Adic Systems and Symbolic Dynamics

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- Thanks to Xavier and Sarah for many of the pictures (as well as results).

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- Incidence matrices describe the numbers of edges connecting vertices on levels n and $n+1$.

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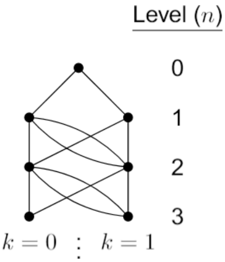

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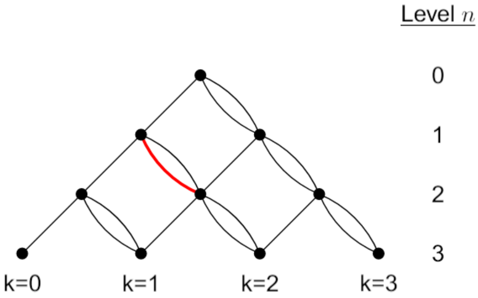
- A cylinder set $C = [c_0c_1c_2\ldots c_n]$ is a clopen set such that $x \in C$ implies $x = c_0c_1\ldots c_nx_{n+1}\ldots$.

Edge Ordering



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- The survey by Durand (in Berthé-Rigo 2010) is highly recommended.

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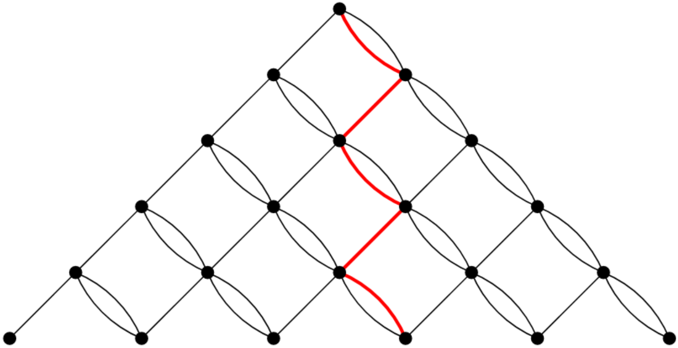

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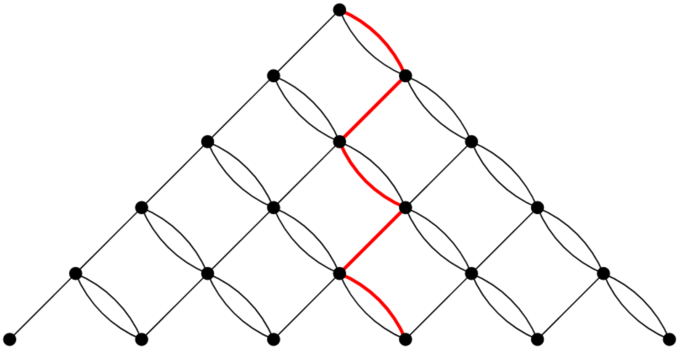

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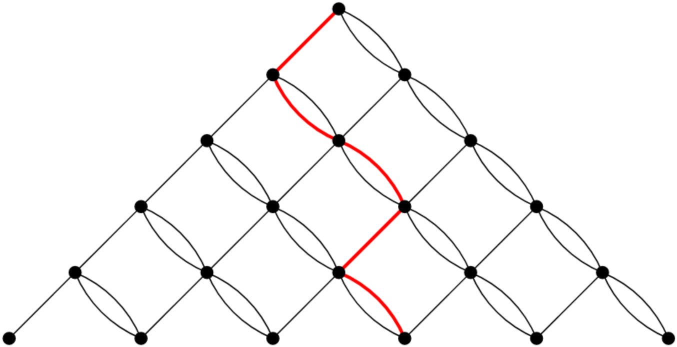

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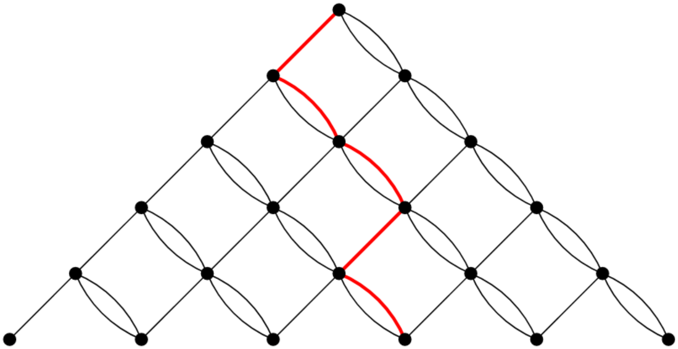

Adics and shifts

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- The maps T and σ are transverse, satisfying $\sigma T = T^2 \sigma$, same as $2(x+1) = 2x + 2$.
- This is analogous to $h_{se^{-t}} g_t = g_t h_s$ for the horocycle and geodesic flows.

Invariant measures for shift vs. adic on SFT

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- The unique invariant measure for the adic on a SFT Σ_M assigns equal measure to all cylinder sets determined by paths from the root to a selected vertex.
- The measure of maximal entropy on Σ_M assigns pretty much the same measure to all cylinder sets of a fixed length.

Zeckendorf Representation

Consider the Fibonacci sequence

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Every $x \in \mathbb{N}$ has a *unique* representation $x = \sum_{i=0}^{k} x_i f_i$ with no $x_i x_{i+1} = 11$.

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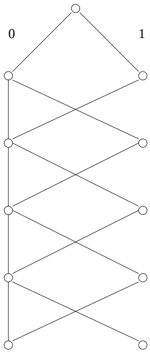
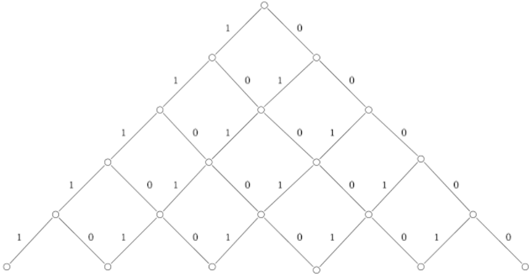


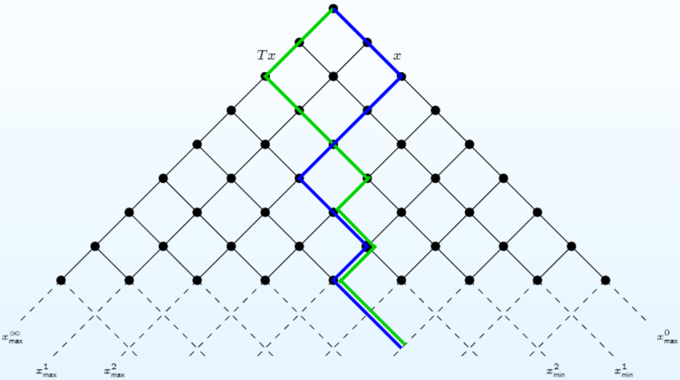
Pascal adic



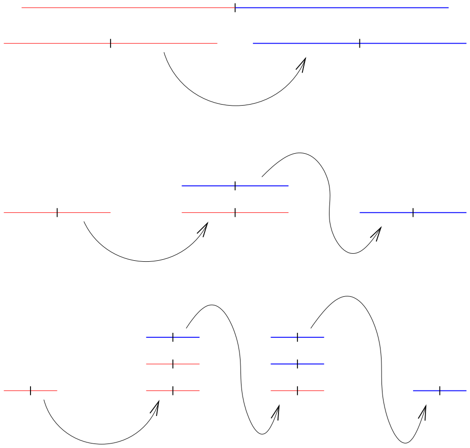
space of infinite downward paths $X \cong \{0,1\}^{\mathbb{N}}$

$$T(0^p 1^q 10*) = 1^q 0^p 01*, \quad p, q \geq 0$$

Action of the Pascal adic



Pascal by cutting and stacking



Pascal as a subshift

$$b \quad a$$

$$b \quad ba \quad a$$

$$b \quad b^2a \quad ba^2 \quad a$$

$$b \quad b^3a \quad b^2aba^2 \quad ba^3 \quad a$$

$$b \quad b^4a \quad b^3ab^2aba^2 \quad b^2aba^2ba^3 \quad ba^4 \quad a$$

$$b \quad b^5a \quad b^4ab^3ab^2aba^2 \quad b^3ab^2aba^2b^2aba^2ba^3 \quad b^2aba^2ba^3ba^4 \quad ba^5 \quad a$$

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- Strengthenings of ergodicity (such as weak mixing) would therefore imply new results in probability.

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- The adic representation suggests the use of C^* ideas such as dimension groups (Elliott 1976 and 1993, Effros-Handelman-Shen 1980)

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- A version in symbolic dynamics called the Marker Lemma is used to prove the Krieger Embedding Theorem for SFT's: There is an embedding $X \to Y$ if and only if $h(X) < h(Y)$ and the periodic points of X embed in those of Y—see Lind-Marcus (1995, Lemma 10.1.8, p. 343).

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- They are topologically strongly orbit equivalent if and only if their dimension groups are isomorphic as ordered groups with distinguished order units.
- (X_1, T_1) and (X_2, T_2) are strongly orbit equivalent if there is a homeomorphism $h : X_1 \to X_2$ such that the time change cocycles $a(x)$ and $b(x)$ defined by

$$hT_1^{a(x)}h^{-1}x = T_2x, \quad h^{-1}T_2^{b(x)}h(x) = T_1(x).$$

have at most one point of discontinuity each.

Dimension group

The dimension group G of an adic system is the direct limit of

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- Namely, the quotient of

$$P = \{v \in \prod_{n \geq 0} \mathbb{Z}^{|\mathcal{V}(n)|} : M_n v_n = v_{n+1} \quad \text{for all large enough } n\}$$

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- It is isomorphic to $\mathcal{C}(X,\mathbb{Z})/\partial\mathcal{C}(X,\mathbb{Z})$, where $\partial\mathcal{C}(X,\mathbb{Z}) = \{f - fT : f \in \mathcal{C}(X,\mathbb{Z})\}$.

Measures and infinitesimals

- T-invariant measures on X correspond to *states* or *traces*—group homomorphisms $\phi : G \to \mathbb{R}$ such that $\phi(G^+) \subset [0, \infty]$ and $\phi(u) = 1$ by $\phi_\mu[f] = \int_X f d\mu \quad (f \in \mathcal{C}(X, \mathbb{Z})$.

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- The *infinitesimals* in G are

$$\begin{aligned} \text{Inf}(G) &= \{g \in G : ng < u \text{ for all } n \in \mathbb{Z}\} \\ &= \{g \in G : \phi(g) = 0 \text{ for all traces } \phi\} \\ &= \{[f] : \quad \int_X f d\mu = 0 \text{ for every invariant measure } \mu\}. \end{aligned}$$

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- The reduced dimension group is $G/\text{Inf}(G)$.

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- a Cantor minimal system (X, T_2) which is topologically strongly orbit equivalent to (X, T_1) and has an invariant measure μ_2 that makes it measure-theoretically isomorphic to the given (Y, S, ν).
- Ormes' Strong Orbit Realization Theorem says that this is possible exactly when the continuous rational point spectrum of T_1 is contained in the point spectrum of (Y, S, ν).

