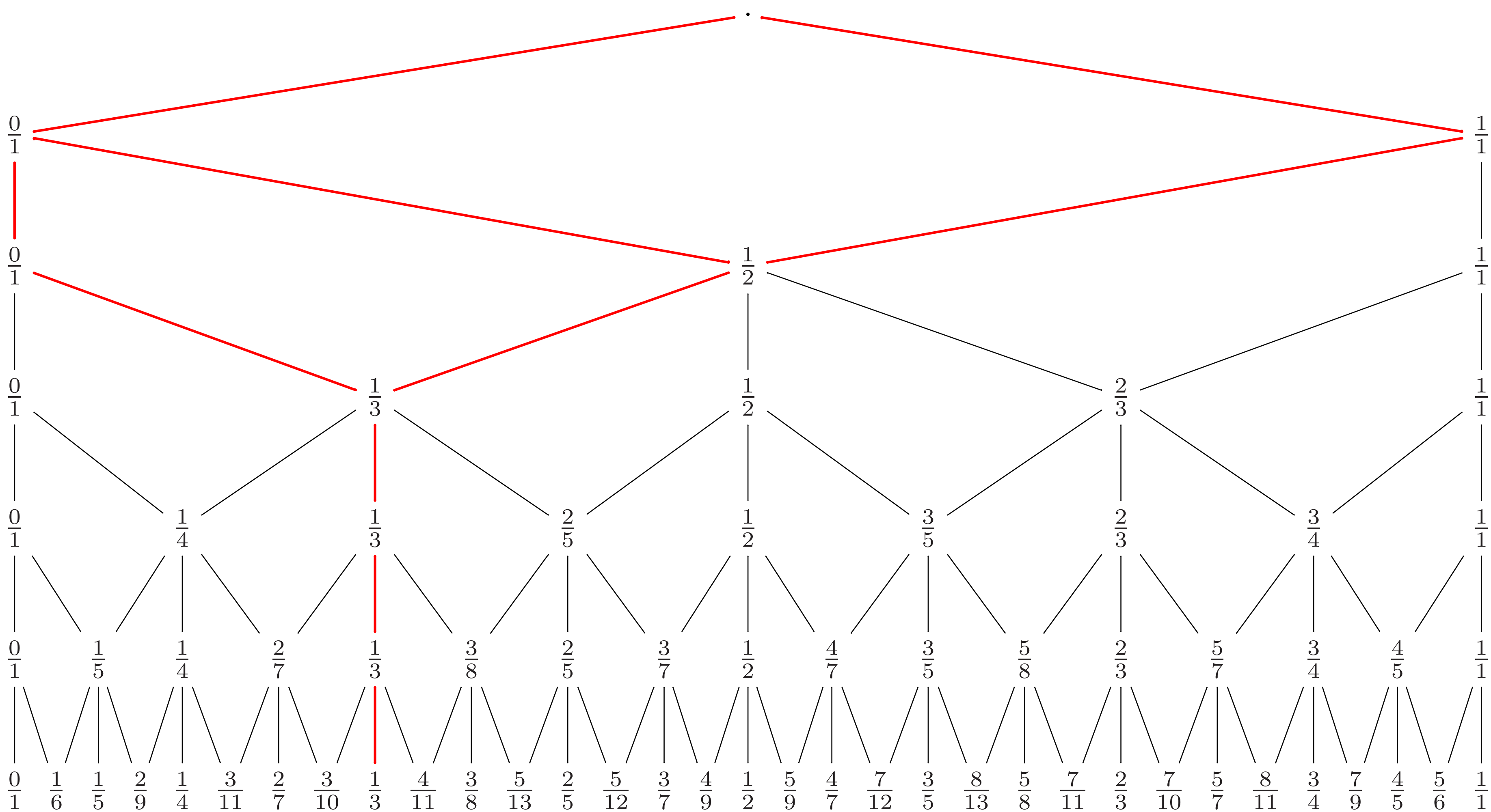

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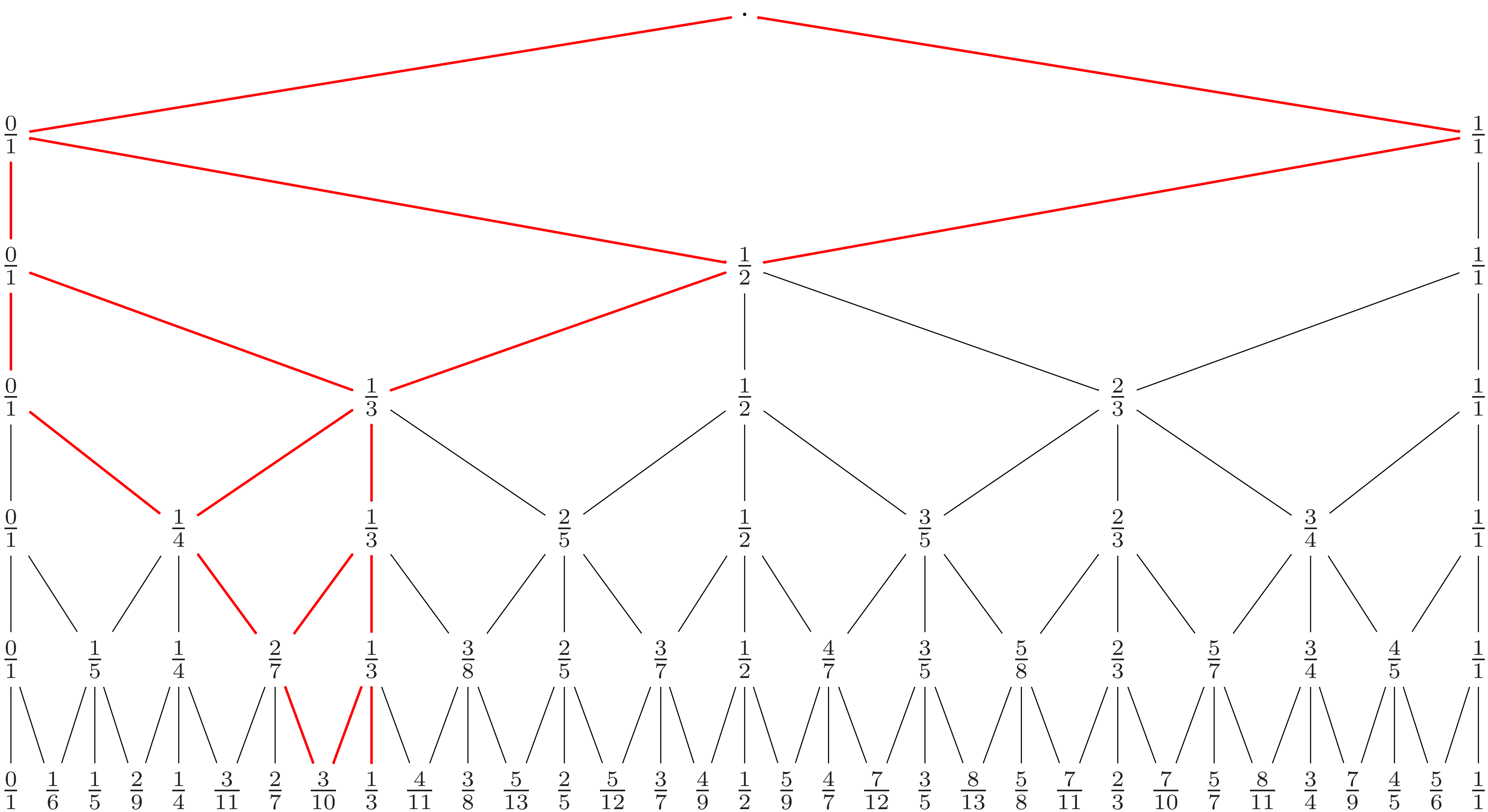
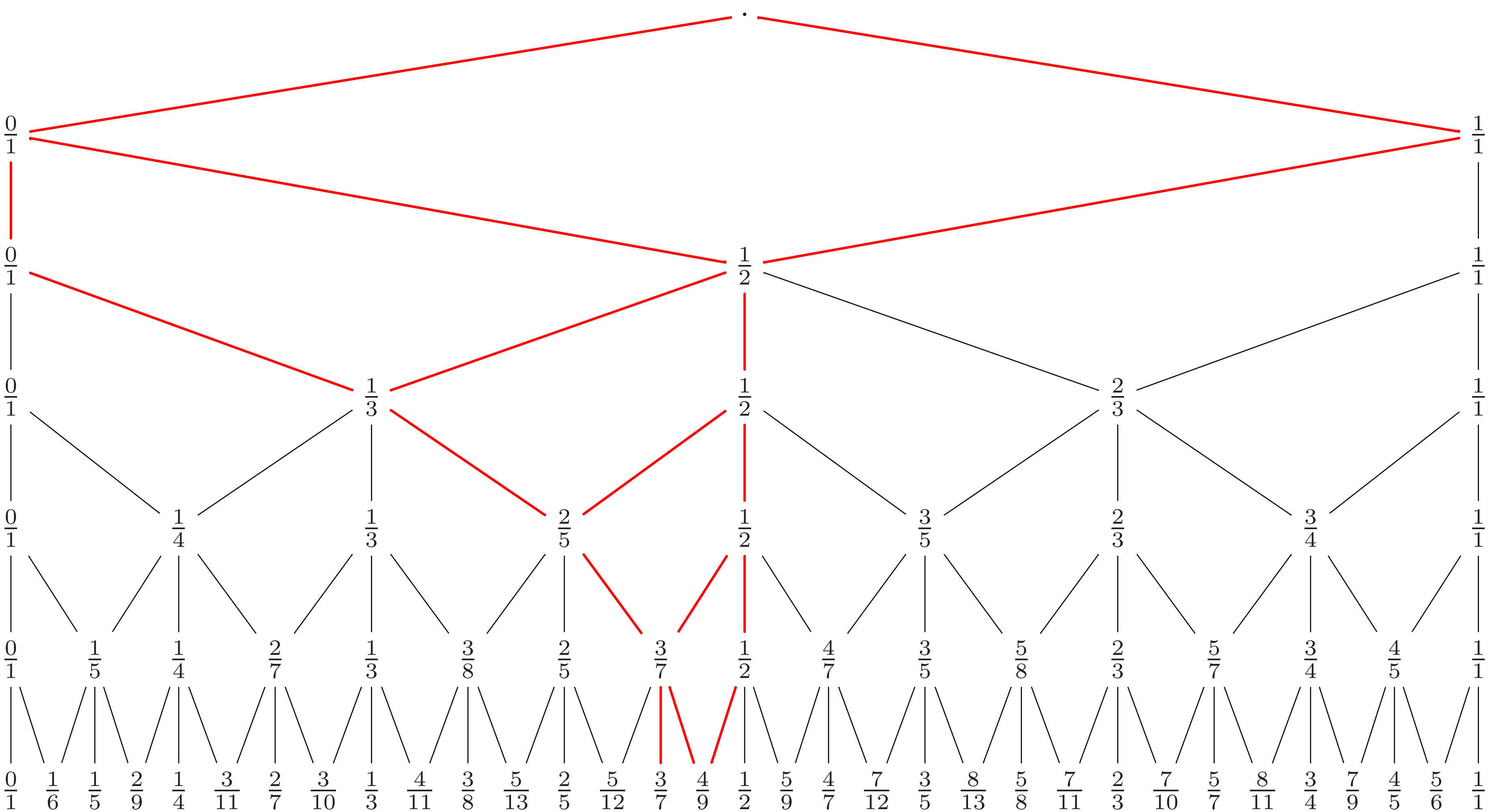


Ideal and orbit closure for $\theta = [2, 3, 2, 4, \dots]$



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- Otherwise there is a first i for which $T_\beta^i 1 = n \in \mathbb{N}$, and then we put $e_\beta(1) = [a_1 \dots a_{i-1}(n-1)]^\infty$.

β-shifts and lexicographic order

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- We define $L : (0, 1] \to (0, 1]$ by $L(\theta) = \beta(\theta) - 1$.

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- I recently found out that in recent papers and preprints, DoYong Kwon has defined and studied essentially the same function.