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Markov diagrams for some non-Markovian systems

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Abstract. Markov diagrams provide a way to understand the structures of topological dynamical systems. We examine the construction of such diagrams for subshifts, including some which do not have any nontrivial Markovian part, in particular Sturmian systems and some substitution systems.

1. Introduction

F. Hofbauer [**12**] and J. Buzzi [**2**] defined Markov diagrams in order to study the structures and invariant measures of dynamical systems, especially those with a Markovian aspect, for example piecewise monotonic interval maps and other possibly nonunifomly expanding maps. Here we examine further the construction of these diagrams for subshifts, including some that are minimal and have zero entropy. Such subshifts may be considered to be highly non-Markovian, since they have some long-range order, indeed infinite memory. We hope that Markov diagrams will be useful also for understanding and classifying such systems, for example besides helping to identify measures of maximal entropy as in [**3**, **12**, **13**] also to determine complexity functions, estimate return times to cylinders, and so on.

In Sections 3 and 4 we provide a construction of Hofbauer-Buzzi Markov diagrams for Sturmian systems. In particular, in Theorem 4.13 we show that the Hofbauer-Buzzi Markov diagram of a Sturmian system can be constructed solely from its left special sequence. In Section 5 we discuss properties of Hofbauer-Buzzi Markov diagrams that hold for any subshift. We show that given a one-sided subshift X^+ there is a correspondence between those paths on the Hofbauer-Buzzi Markov diagram of X^+ that start with a vertex of length one and points in X^+ (Theorem 5.8). Corollary 5.9 relates the number of such paths to the complexity function of X^+. We prove that the eventually Markov part of the natural extension of any one-sided subshift is empty provided that the natural extension is an infinite minimal subshift (Proposition 5.13). In Section 6 we construct the Hofbauer-Buzzi Markov diagram for the Morse minimal subshift by showing that the vertices are precisely those blocks in the language of the subshift that are of the form 0 or 1 followed by a block that can be extended to the left in two ways. Theorem 7.1 applies the Hofbauer-Buzzi Markov diagrams of Sturmian subshifts to identify their lexicographically maximal elements.

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2. Background

We recall some of the basic terminology and notation from topological and symbolic dynamics; for more details, see for example [**17**] or [**21**]. A *topological dynamical system* is a pair (X, T), where X is a compact Hausdorff space (usually metric) and $T : X \to X$ is a continuous mapping. We focus on topological dynamical systems which are *shift dynamical systems*. Let $\mathcal{A}$ be a finite set, called an *alphabet*, whose elements are called *symbols*. For us often $\mathcal{A} = \{0, 1, \dots, n-1\}$, in fact often $\mathcal{A} = \{0, 1\}$. A *sequence* is a one-sided infinite string of symbols (a function $\mathbb{N} \to \mathcal{A}$) and a *bisequence* is an infinite string of symbols that extends in two directions (a function $\mathbb{Z} \to \mathcal{A}$). We will use the word "sequence" to apply also to bisequences, depending on the context to clarify the meaning. The *full n-shift* is $\Sigma_n = \{0, 1, \dots, n-1\}^{\mathbb{Z}}$, the collection of all bisequences of symbols from $\mathcal{A} = \{0, 1, \dots, n-1\}$. The *one-sided full n-shift* is $\Sigma_n^+ = \{0, 1, \dots, n-1\}^{\mathbb{N}}$. We also define the *shift transformation* $\sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{A})$ and $\Sigma^+(\mathcal{A}) \to \Sigma^+(\mathcal{A})$ by $(\sigma x)_i = x_{i+1}$ for all i. The pair (Σ_n, σ) is called the *n-shift dynamical system*. We give $\mathcal{A}$ the discrete topology and $\Sigma(\mathcal{A})$ and $\Sigma^+(\mathcal{A})$ the product topology. The topologies on $\Sigma(\mathcal{A})$ and $\Sigma^+(\mathcal{A})$ are compatible with the metric $d(x, y) = 1/2^n$, where $n = \inf\{|k| \mid x_k \neq y_k\}$. A *subshift* is a pair (X, σ) (or (X^+, σ)), where $X \subset \Sigma_n$ (or $X^+ \subset \Sigma_n^+$) is a nonempty, closed, shift-invariant set. A finite string of letters from $\mathcal{A}$ is called a *block* and the length of a block B is denoted $|B|$. Furthermore, a block of length n is an n-block. Given a subshift (X, σ) of a full shift, $\mathcal{L}_n(X)$ denotes the set of all n-blocks that occur in points in X. The *language* of X is the collection $\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X)$. A *shift of finite type* is a subshift consisting of all sequences none of whose subblocks are in some finite collection of blocks. A topological dynamical system is *minimal* if every orbit is dense. The orbit closure of a sequence is minimal if and only if the sequence is *syndetically recurrent*: every block that appears in the sequence appears with bounded gaps see [**21**, p. 136]. The *complexity function* of a sequence u, denoted p_u, maps each natural number n to the number of blocks of length n that appear in u. If X is a subshift, then $p_X(n)$ is the number of blocks of length n that appear in $\mathcal{L}(X)$.