- Moreover, given also an ergodic T_1-invariant measure μ_1, one can arrange that the o.e. mapping h between T_1 and T_2 is the identity, and $\mu_2 = \mu_1$.

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- For topological orbit equivalence, the condition about embedding of rational point spectrum is not needed: Ormes' Orbit Realization Theorem says that, given (X, T_1) and (Y, S, ν) as above, and a T_1-invariant measure μ_2, there is a Cantor minimal (X, T_2) that is topologically orbit equivalent to (X, T_1) and such that (X, T_2, μ_2) is measure-theoretically isomorphic to (Y, S, ν).

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- Strong orbit equivalence is achieved under conditions as before on rational point spectrum.

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- They also extended the Downarowicz-Maass result to aperiodic Cantor systems of finite rank, proving that either they are expansive or else all of their minimal components are topologically conjugate to odometers.

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- Gjerde and Johanssen showed that a minimal subshift is a Toeplitz system if and only if it is topologically conjugate to an expansive adic system that has the *equal path number property*: for all $n \geq 1$, each vertex in $\mathcal{V}(n)$ has the same number of entering edges from $\mathcal{V}(n-1)$. (But the EPN property does not imply expansive nor equicontinuous.)

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- Yassawi and Janssen (2014) produce a class of infinite rank diagrams similar to those of Toeplitz systems for which $J = \infty$.

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- Frank-Sadun (in progress) define "fusion tiling systems", which can be viewed as generalized higher-dimensional adic actions—analogous to the translation action of $\mathbb{R}$ on the space of tilings of the line generated by a substitution such as $0 \to 01, 1 \to 0$.

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- "Nearly continuous" or "almost continuous" means that the map is a homeomorphism once restricted to invariant G_δ sets of full measure (in a Polish space).

Some current work on arbitrary orderings

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- We think that now we can prove that the Pascal with any ordering is essentially expansive in this sense.
- There are orderings of the Pascal graph with uncountably many maximal and minimal paths, although for each ordering and each invariant probability the set of maximal and minimal paths has measure 0.

- We are also trying to estimate the asymptotic complexity for the coding of each ordering, as well as the expected asymptotic complexity—more about this later.

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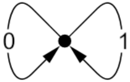
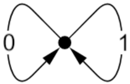
- We are also starting to study the “large subshift”: the closure of the union of the subshifts from codings of all adics coming from orderings of the Pascal graph.

Stationary adics



- We describe first the stationary adic system (on an infinite downward directed graph) that arises from a finite directed graph.
- Vertices divided into levels, with the number on each level equal to the total number in the finite graph
- Each vertex on level n corresponds to a vertex in the finite graph. Connect the edges according to the allowed transitions in the SFT.

The full shift on $\{0, 1\}$ generates the binary odometer



$\gamma = .01011\ldots$

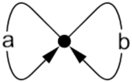
$\sigma(\gamma) = .1011\ldots \qquad S(\gamma) = .11011\ldots$

$\sigma^2(\gamma) = .011\ldots \qquad S^2(\gamma) = .00111\ldots$

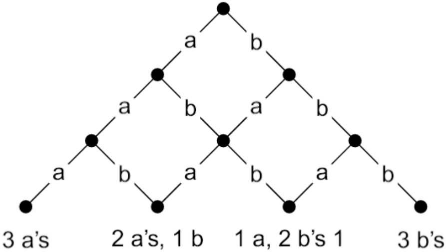
This stationary adic is the dyadic odometer.

The adic transformation is transverse to the shift.
The translation action in a self-similar tiling system is like the adic, while the shift is like a change of scale (action of a substitution).
This is similar to transverse actions of horocycle and geodesic flows.

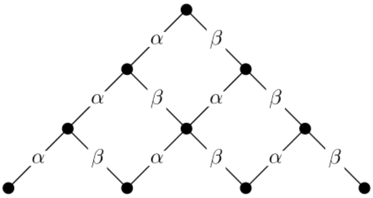
Symbol count adics



- Keeps track of symbol counts
- Regardless of path to vertex, same symbol counts vector
- Each path in the symbol count adic gives the history from time 0 of the random walk on the labeled edge graph



The symbol count adic for the full shift is the Pascal adic



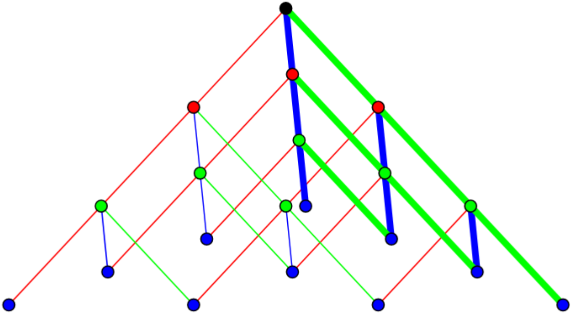
These are the adic invariant, fully-supported ergodic probability measures on the Pascal adic (Hewitt-Savage, de Finetti).
Cylinders are given measures by multiplying the weights on their edges.
The diagram defines the “CCR" C^* algebra, found already in Bratteli's 1972 paper.

Higher-dimensional Pascal

- We can think of walks in higher dimensions. $p(x, y, z) = x + y + z$
- The number of paths from $(0, 0, 0)$ to (a, b, c) is the coefficient of $x^a y^b z^c$ in $(p(x, y, z))^{a+b+c}$
- the three-dimensional Pascal has three "normal" Pascal adics as invariant sets.
- The ergodic invariant measures are given by weights α, β, γ on the edges.

Higher dimensional Pascal

- We can think of polynomials in more variables, $p(x, y, z) = x + y + z$
- The number for paths from $(0, 0, 0)$ to (a, b, c) is the coefficient of $x^a y^b z^c$ in $(p(x, y, z))^{a+b+c}$
- Has three "normal" Pascal adics as invariant sets



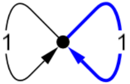
Reinforced random walk (or urn model)

- Reinforcement scheme on a finite directed graph: For each edge e we have $v_e \in \mathbb{Z}_+^2$ that tells what to add to the weights on the edges.
- Start with initial vector $v_i = (s, s)$, corresponding to equal probability of each edge
- As edge e is traversed, add v_e to the accumulated sum of the v_i and normalize to obtain the probabilities of taking each edge. This defines the *walk measure*.

Reinforced random walk

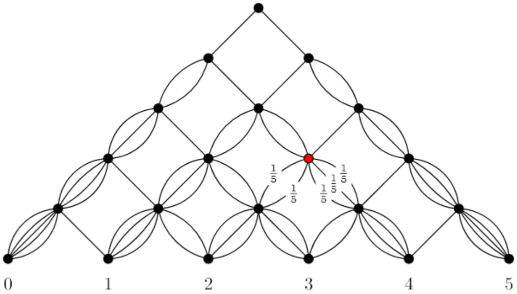
- For each edge e let $v_e \in \mathbb{Z}_+^2$
- Start with initial vector $v_i = (s, s)$, corresponding to equal probability of each edge
- As edge e is traversed, add v_e to v_i and normalize to obtain the probabilities of taking each edge.

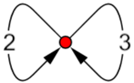
$v_L = (0, 1)$



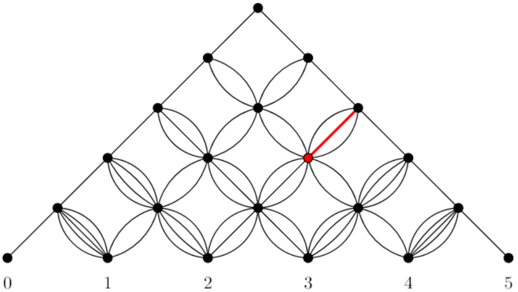
$v_R = (1, 0)$

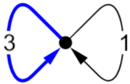
Positively reinforced random walk on two loops: the reverse Euler adic





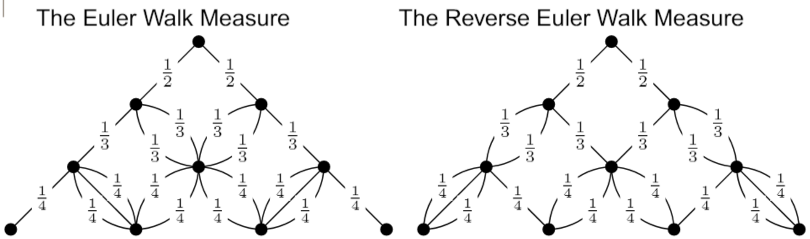
Negatively reinforced random walk on two loops: the Euler adic





$v_0 = (0, 1)$
and
$v_1 = (1, 0)$

The Walk Measures (not necessarily invariant or ergodic)



- Adic invariant
- Gives each edge connecting level n to level $n+1$ weight $\dfrac{1}{n+2}$.
- Gives each cylinder of length n measure $\dfrac{1}{(n+1)!}$

Counting paths: $\dim(C, x_n)$ and $\dim(x_n)$

- Let $\dim(x_n)$ be the number of finite paths from the root vertex to vertex through which the path x passes at level n.
- For any cylinder C, let $\dim(C, x_n)$ be the number of paths in C that agree with x_n after level n.

- **Theorem (Vershik)**

  If (X, T) is a Bratteli-Vershik system and μ is an ergodic, T-invariant measure on X, then for any cylinder $C \subset X$ and μ-a.e. x,

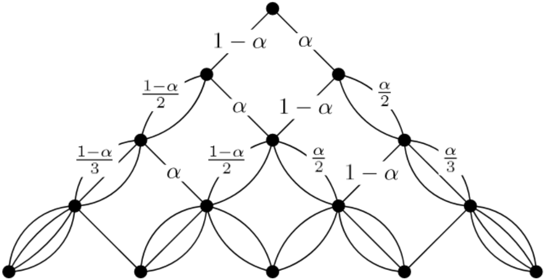
$$\lim_{n\to\infty} \frac{\dim(C, x_n)}{\dim(x_n)} = \mu(C).$$

- Proved by Ergodic Theorem or Reverse Martingale Theorem.
- Also can use generalized Perron-Frobenius Theorem, as with substitutions.

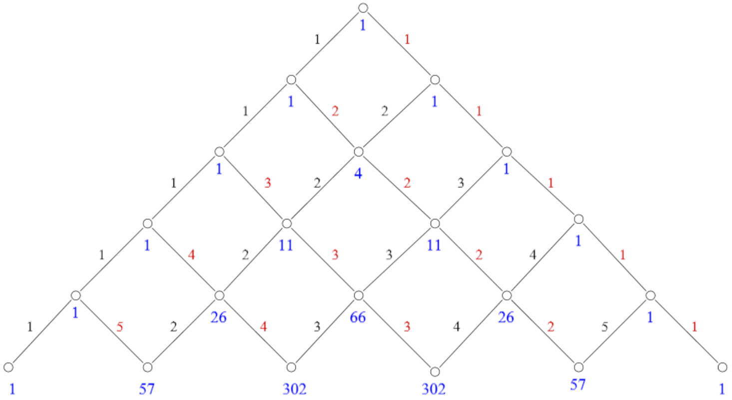
Ergodic Measures on the Reverse Euler

- Let μ be an ergodic measure on X_R and C a cylinder in X_R, for μ-a.e. $x \in X_R$,

$$\mu[C] = \lim_{n\to\infty} \frac{\dim(C,(n,k))}{n!} = \frac{(\alpha)^{k_0}(1-\alpha)^{n_0-k_0}}{k_0!(n_0-k_0)!}.$$



Dimensions of vertices in the Euler graph: Eulerian numbers



Ergodicity of the walk measure on the Eulerian adic

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- A second proof of unique fully supported ergodicity was found by KP and A. Varchenko with hopes to extend it to more dimensions.
- This approach, via a formula for generalized Eulerian numbers, also identifies the generic paths for η and yields convergence of the dimension quotients in *sectors* rather than along a.e. path.

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The symmetric measure η is ergodic.

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By the Ergodic Theorem, or Reverse Martingale Theorem (Vershik), for each cylinder set C ending in a fixed vertex λ,

$$\frac{\dim(\lambda, t_j)}{\dim(t_j)} \to E(\chi_C | \mathcal{I}) \text{ a.e.}$$

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For the Eulerian numbers, it's much harder.
Instead we adapted Mike Keane's approach to prove ergodicity of the Bernoulli $1/2, 1/2$ measure for the Pascal adic.
(Previous proofs for the Pascal were given by Hajian-Ito-Kakutani (1972), and Vershik.)

Collision property

Proposition

For $\eta \times \eta$-almost every $(x, y) \in X \times X$, there are infinitely many n such that the cylinders $I_n(x)$ and $I_n(y)$ end in the same vertex of the Euler graph.

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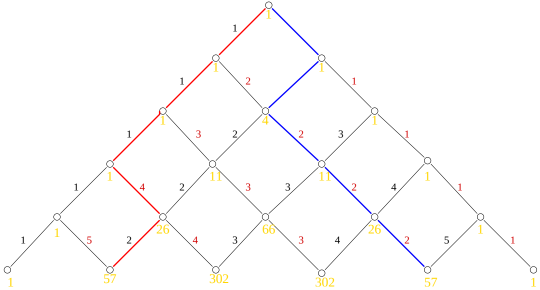

Lemmas for the Proposition

Lemma

On $(X \times X, \eta \times \eta)$ *let* $D_n(x, x') = |k_n(x) - k_n(x')|$. *Let* $\mathcal{F}_n = \mathcal{B}((x_1, x_1'), \ldots, (x_n, x_n'))$ *denote the* σ*-algebra generated by* $(x_1, x_1'), (x_2, x_2'), ..., (x_n, x_n')$. *Then* (D_n) *is a supermartingale with respect to* $(\mathcal{F}_n)$.

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The proof is by direct computation, using the weights on the edges that determine the measure η.

This lemma expresses the central tendency of the infinite paths in the Euler graph: paths close to the edge tend toward the center with greater probability the closer they are to the edge.

Convergence of probabilities

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Proof: Direct computation of the variance, Chebyshev's Inequality.

Proof of the Theorem

Suppose that $A \subseteq X$ is measurable and T-invariant and that $0 < \eta(A)\eta(A^c) < 1$.

Pick an $n_0 = n_0(x, y)$ such that for all $n \geq n_0$, and $\eta \times \eta$-a.e. $(x, y) \in A \times A^c$,

$$\frac{\eta(A \cap I_n(x))}{\eta(I_n(x))} > \frac{1}{2} \text{ and } \frac{\eta(A^c \cap I_n(y))}{\eta(I_n(y))} > \frac{1}{2}. \tag{2}$$

Then, by Proposition 2, we can choose $n \geq n_0$ such that $I_n(x)$ and $I_n(y)$ end in the same vertex and hence there is $j \in \mathbb{Z}$ such that $T^j(I_n(x)) = I_n(y)$.
Since A is T-invariant, this contradicts (2)—most of $I_n(x)$ is made up of A, while most of $I_n(y)$ is made up of A^c.

Then we must have $\eta(A) = 0$ or $\eta(A) = 1$.

Uniqueness

Theorem (Bailey-KP)

The symmetric measure η is the only fully supported ergodic invariant Borel probability measure for the Euler adic transformation.

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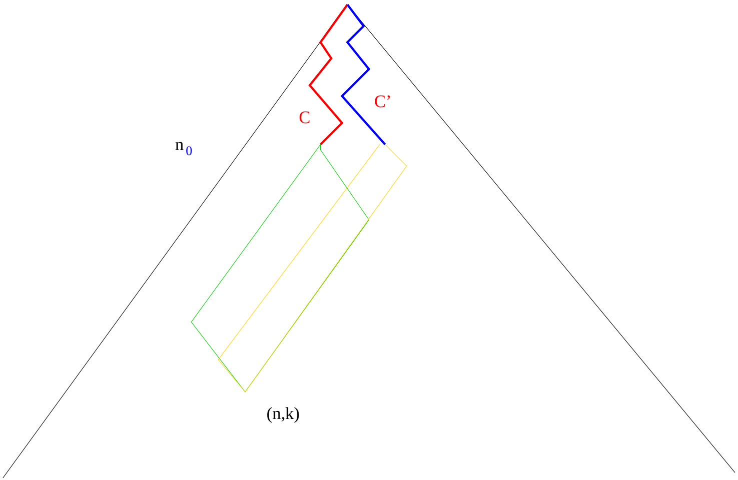
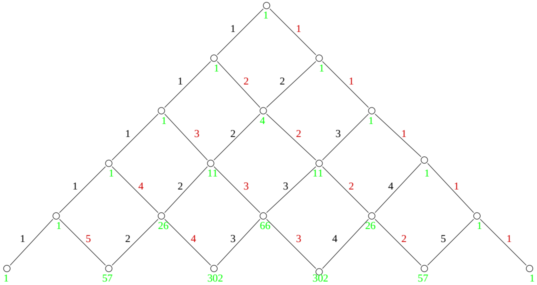
Interpretation: If any two permutations of the same length *with the same number of rises* are equally likely, and every permutation has positive probability, then *all* permutations of a given length are equally likely.

Counting paths



The Euler graph and random permutations



Path leaving (n, k) to the *left* $\sim$ inserting $n + 2$ at a place where it creates a new *fall*.

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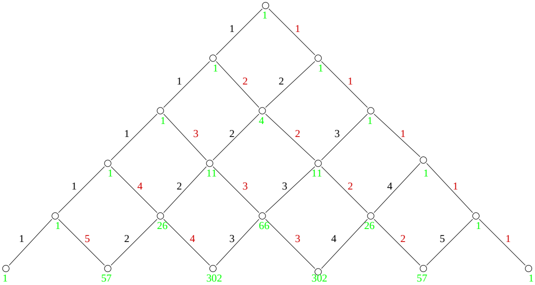

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$$A(n + 1, k) = (n - k + 2)A(n, k - 1) + (k + 1)A(n, k).$$

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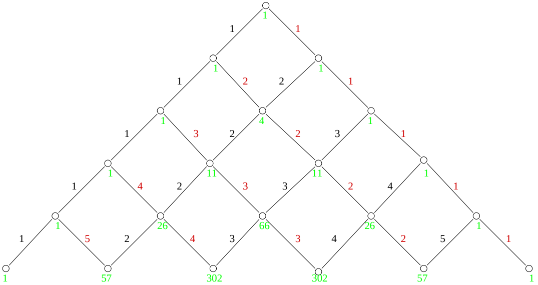

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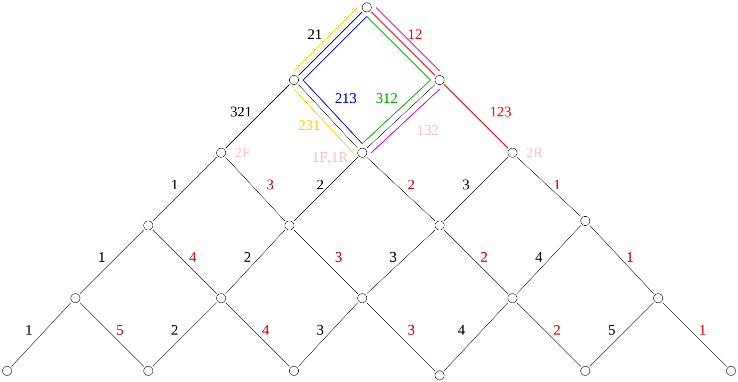
The space X of infinite paths $\sim$ the set of all linear orderings of $\mathbb{N}$.

Cylinders and permutations



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They correspond to permutations $\pi(C_1)$ and $\pi(C_2)$ of $1, 2, \ldots, n_0 + 1$, and to paths of length n_0 down from the root.

Dimensions

$\dim(x_n) =$ the number of paths from the root to the vertex $(n, k_n(x))$
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We know that

$$\frac{\dim(C_1, x_n)}{\dim(x_n)} \to E_\mu(\chi_{C_1}|\mathcal{I})(x) = \mu(C_1) \quad \text{a.e.,}$$

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This does involve asymptotics of the Eulerian numbers $A(n, k)$, but we claim we can get the result without knowing too much.

Permutations

- Each path from the end of C_1 to (n, k) corresponds to a permutation of $1, \dots, n+1$ in which $1, \dots, n_0+1$ appear in the order $\pi(C_1)$.
- We obtain each such permutation by starting with a permutation of $n_0+2, \dots, n+1$ and inserting $1, \dots, n_0+1$ in the order prescribed by $\pi(C_1)$.
- And we are supposed to end up with a permutation of $1, \dots, n+1$ which has exactly k rises and $n-k$ falls.
- If no two elements of $1, \dots, n_0+1$ are placed consecutively, we have n_0+1 choices for where to put them (in a rise, in a fall, at the beginning or end).
- And the *effect* on the number of rises and falls is the *same* for $\pi(C_1)$ as for $\pi(C_2)$—putting any $i \le n_0+1$ into a fall or at the beginning produces a new *rise*, putting it into a rise or at the end produces a new *fall* (and the number of rises does not change).

Asymptotics

- In counting $\dim(C_1, x_n)$ we see Eulerian numbers $A(n-(n_0+1), j)$, with coefficients of various degrees in k and $n-k$.
- For example, if all of $1, \dots, n_0+1$ are to be inserted into rises or at the end, there are $C(k+1, n_0+1) = (k+1)k \cdots (k-n_0+1)/(n_0+1)!$ choices for the set of places, and the number of rises will stay fixed at k.
- Similarly, if we insist that a certain number of $1, \dots, n_0+1$ be placed into separate rises or at the end, and the rest into separate falls or at the beginning, we again find a product of n_0+1 factors on the order of k or $n-k$.
- But if we allow some of $1, \dots, n_0+1$ to be placed adjacently, we will obtain a lower degree product.
- Thus the coefficients of each $A(n-(n_0+1), j)$ of highest degree (in k and $n-k$) are *the same* for $\pi(C_1)$ and $\pi(C_2)$, and so $\dim(C_1, x_n)/\dim(C_2, x_n) \to 1, \quad$ provided that $k, n-k \to \infty$.