The construction of Hofbauer-Buzzi Markov diagrams involves the use of *follower sets*. There are several ways to define follower sets. The (*block to block*) *follower set* of a block $w \in \mathcal{L}(X)$ is $F_X(w) = \{v \in \mathcal{L}(X) | wv \in \mathcal{L}(X)\}$. Another approach defines the *future* F_X of a left-infinite sequence λ in X to be the collection of all right-infinite sequences ρ such that $\lambda\rho \in X$. This is a *ray to ray* follower set. It is also possible to define *block to ray* or *ray to block* follower sets. The definition of follower set (3.2) used in constructing Hofbauer-Buzzi diagrams is slightly different from both of these.

Follower sets have been particularly useful in examining sofic systems. A *sofic shift* is a shift space that is a factor of a shift of finite type [**25**]. Alternatively, a sofic shift consists of all sequences that are labels of infinite walks on a finite graph with labeled edges (see [**17**]). Fischer [**8**] and Krieger [**15**] used follower sets to construct covers for sofic shifts. A *presentation* of a sofic shift X is a finite labeled graph $\mathcal{G}$ for which $X_{\mathcal{G}} = X$. A presentation is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving presentation of X having the fewest vertices among all right-resolving presentations of X. Fischer proved that any two minimal right-resolving presentations are isomorphic as labeled graphs; the minimal right-resolving presentation of a sofic shift X is called the *Fischer cover* [**8**, **17**].

Given an irreducible (topologically transitive) sofic shift X over a finite alphabet $\mathcal{A}$, the Fischer cover can be constructed using the follower sets defined above. Let $\mathcal{C}_X$ be the collection of all (block to block) follower sets in X. We write $\mathcal{C}_X = \{F_X(w) | w \in \mathcal{L}(X)\}$ a finite set since X is sofic. Now construct a labeled graph $\mathcal{G} = (G, L)$ as follows. The vertices of G are the elements in $\mathcal{C}_X$. Let $c = F_X(w)$ be an element in $\mathcal{C}_X$ and $a \in \mathcal{A}$. If $wa \in \mathcal{L}(X)$, let $c' = F_X(wa) \in \mathcal{C}_X$ and draw an edge labeled a from c to c'. If $wa \notin \mathcal{L}(X)$, do nothing. Continuing this process for all elements in $\mathcal{C}_X$ yields a labeled graph $\mathcal{G}$ called the *follower set graph*. The Fischer cover of X is the labeled subgraph of the follower set graph formed by using only the follower sets of intrinsically synchronizing blocks. Here a block w in $\mathcal{L}(X)$ is *intrinsically synchronizing* if whenever $uw, wv \in \mathcal{L}(X)$ then $uwv \in \mathcal{L}(X)$ [**17**].

The Krieger cover is constructed using the *futures*, as defined above, of left-infinite sequences in X. We define the *future cover* as follows. Let $\mathcal{G}$ be the labeled graph whose vertices are the futures of left-infinite sequences. For a in $\mathcal{A}$, if λ and λa are left-infinite sequences in X, then there is an edge labeled a from $F_X(\lambda)$ to $F_X(\lambda a)$. The graph $\mathcal{G}$ is the *future* or Krieger cover of the subshift X [**15**, **17**]. The Krieger cover can be constructed for any subshift X, but it usually leads to non-irreducible and often uncountable graphs. Nevertheless, the Krieger cover is canonically associated to the subshift X. This is proved for the sofic case in [**16**] and in general in [**6**].

3. Hofbauer-Buzzi Markov diagrams

Franz Hofbauer [**12**] constructed Markov diagrams to determine measures of maximal entropy for piecewise monotone functions on the interval. In 1997, Buzzi extended the construction of the Hofbauer Markov diagram to arbitrary smooth interval maps [**1**], and in 2010 to any subshift [**2**]. The Hofbauer-Buzzi Markov diagram is a slight variation of Hofbauer's original Markov diagram. We will refer to such diagrams as *HB diagrams*. In order to describe the construction, we introduce the following definitions from [**2**].