Comparing $\mu(C_1)$ and $\mu(C_2)$

- We will compare $\dfrac{\mu(C_1)}{\mu(C_2)}$ when C_1 and C_2 are of the same length, n_0.
- $\dim(C_1, (n, k))$ is dominated by permutations in which $\pi(C_1)$ is "broken up"
  - $\pi(C_1)$ is "broken up" in 41752638
- This term is the same for cylinders of the same length.
- Hence $\dfrac{\mu(C_1)}{\mu(C_2)} = 1$, and μ must be the symmetric measure. $\diamond$
  - 41752638 and 43751628 have the same number of rises.

Conclusion of the proof that η is ergodic

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This constitutes a proof by the asymptotics of the Eulerian numbers, different from the random-walk and supermartingale one.

A formula generalizing the one for Eulerian numbers.

Theorem (KP-A. Varchenko)

For $p \geq 0$*,* $q \geq 1$*, and* $i, j \geq 0$*, let* $B_{p,q}(j+i, i)$ *denote the number of paths in the Euler graph from the vertex* $(p+q, q)$ *to the vertex* $(p+j+q+i, q+i)$*. Then for all* p, q, i, j *we have*

$$B_{p,q}(j+i,i) \; = \; \sum_{t=0}^{i} (-1)^{i-t} \binom{p+q+t}{t} \binom{p+q+j+i+1}{i-t} (q+t)^{j+i}.$$

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Then we got a much shorter argument, satisfying boundary conditions for a recurrence equation by checking equality of two degree i polynomials in p at $i+1$ points.

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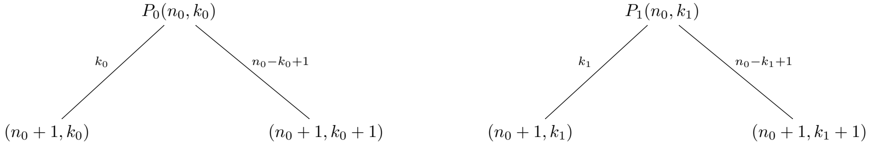

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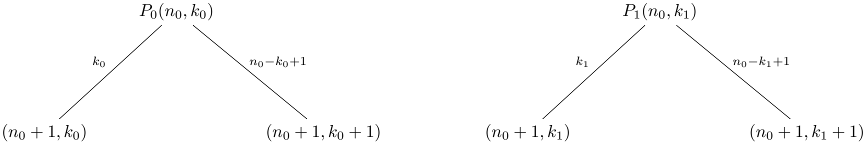

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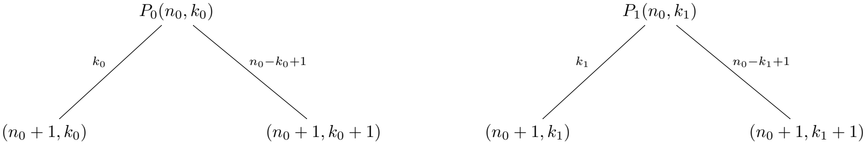

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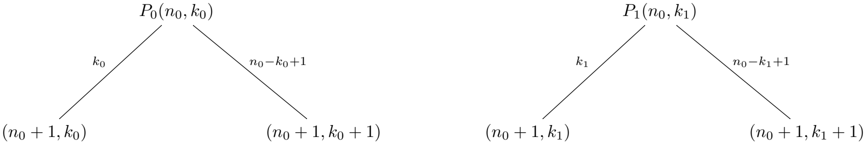

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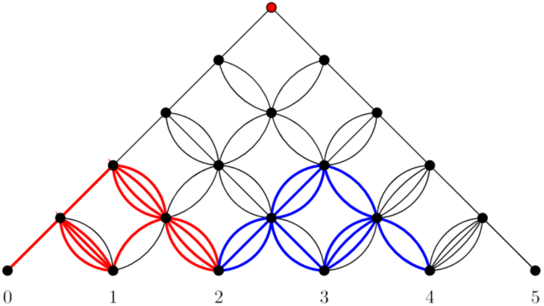
So we set up a *dynamic labeling* of paths below P_0 and P_1 that effectuates a one-to-one correspondence between such *good paths*.

Pairs of paths in the Euler graph.



From each vertex below each of P_0, P_1, $n_0 + 1$ paths are colored and numbered $1, 2, \ldots, n_0 + 1$ from left to right.
The other vertices are numbered as falls or rises.
When a colored edge is used, we note the label, remove the color, relabel it at all vertices below as a fall or rise.
We have to show that *most* paths from P_0 and P_1 correspond by means of their labels.

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But to conclude, we need for example that the ratios *increase along rows*.

Double induction with multiples

Let us consider four adjacent vertices in the graph,

$$y = (p + j + q + i, q + i), \quad Q = (p + j + q + i + 1, q + i + 1),$$
$$x = (p + j + 1 + q + i, q + i), \quad P = (p + j + 1, q + i + 1, q + i + 1).$$

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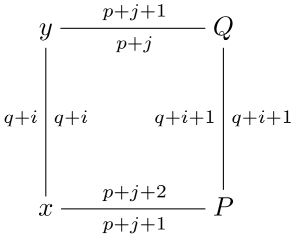

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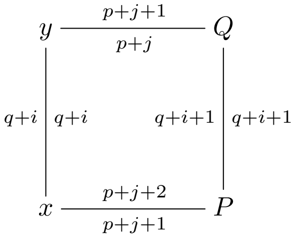

Proposition

For all $i, j \geq 0$, we have

$$\frac{B_Q}{B'_Q} \geq \frac{p+j+1}{p+j}\frac{B_y}{B'_y} \quad \textit{and} \quad \frac{B_x}{B'_x} \leq \frac{B_y}{B'_y}.$$

Asymmetric reinforcement

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- Consider now a random walk on the graph G_{n+1} consisting of $n+1$ loops at a single vertex, with a different kind of opposite reinforcement.
- When the walker takes a step in the direction of e_{n+1}, the numbers of edges in all the other directions (e_j for $j = 1, \dots, n$) are incremented by 1;
- but when the walker takes a step in the direction of e_j for some $j = 1, \dots, n$, *only* the number of edges in the direction of e_{n+1} is incremented by 1.

Asymmetric reinforcement

- Consider now a random walk on the graph G_{n+1} consisting of $n+1$ loops at a single vertex, with a different kind of opposite reinforcement.
- When the walker takes a step in the direction of e_{n+1}, the numbers of edges in all the other directions (e_j for $j = 1, \ldots, n$) are incremented by 1;
- but when the walker takes a step in the direction of e_j for some $j = 1, \ldots, n$, *only* the number of edges in the direction of e_{n+1} is incremented by 1.
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- Studying the asymptotic growth rate of path counts in the resulting adic diagram leads us to an identity involving two special kinds of polynomials.
- This in turn has as a corollary an identity relating Stirling numbers of the first and second kinds:

An identity involving Stirling numbers

For $1 \leq k \leq n$, $0 \leq r \leq k$,

$$\binom{r+n-k-1}{r} s_1(n, r+n-k) =$$

$$\sum_{m=0}^{k} \binom{m+n-k}{m+1} \sum_{i=0}^{r} \binom{i+n-k+m-1}{i} \frac{(-1)^{m+r-i}}{n^{r-i+1}} (r-i+1)! \times$$

$$s_2(m+1, r-i+1) s_1(n, i+n-k+m),$$

Hitting densities (boundaries)

- Ergodic decomposition of the walk measure is related to asymptotic edge traversal frequencies.
- Sometimes this has a *density* on the simplex—e.g., for positive reinforcement (Coppersmith-Diaconis, Keane-Rolles).
- Other reinforcement schemes lead to other interesting examples, such as the *Stirling system* that comes from always reinforcing to the left (Salama).
- More complicated graphs
- Shift-of-finite-type restrictions
- Applications back to random walks?

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- and gave a formula for the density.
- We can interpret this formula in terms of the ergodic decomposition of the walk measure, when it is adic-invariant.

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$$w(n,k) = \{s(a+s)\dots[(n-k-1)a+s] \cdot s(a+s)\dots[(k-1)a+s]\}^{-1}$$
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Use approximate identity (via Euler's beta integral)

$$p_{n,k}(\alpha) = (n+1)C(n,k)\alpha^k(1-\alpha)^{n-k}$$

(peaks at α_0 as $k/n \to \alpha_0$)
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Formula for the density when $v_L = (a, 0), v_R = (0, a)$

$$f_{a,s}(\alpha) = \frac{s}{\sqrt{\pi}} \, e^{a/8 - c_{a,s}} 2^{2s/a-1} \sqrt{\frac{1}{s\alpha(1-\alpha)a}} \, [\alpha(1-\alpha)]^{s/a-1/2}.$$

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- The time-0 partition of Ω is a generator for the m.p. system (Ω, μ, σ).

Tail fields

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- When α is a generator, $\mathcal{T}^+(\alpha)$ is the *Pinsker algebra* of $(X, \mathcal{B}, \mu, T)$.

The K property

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- **Rohlin-Sinai, 1961:** $(X, \mathcal{B}, \mu, T)$ is K if and only if it has *completely positive entropy*, i.e. every nontrivial factor has positive entropy.
- Therefore, for any partition α, $\mathcal{T}^-(\alpha)$ is trivial if and only if $\mathcal{T}^+(\alpha)$ is trivial (because for any $\beta \leq \alpha, h_\mu(T, \beta) = h_\mu(T^{-1}, \beta)$).

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- $\psi_m^n(x) = \psi(T^m x) \cdots \psi(T^n x)$, in abelian case $\sum_{k=m}^{n} \psi(T^k x)$

Fine tail fields 2

- E.g., if $\psi : \Omega \to \mathbb{Z}^d$ is defined by $\psi(\omega) = e_i \in \mathbb{Z}^d$ if $\omega_0 = a_i$, then $\psi_0^{n-1}(\omega)$ gives in each entry i the number of times that a_i appears in the first n entries in ω: this ψ is the *symbol-counting cocycle*.

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- When ψ is the symbol-counting cocycle, these equivalence relations are the orbit relation of the group of *finite coordinate permutations*.

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- For example, Bernoulli processes are super-K^+, super-K^-, and super-$K^\pm$ (Hewitt-Savage, 1988).
- There are also such results for the 2-sided case by Blackwell-Freedman for Markov processes, Georgii for Gibbs states, Berbee-den Hollander for integer-valued processes, and others.

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- And unlike the K property, super-K depends on the choice of generating partition.
- We can have $\mathcal{F}_\psi^+(\alpha)$ trivial and find a refinement $\beta \geq \alpha$ with $\mathcal{F}_\psi^+(\beta)$ nontrivial (in fact equal to $\mathcal{B}$).

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- **K. Schmidt, 1999:** If $(X, \mathcal{B}, \mu, T)$is ergodic and $\psi : X \to G$ (as above) is Borel, then $\mathcal{F}_\psi^\pm(\alpha)$=$\mathcal{T}^\pm$.

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Triviality of two-sided fine tails

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- Interpretation: History is useless and science is impossible.
- **Corollary:** Any process (could be countable-state) with 2-sided trivial tail field $\mathcal{T}^\pm$ is super-$K^\pm$: $\mathcal{F}_\psi^\pm(\alpha)$ is trivial.

Super-K^+ generators

- **JPT-KP, 2004:** If an ergodic system $(X, \mathcal{B}, \mu, T)$, with generator α, is isomorphic to the direct product of a positive-entropy Bernoulli system (B, σ) and some other system (Y, S), then there is a generator β for $(X, \mathcal{B}, \mu, T)$ such that $\mathcal{F}^+(\beta) = \mathcal{T}^+(\beta) = \mathcal{T}^+$.

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- Consequently, every K process with a direct Bernoulli factor has a super-K^+ generator (since then $\mathcal{T}^+$, the Pinsker algebra, is trivial).
- The idea of the proof is to construct a generating partition β with $\mathcal{F}^+(\beta) \subset \mathcal{T}^+(\beta)$, so that no new information is provided by counting β-symbols.

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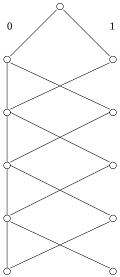
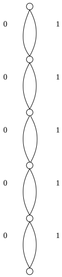
- 



Graphs for the fine tail fields

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- x_k labels the edge from $s_{k-1}(x)$ to $s_k(x)$.

Adic systems present tail fields

- The *fine tail equivalence relation* on $A^{\mathbb{N}}$ has $x \sim y$ if there is N such that $s_n(x) = s_n(y)$ for all $n \geq N$: the paths are *cofinal*—eventually coincide.

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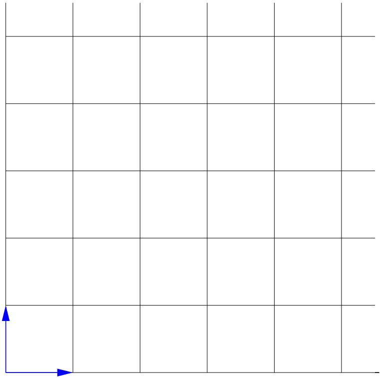
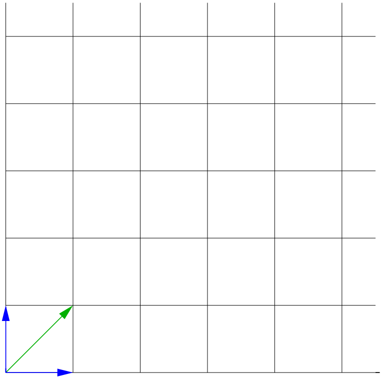
- Thus these systems visually present the future fine tail fields—we can see the corresponding equivalence relations.

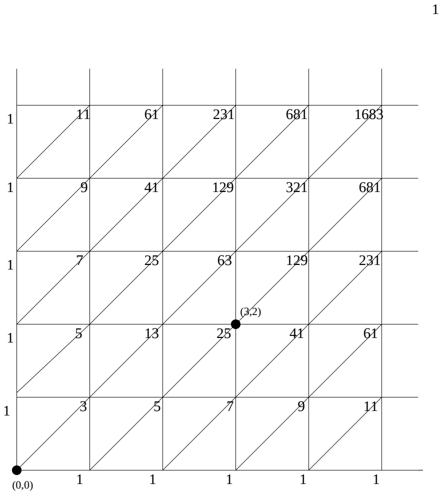
The Pascal walk



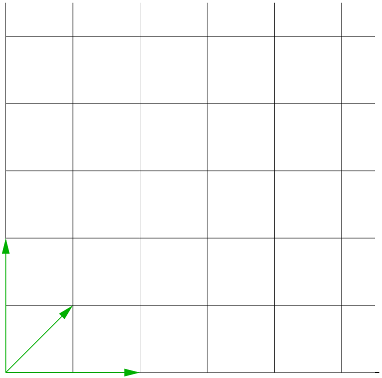
The Delannoy walk



The Delannoy graph



Xavier Méla's X_3 walk



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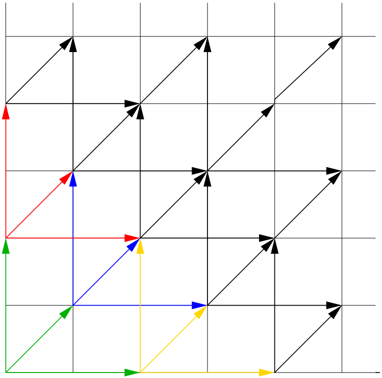
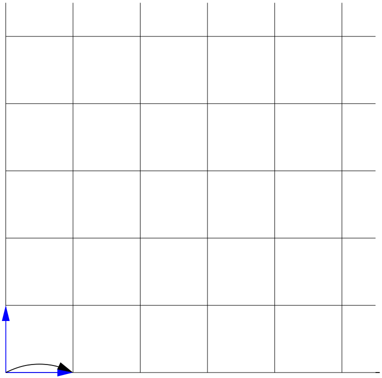
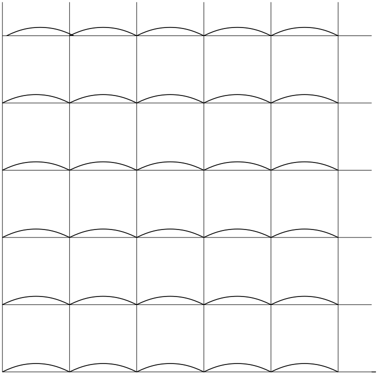


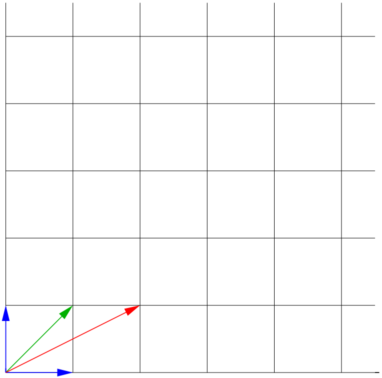
Frick's $2x+1$ walk



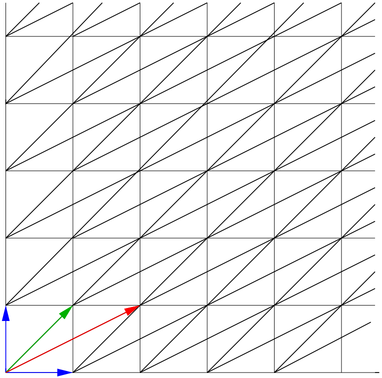
Frick's $2x + 1$ system



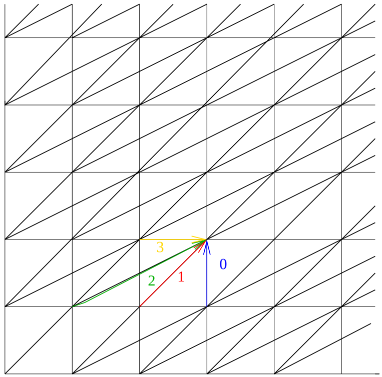
A walk with 4 vectors



An isotropic adic system based on a walk with 4 vectors



Ordering incoming edges to define the transformation



Ergodic measures

Identifying the invariant measures depends on knowing the *path counts* $\dim(v, w)$ = number of paths from v to w.

For Pascal, $\binom{n-n_0}{k-k_0}$.

For Delannoy, $D(i,j) = \sum_{d=0}^{j} 2^d \binom{i}{d}\binom{j}{d}$.

Recurrence formula and generating function for Delannoy numbers

- $$D(n,0) = D(0,n) = 1 \text{ for all } n \geq 0;$$
$$D(n,k) = 0 \text{ if either } n \text{ or } k < 0;$$
$$D(n,k) = D(n,k-1) + D(n-1,k-1) + D(n-1,k) \text{ for all } n,k.$$

- $$\sum_{n,k\geq 0} D(n,k)x^n y^k = \frac{1}{1-(x+y+xy)}$$

Various formulas for Delannoy numbers

Assuming $n \geq k$,

- $$D(n,k) = \sum_{d=0}^{k} \binom{k}{d}\binom{n+k-d}{k} = \sum_{d=0}^{k} 2^d \binom{n}{d}\binom{k}{d}$$
- $$= \sum_{d=0}^{k} \binom{k}{d}\binom{n+d}{k} = \sum_{d=0}^{k} \binom{k}{k-d}\binom{n+d}{k}$$
- $$= \sum_{d=0}^{k} \binom{n+k-d}{k-d}\binom{n}{d} = \sum_{d=0}^{k} \binom{n+d}{d}\binom{n}{k-d}.$$

Asymptotics of Delannoy numbers on the diagonal

$$D(n,n) \sim (3+2\sqrt{2})^n(.57\sqrt{n} - .067n^{-3/2} + .006n^{-5/2} + \dots).$$

Invariant measures for the Delannoy adic

Theorem

The non-atomic ergodic (invariant probability) measures for the Delannoy adic dynamical system are a one-parameter family $\{\mu_\alpha : \alpha \in [0,1]\}$ *given by choosing nonnegative* α, β, γ *with* $\alpha + \beta + \gamma = 1$ *and* $\beta\gamma = \alpha$ *and then putting weight* β *on each horizontal edge, weight* γ *on each vertical edge, and weight* α *on each diagonal edge. (The measure of any cylinder set is then determined by multiplying the weights on the edges that define it.)*

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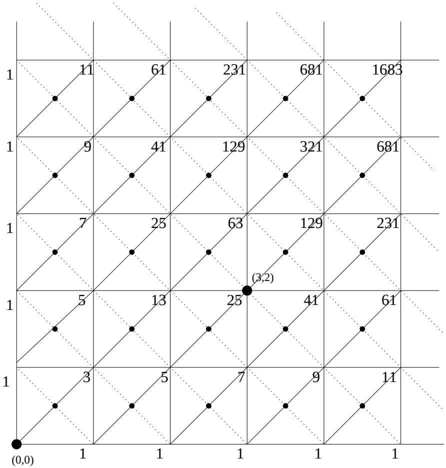

Ingredients of the proof

- Pemantle-Wilson asymptotics for the Delannoy numbers:

$$D(n,k) \sim \left(\frac{\sqrt{n^2+k^2}-k}{n}\right)^{-n} \left(\frac{\sqrt{n^2+k^2}-n}{k}\right)^{-k} \times$$
$$\sqrt{\frac{1}{2\pi}}\sqrt{\frac{nk}{(n+k-\sqrt{n^2+k^2})^2\sqrt{n^2+k^2}}},$$

  uniformly if n/k and k/n are bounded.
- Collision argument based on recurrence of symmetric random walk in $\mathbb{Z}^2$
- X. Méla's isotropy argument

Total ergodicity of the Delannoy adics

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With respect to each of its ergodic (invariant probability) measures, the Delannoy adic dynamical system is totally ergodic (i.e., has among its eigenvalues no roots of unity besides 1*).*

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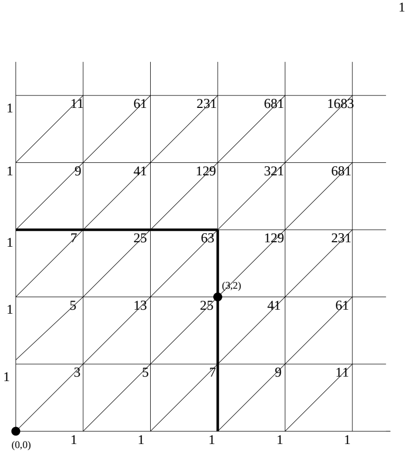
For p *prime,* $r \geq 0$*, and* $n = 0, 1, 2, \dots$*,*

$$D(n, p^r - 1) \equiv_p (-1)^{(n \mod p^r)}.$$

The Delannoy graph with a "blocking set"



Remarks on the Delannoy system

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- We do not know about limit laws for return times, weak mixing, multiplicity of the spectrum, or joinings.
- But there is some progress on the complexity ($n^3/24$ for the Delannoy, by Sarah Bailey Frick) and on generalizing these considerations to a class of systems.