Let $\mathcal{A}$ be a finite alphabet and $X^+ \subset \mathcal{A}^{\mathbb{N}}$ a one-sided subshift. Furthermore, let $X^{+-} \subset \mathcal{A}^{\mathbb{Z}}$ be its natural extension

$$X^{+-} = \{x \in \mathcal{A}^{\mathbb{Z}} | \text{ for all } p \in \mathbb{Z} \;\; x_p x_{p+1} ... \in X^+\},$$

with the action of the shift σ defined by $(\sigma x)_n = x_{n+1}$ for all $x \in X, n \in \mathbb{Z}$.

DEFINITION 3.1. Let π_{X^+} denote the continuous shift commuting projection from X^{+-} to X^+ defined by

$$\pi_{X^+}(x) = x_0 x_1 x_2 ...,$$

where $x = ...x_{-1}.x_0 x_1 x_2 ....$

DEFINITION 3.2. The *follower set* of a block $w = a_{-n} a_{-n+1} ... a_0$ in $\mathcal{L}(X^{+-})$ is

$$\text{fol}(a_{-n} a_{-n+1} ... a_0) = \{b_0 b_1 ... \in X^+ | \text{ there exists } b \in X^{+-} \text{ with } b_{-n} ... b_0 = a_{-n} ... a_0\}.$$

In other words, the follower set of a block $a_{-n} a_{-n+1} ... a_0$ consists of all one-sided rays $b_0 b_1 \ldots$ which have an extension to a two-sided sequence $b = \ldots b_{-n} \ldots b_0 b_1 \ldots$ in the natural extension X^{+-} of X^+ such that $b_{-n} \ldots b_0 = a_{-n} \ldots a_0$.

DEFINITION 3.3. A *significant block* of X^{+-} is $a_{-n}a_{-n+1}...a_0$ such that

$$\text{fol}(a_{-n}a_{-n+1}...a_0) \subsetneq \text{fol}(a_{-n+1}a_{-n+2}...a_0).$$

DEFINITION 3.4. The *significant form* of $a_{-n}a_{-n+1}...a_0$ in X^{+-} is

$$\text{sig}(a_{-n}...a_0) = a_{-k}...a_0,$$

where $k \leq n$ is maximal such that $a_{-k}...a_0$ is significant.

It is apparent that these definitions are tailored for one-sided subshifts. However, we can easily extend such definitions to an arbitrary two-sided subshift $X \subset \Sigma_n$ by letting X^+ denote the set of right rays that appear in points in X. Then X is equal to the natural extension X^{+-} of X^+.

We define the *HB diagram* $\mathcal{D}$ of a one or two-sided subshift X with natural extension X^{+-} to be the oriented graph whose vertices are the significant blocks of X^{+-} and whose arrows are defined by

$$a_{-n}...a_0 \to b_{-m}...b_0$$

if and only if $a_{-n}...a_0b_0 \in \mathcal{L}(X^{+-})$ and

$$b_{-m}...b_0 = \text{sig}(a_{-n}...a_0b_0).$$

(In Hofbauer's construction of Markov diagrams the vertices are the follower sets, not the significant blocks [**2**, **12**].)

Let $\mathcal{D}$ be the HB diagram of any one or two-sided subshift X. The following definitions from [**2**] relate $\mathcal{D}$ to X^{+-}.

DEFINITION 3.5. Given an HB diagram $\mathcal{D}$ of a subshift X with vertex set $V_{\mathcal{D}}$ (which may be infinite), the corresponding *Markov shift* is the set of all bi-infinite paths that occur on $\mathcal{D}$,

$$X(\mathcal{D}) = \{\alpha \in V_{\mathcal{D}}^{\mathbb{Z}} \mid \text{for all } p \in \mathbb{Z} \;\; \alpha_p \to \alpha_{p+1} \text{ on } \mathcal{D}\} \subset (V_{\mathcal{D}})^{\mathbb{Z}},$$

together with the shift map σ.

Note that the alphabet $V_{\mathcal{D}}$ may be infinite, and the HB diagram of an arbitrary subshift may not have paths that continue infinitely in two directions.

We relate $X(\mathcal{D})$ to X^{+-} as follows.