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- Recall that Bezuglyi, Kwiatkowski, Yassawi have investigated the probability that an order is "perfect", i.e. admits the Vershik map as a homeomorphism.
- They also showed that for a fixed finite rank diagram there is a number J such that with probability 1 there are J maximal paths and J minimal paths.

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- And a couple of their questions have been answered by Behrend, Boshernitzan, and Kolesnik:
- (1) $\{\lambda^{d_n(x)}\}$ is uniformly distributed in S^1 for *every* λ not a root of unity if and only if x is eventually diagonal.

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- Possible new avenues toward proving weak mixing of the Pascal are being explored by A. Prikhodko and A. Vershik.

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- The joint action of the shift and 2-odometer is that of the step-2 Baumslag-Solitor group: the only relation is $\sigma T = T^2\sigma$.

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- We consider some recent developments regarding periodic and *nonperiodic* “Sturmian” sequences, involving lexicographic order, Farey diagrams, and adic transfomations.

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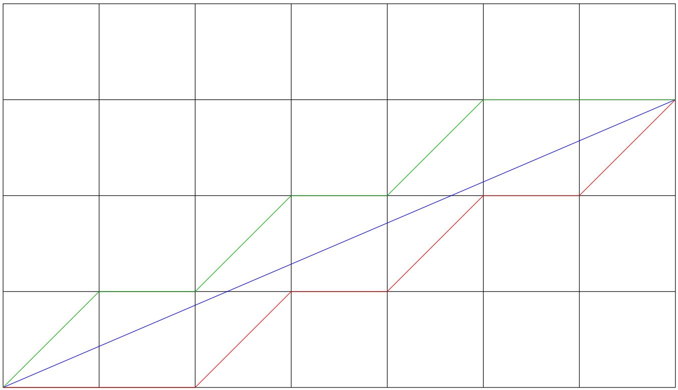
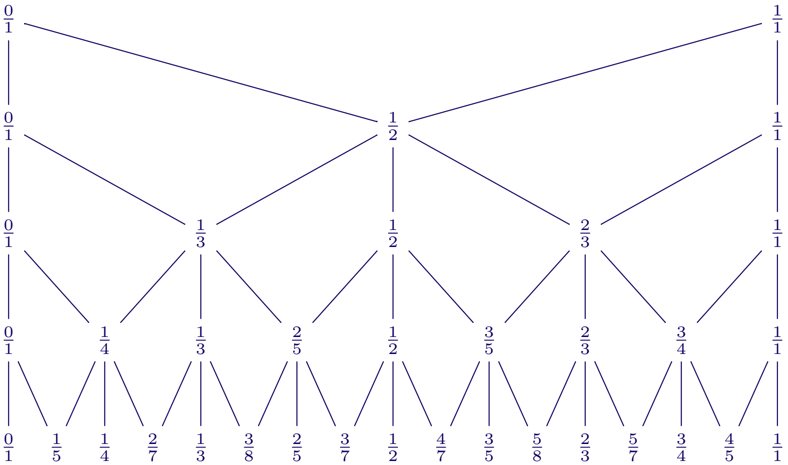
- Staircase coding: There are x and irrational θ such that for all n, $\omega(n) = \lfloor x + (n+1)\theta \rfloor - \lfloor x + n\theta \rfloor$ or for all n, $\omega(n) = \lceil x + (n+1)\theta \rceil - \lceil x + n\theta \rceil$. (Look at jumps between lattice points above or below line through origin of slope θ. Get jump (of floor) when $n\theta$ is in $[1 - \theta, 1)$.)

Lower staircase coding of 3/7



0 0 1 0 1 0 1

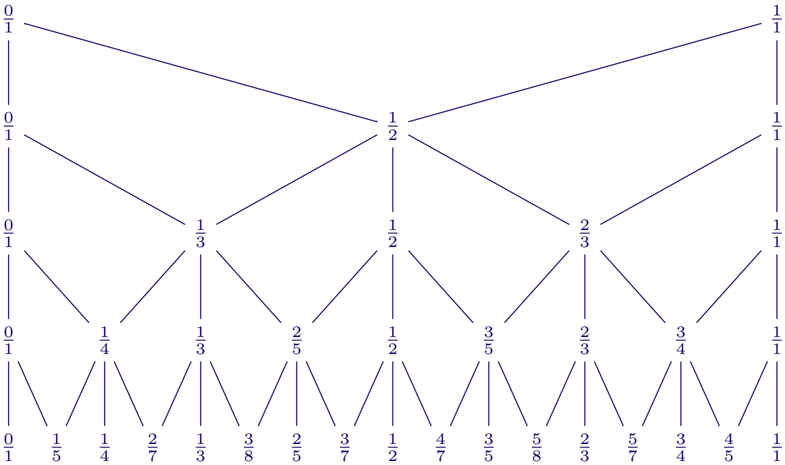
Farey or Stern-Brocot Diagram



Properties of Farey diagram

- Generated by adding numerators and denominators.
- Every rational in $[0,1]$ appears, generated exactly once, automatically in lowest terms.
- Two Farey neighbors, p/q and p'/q', satisfy $p'q - q'p = \pm 1$.
- Infinite paths give best one-sided approximations to irrationals. When switch sides, have best two-sided approximations, the ordinary continued fractions.

Farey Diagram



Ordinary and intermediate continued fractions

- Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$
- Ordinary continued fractions for $x = [a_1, a_2, \dots]$:

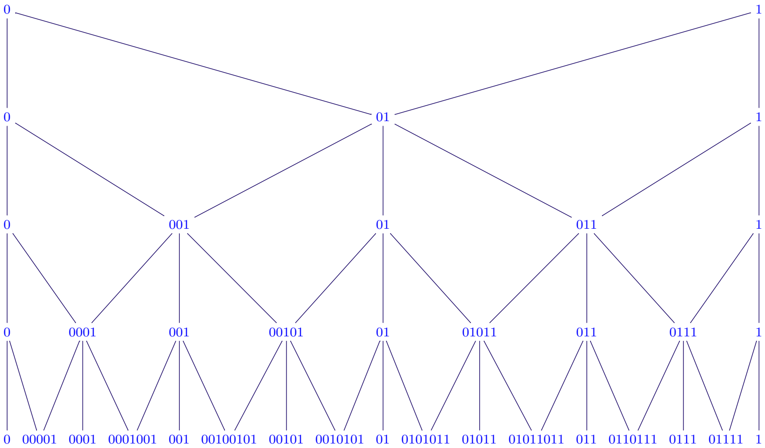
$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

-
$$= B\,A^{a_1-1}\,BA^{a_2-1}\dots B\,A^{a_n-1}$$

- The intermediate products give the intermediate, Farey, approximations.
-
$$x = [2, 3, 2, 4, \dots] \approx 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{7}{16}, \frac{10}{23}, \frac{17}{39}, \frac{24}{55}, \frac{31}{71}, \dots$$

- I learned about the Farey shift from papers of Jeff Lagarias.

Farey Diagram of Blocks



Balanced periodic sequences

- The word at position corresponding to fraction $p/(p+q)$ has p 1's and q 0's (hence length $p+q$).
- The periodic sequence formed by each of these words is balanced.
- These words are Lyndon words—primitive and lexicogaphically minimal among their rotations.
- They also increase lexicographically left to right in each row.
- Every balanced word of length $p+q$ with exactly p 1's is a rotation of the word in the Farey diagram that corresponds to $p/(p+q)$. There are exactly $p+q$ of them.
- Infinite nonperiodic Sturmian sequences are found as "ends" of infinite paths in the Farey diagram.

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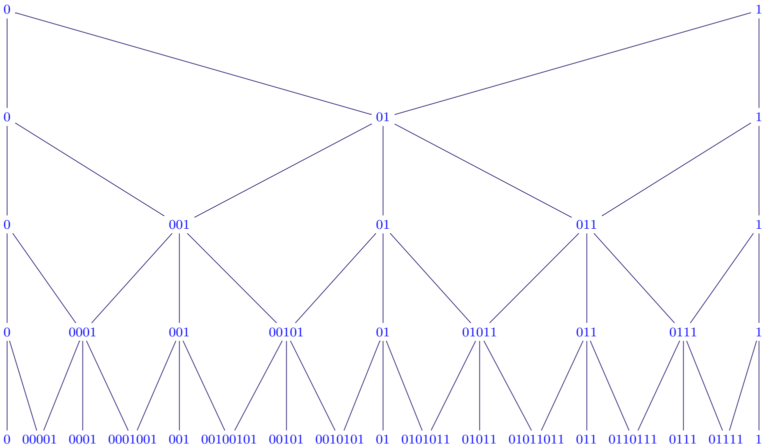

Times 2 map

- Viewed as dyadic expansions, the words in the Farey diagram correspond to periodic orbits under the map $Tz = z^2$ on the circle. Each orbit is contained in a closed semicircle, and T preserves the cyclic order on the circle.

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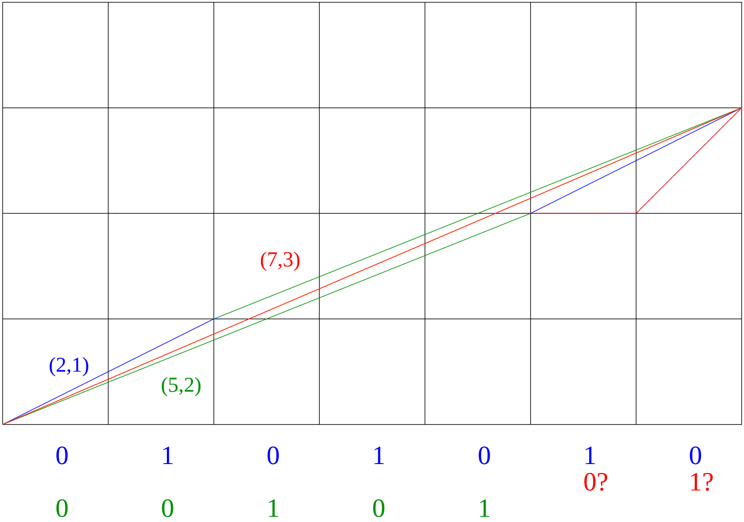
- The invariant measures coming from Sturmian minimal sets minimize the integrals of strictly convex functions (over all T-invariant measures with a fixed frequency of 1's) (Jenkinson 2007).
- Besides Coven-Hedlund (1973) and Hedlund-Morse (1940), we should also mention Jenkinson-Zamboni (2004), Arnoux (2002—in Pytheas Fogg), Berstel-Séébold (2002—in Lothaire), Jenkinson (1996–), Bullett-Sentenac (1994), Borel-Laubie (1993), Rauzy (1985), Gambaudo-Lanford-Tresser (1984), Hedlund (1944), Christoffel (1875), J. Bernoulli (1772), and probably others.