DEFINITION 3.6. Let $\hat{\pi}$ denote the natural continuous projection defined by

$$\hat{\pi} : \alpha \in X(\mathcal{D}) \mapsto a \in X^{+-}$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{Z}$.

DEFINITION 3.7. Let $X(\mathcal{D})_v^+$ denote the space of one-sided infinite paths starting at vertex v on $\mathcal{D}$, and let $X(\mathcal{D})^+$ denote the space of one-sided infinite paths starting at a vertex v of length 1 on $\mathcal{D}$.

DEFINITION 3.8. Let $\hat{\pi}^+$ denote the projection defined by

$$\hat{\pi}^+ : \alpha \in X(\mathcal{D})^+ \mapsto a \in X^+$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{N}$.

In case we want to project a finite path $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$ on $\mathcal{D}$ to a block in $\mathcal{L}(X^+)$, we write $\hat{\pi}(\alpha_0...\alpha_n) = a_0...a_n$, where a_i is the last letter of α_i.

We state a few preliminary results that apply to any subshift.

LEMMA 3.9. *If $a_1...a_n$ is a signficant block of a subshift X, then $a_1...a_{n-1}$ is also a significant block of X.*

Proof. If $a_1 \dots a_{n-1}$ is not significant, then $\text{fol}(a_1 \dots a_{n-1}) = \text{fol}(a_2 \dots a_{n-1})$. We want to show that $\text{fol}(a_1 \dots a_n) = \text{fol}(a_2 \dots a_n)$, so that $a_1 \dots a_n$ is not significant. Let $b_n b_{n+1} \dots \in \text{fol}(a_2 \dots a_n)$, so that there is $b \in X^{+-}$ with $b_2 \dots b_n = a_2 \dots a_n$. Then $b_{n-1} b_n b_{n+1} \dots \in \text{fol}(a_2 \dots a_{n-1}) = \text{fol}(a_1 \dots a_{n-1})$, so there is $c \in X^{+-}$ with $c_{n-1} c_n c_{n+1} \dots = b_{n-1} b_n b_{n+1} \dots$ and $c_1 \dots c_{n-1} = a_1 \dots a_{n-1}$. Therefore $c_n c_{n+1} \dots = b_n b_{n+1} \dots \in \text{fol}(a_1 \dots a_n)$. □

The following Proposition is an immediate consequence of Lemma 3.9.

Proposition 3.10. *If there are infinitely many significant blocks of a subshift X, then for all $n \in \mathbb{N}$ there exists a significant block of X of length n.*

4. Markov diagrams for one-sided Sturmian systems

4.1. Basic properties of Sturmian sequences. We recall the definition and basic properties of Sturmian sequences; see [**9**, Ch. 6] for details. A one or two-sided sequence u with values in a finite alphabet is called *Sturmian* if it has complexity function (defined above) $p_u(n) = n + 1$ for all n. If u is Sturmian, then $p_u(1) = 2$. This implies that Sturmian sequences are over a two-letter alphabet, so we fix the alphabet $\mathcal{A} = \{0, 1\}$. Given a one-sided Sturmian sequence u, we let X_u^+ be the closure of $\{\sigma^n u | n \in \mathbb{N}\}$. Then (X_u^+, σ) is the *Sturmian system* associated with u.

Example 4.1. The Fibonacci substitution is defined by:

$$\phi : 0 \mapsto 01$$
$$1 \mapsto 0.$$

The fixed point of the Fibonacci substitution, f=0100101001001010010100100101..., is a Sturmian sequence, and (X_f^+, σ) is the Sturmian system associated with f (see [**18**]).

An infinite sequence u is *periodic* (respectively *eventually periodic*) if there exists a positive integer M such that for every n, $u_n = u_{n+M}$ (respectively there exists $m \in \mathbb{N}$ such that for all $|n| \geq m$, $u_n = u_{n+M}$). (Note that there is some possibility for variation in this definition). A set S of blocks is *balanced* if for any pair of blocks u, v of the same length in S, $||u|_1 - |v|_1| \leq 1$, where $|u|_1$ is the number of occurrences of 1 in u and $|v|_1$ is the number of occurrences of 1 in v. It follows that if a sequence u is balanced and not eventually periodic then it is Sturmian. This is a result of the fact that if u is aperiodic, then $p_u(n) \geq n+1$ for all n, and if u is balanced then $p_u(n) \leq n+1$ for all n. In fact, it can be proved that a sequence u is balanced and aperiodic if and only if it is Sturmian [**18**]. Furthermore, any shift of a Sturmian sequence is also Sturmian.