Why does the concatenation work?

- *Prop:* If $u < v$ are Lyndon words, then uv is Lyndon.

- The following are equivalent:
  - Two integer vectors (q, p) and (q', p') span the integer lattice $\mathbb{Z}^2$.
  - $pq' - qp' = \pm 1$.
  - The parallelogram spanned by the vectors (q, p) and (q', p') has no point of the integer lattice $\mathbb{Z}^2$ in its interior.

(7,3) is Farey child of (5,2) and (2,1)



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- These observations were stimulated by a talk by O. Jenkinson, are based on papers by O. Bratteli and F. Boca, and were developed in conversations with T. de la Rue and E. Janvresse.

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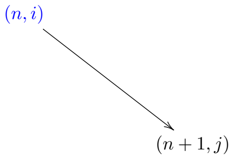

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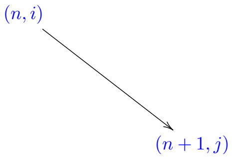

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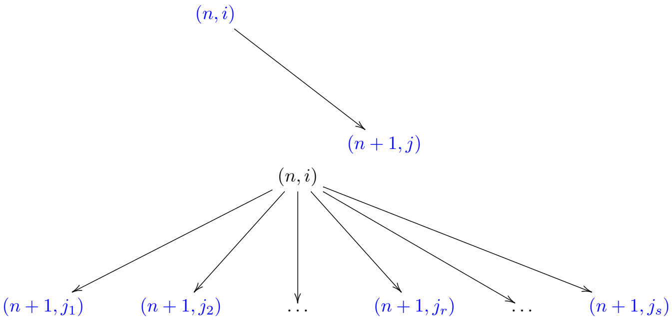

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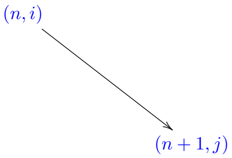

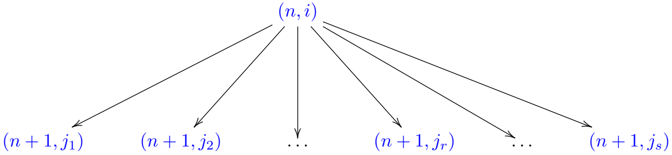

Primitive ideals in $\mathcal{A}$

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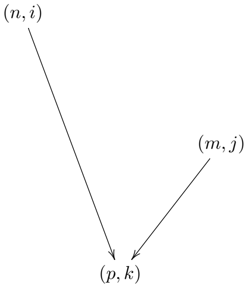

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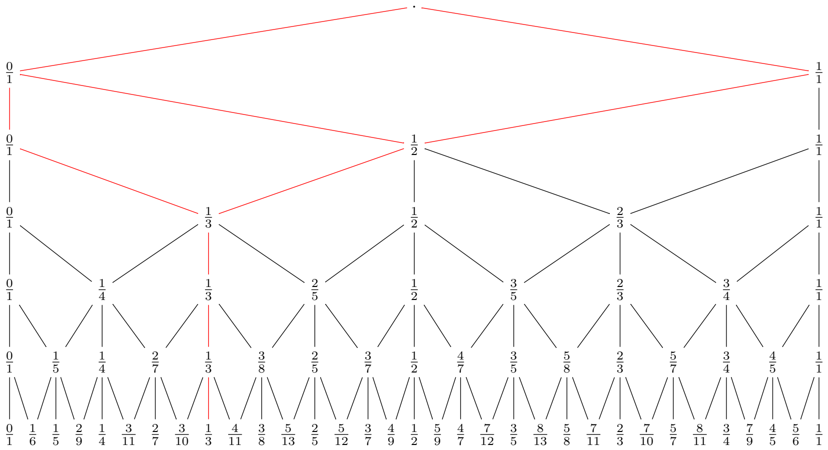
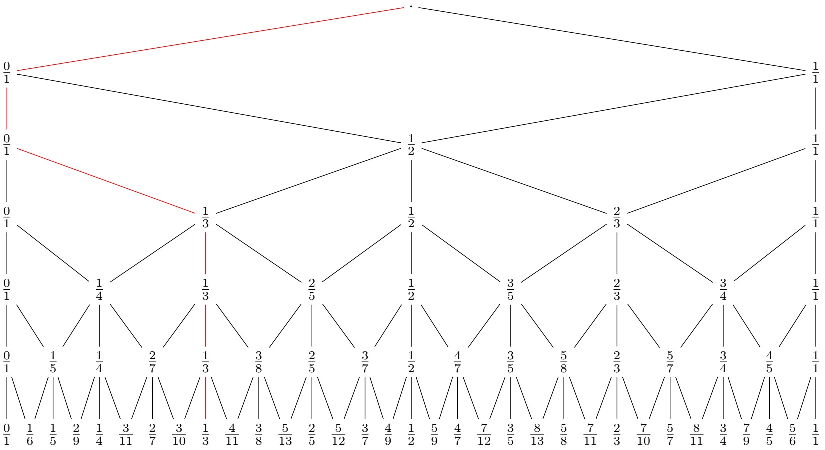
The orbit of $1/3 \sim 001001001001\ldots$



Mapping $1/3 \sim 001001001001\ldots$



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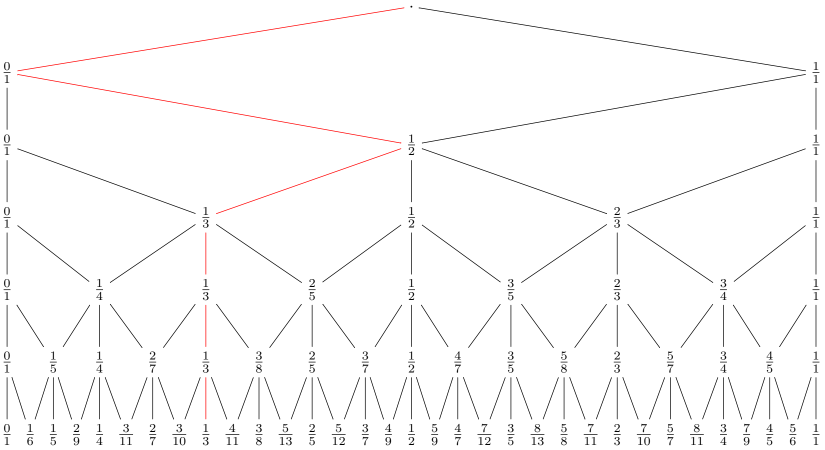
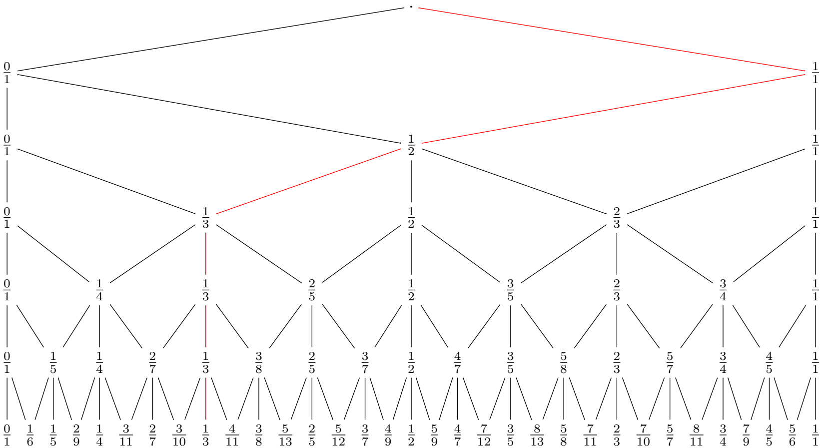
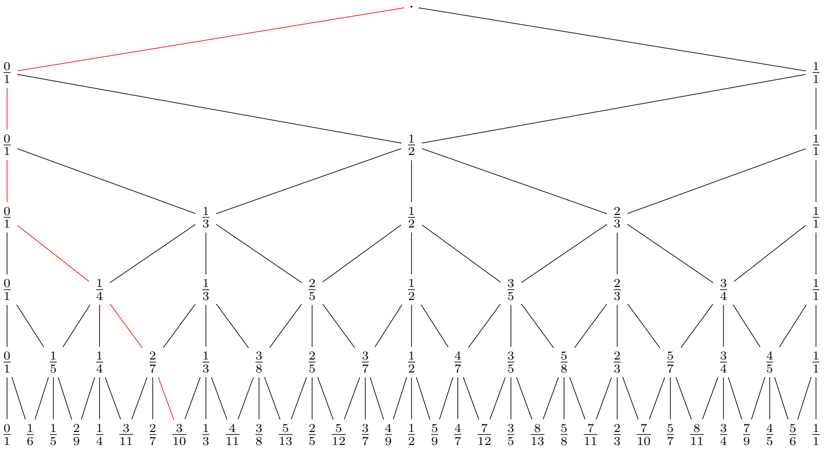


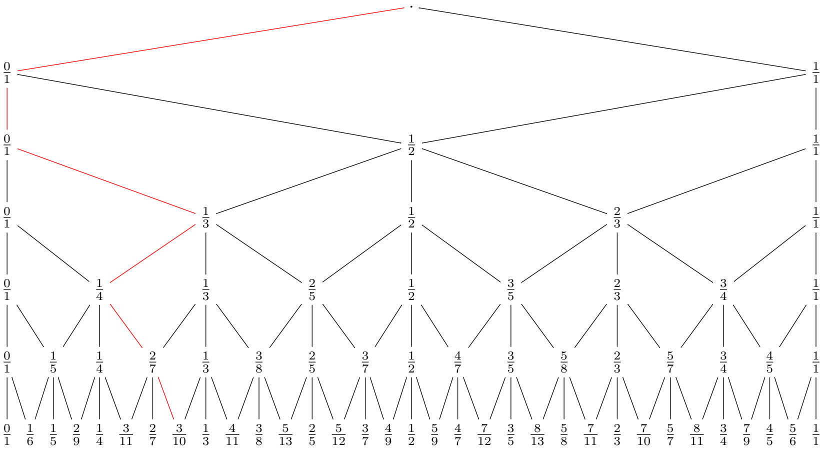
Mapping $1/3 \sim 001001001001\ldots$



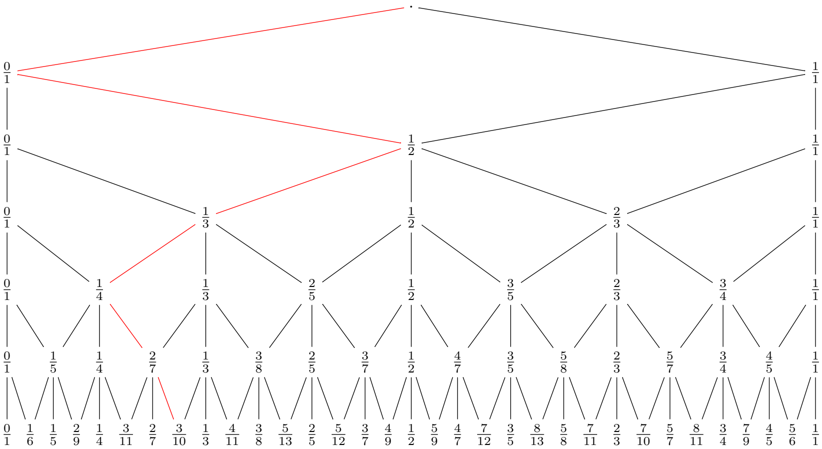
An orbit forward asymptotic to that of $1/3$



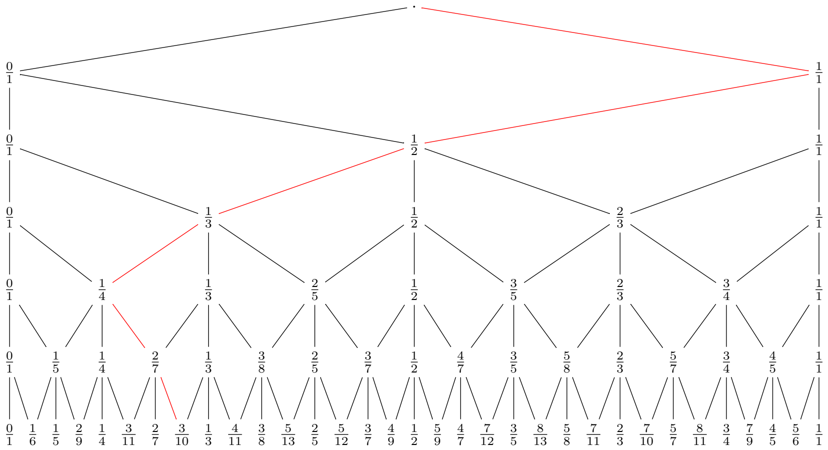
An orbit forward asymptotic to that of $1/3$



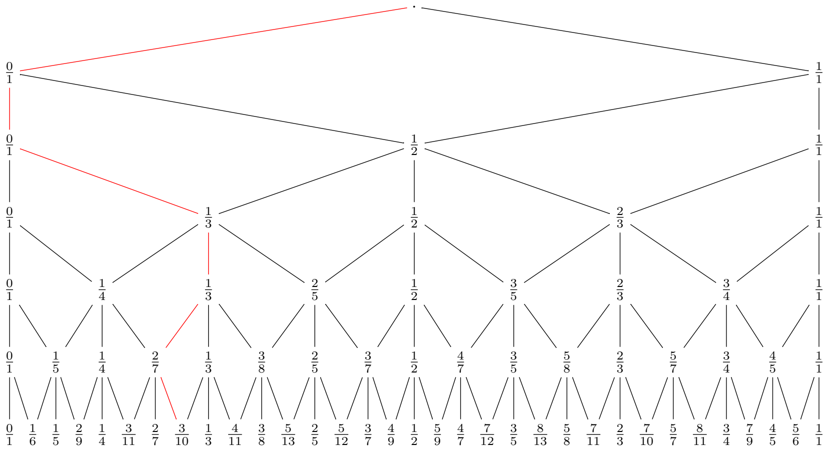
An orbit forward asymptotic to that of $1/3$



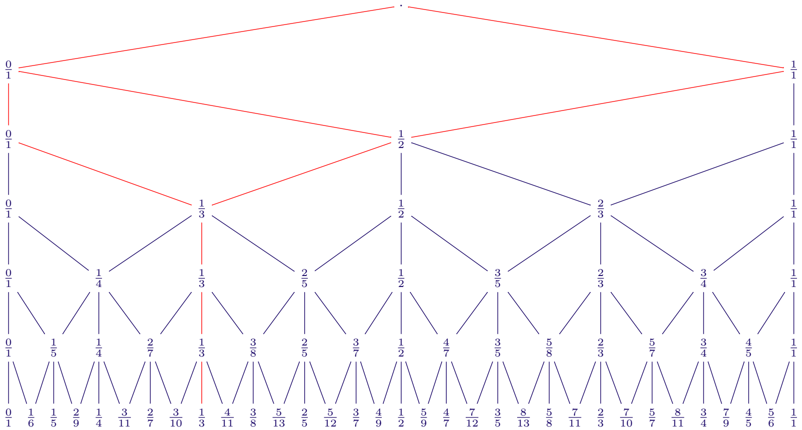
An orbit forward asymptotic to that of $1/3$



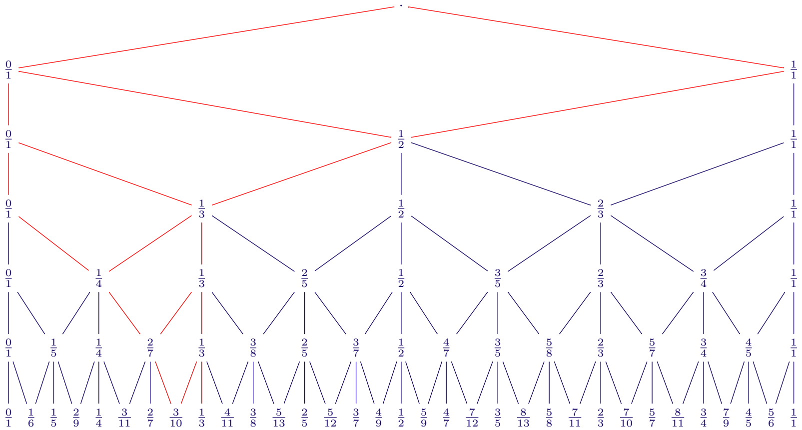
An orbit forward asymptotic to that of $1/3$



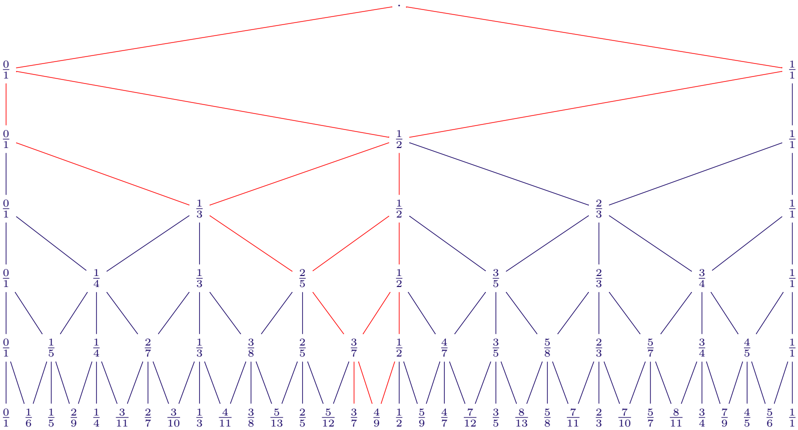
The diagram (non-red) of one ideal for $1/3 \sim 001$



The diagram (non-red) of another ideal for $1/3 \sim 001$



Ideal and orbit closure for $\theta = [2, 3, 2, 4, \dots]$



- P. Dartnell, F. Durand, and A. Maass (Studia Math.2000) computed the dimension groups of Sturmian subshifts and showed that two Sturmian subshifts are orbit equivalent if and only if they are topologically conjugate.

- P. Dartnell, F. Durand, and A. Maass (Studia Math.2000) computed the dimension groups of Sturmian subshifts and showed that two Sturmian subshifts are orbit equivalent if and only if they are topologically conjugate.
- What further insight into the much-studied class of Sturmian subshifts might be gained from the adic viewpoint?

β-shifts

- Fix $\beta > 1$, let $d = \lceil \beta \rceil$, and $D = \{0, 1, \dots, d-1\}$.
- Let $\Sigma_\beta^+ \subset D^{\mathbb{N}}$ denote the closure of the set of all greedy expansions base β of all $x \in [0,1]$,

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$$

- (Σ_β^+, σ) is a symbolic coding (lift) of the β-transformation $T_\beta : [0,1] \to [0,1]$ defined by $T_\beta x = \beta x \mod 1$.
- If the expansion $a_1 a_2 \dots$ of 1 base β is nonterminating, we put $e_\beta(1) = a_1 a_2 \dots$.
- Otherwise there is a first i for which $T_\beta^i 1 = n \in \mathbb{N}$, and then we put $e_\beta(1) = [a_1 \dots a_{i-1}(n-1)]^\infty$.

β-shifts and lexicographic order

- A sequence $a = a_1 a_2 \cdots \in D^{\mathbb{N}}$ is in Σ_β^+ if and only if $\sigma^k x \leq e_\beta(1)$ for all $k \geq 0$.
- A sequence $a = a_1 a_2 \cdots \in D^{\mathbb{N}}$ is $e_\beta(1)$ for some β if and only if it dominates all its shifts: $a \geq \sigma^k a$ for all $k \geq 0$ (Parry, 1960).

A map of the interval

- Return now to a Sturmian symbolic dynamical system with rotation number θ. It also has a lexicographically *maximal* element.
- This maximal element, $M(\theta) = (1_{[0,\theta)}(n(1-\theta)))$, is obtained from the Farey diagram of blocks by switching 0's and 1's (and θ with $1-\theta$).
- Since $M(\theta)$ is lexicographically maximal in a subshift, it dominates all its shifts and hence is the expansion $e_\beta(1)$ of 1 base β for some $\beta = \beta(\theta) \in (1,2)$.
- We define $L : (0,1] \to (0,1]$ by $L(\theta) = \beta(\theta) - 1$.

The map L

- The map $L : (0,1] \to (0,1]$ is strictly increasing.
- This is because $\beta \to e_\beta(1)$ is strictly increasing and each row of the Farey diagram of blocks is strictly increasing.
- Then we switch 0's and 1's, and θ and $1-\theta$.
- For $\theta = 1/3$, the minimal element is $001001001\dots$, the maximal element is $M(\theta) = 100100100\cdots = (1_{[0,1/3)}(n \times 2/3))$, and $\beta(\theta)$ is the reciprocal of the solution of $1 = x + x^4 + x^7 + \dots$, i.e. $1 = x + x^3$.
- For $\theta = 2/3$, the minimal element is $011011011\dots$, the maximal element is $M(\theta) = 110110110\cdots = (1_{[0,2/3)}(n \times 1/3))$, and $\beta(\theta)$ is the reciprocal of the solution of $1 = (x + x^2)(1 + x^3 + \dots)$, i.e. $1 = x + x^2 + x^3$.
- So $\beta(1/3) < \beta(2/3)$

Some values of L

- $L(1/2)$ is the solution α of $x + x^2 = 1$.
- $L(\mathbb{Q}) \subset$ algebraic numbers.
- $M(\alpha) = 1f$, where f is the fixed point of the Fibonacci substitution $0 \to 01, 1 \to 0$.
- The 1999 thesis of Kimberly Johnson gives (among other things) an algorithm for finding the maximal elements in substitution subshifts.
- $L(\alpha)$ is transcendental (Chi and Kwon, 2004).
- Since the mapping L connects the lexicographic order properties of Sturmian systems and β-shifts (and the interval), it may be interesting to develop further its properties and those of the dynamical system it defines.
- In recent papers and preprints, DoYong Kwon has defined and studied essentially the same function.