Sturmian sequences also have a natural association to lines with irrational slope. To see this, we introduce the following definitions. Let α and β be real numbers with $0 \leq \alpha, \beta \leq 1$. We define two infinite sequences $x_{\alpha,\beta}$ and ${x'}_{\alpha,\beta}$ by

$$\begin{aligned} (x_{\alpha,\beta})_n &= \lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor \\ ({x'}_{\alpha,\beta})_n &= \lceil \alpha(n+1) + \beta \rceil - \lceil \alpha n + \beta \rceil \end{aligned}$$

for all $n \geq 0$. The sequence $x_{\alpha,\beta}$ is the *lower mechanical sequence* and ${x'}_{\alpha,\beta}$ is the *upper mechanical sequence* with slope α and intercept β. The use of the words slope and intercept in the above definitions stems from the following graphical interpretation. The points with integer coordinates that sit just below the line $y = \alpha x + \beta$ are $F_n = (n, \lfloor \alpha n + \beta \rfloor)$. The straight line segment connecting two consecutive points F_n and F_{n+1} is horizontal if $x_{\alpha,\beta} = 0$ and diagonal if $x_{\alpha,\beta} = 1$. The lower mechanical sequence is a coding of the line $y = \alpha x + \beta$ by assigning to each line segment connecting F_n and F_{n+1} a 0 if the segment is horizontal and a 1 if the segment is diagonal. Similarly, the points with integer coordinates that sit just above this line are $F'_n = (n, \lceil \alpha n + \beta \rceil)$. Again, we can code the line $y = \alpha x + \beta$ by assigning to each line segment connecting F'_n and F'_{n+1} a 0 if the segment is horizontal and a 1 if the segment is diagonal. This coding yields the upper mechanical sequence [**18**].

A mechanical sequence is *rational* if the line $y = \alpha x + \beta$ has rational slope and *irrational* if $y = \alpha x + \beta$ has irrational slope. In [**18**] it is proved that a sequence u is Sturmian if and only if u is irrational mechanical. In the following example we construct a lower mechanical sequence with irrational slope, thus producing a Sturmian sequence.

EXAMPLE 4.2. Let $\alpha = 1/\tau^2$, where $\tau = (1+\sqrt{5})/2$ is the golden mean, and $\beta = 0$. The lower mechanical sequence $x_{\alpha,\beta}$ is constructed as follows:

$$\begin{aligned}
(x_{\alpha,\beta})_0 &= \lfloor 1/\tau^2 \rfloor = 0\\
(x_{\alpha,\beta})_1 &= \lfloor 2/\tau^2 \rfloor - \lfloor 1/\tau^2 \rfloor = 0\\
(x_{\alpha,\beta})_2 &= \lfloor 3/\tau^2 \rfloor - \lfloor 2/\tau^2 \rfloor = 1\\
(x_{\alpha,\beta})_3 &= \lfloor 4/\tau^2 \rfloor - \lfloor 3/\tau^2 \rfloor = 0\\
(x_{\alpha,\beta})_4 &= \lfloor 5/\tau^2 \rfloor - \lfloor 4/\tau^2 \rfloor = 0\\
(x_{\alpha,\beta})_5 &= \lfloor 6/\tau^2 \rfloor - \lfloor 5/\tau^2 \rfloor = 1\\
&\vdots
\end{aligned}$$

Further calculation shows that $x_{\alpha,\beta} = 0010010100... = 0f$, and $x'_{\alpha,\beta} = 1010010100... = 1f$ (recall (Example 4.1) that f denotes the fixed point of the Fibonacci substitution), hence the fixed point f is a shift of the lower and upper mechanical sequences with slope $1/\tau^2$ and intercept 0.

We now consider the language of a Sturmian sequence u. It is easy to show that while Sturmian sequences are aperiodic, they are syndetically recurrent [**9**]. As a result, any block in $\mathcal{L}_n(u)$ appears past the initial position and can thus be extended on the left. Since there are $n+1$ blocks of length n, it must be that exactly one of them can be extended to the left in two ways. In a Sturmian sequence u, the unique block of length n that can be extended to the left in two different ways is called a *left special block*, and is denoted $L_n(u)$. The sequence $l(u)$ which has the $L_n(u)$'s as prefixes is called the *left special sequence* or *characteristic word* of X_u^+ [**9**, **18**]. Similarly, in a Sturmian sequence u, the unique block of length n that can be extended to the right in two different ways is called a *right special block*, and is denoted $R_n(u)$. The block $R_n(u)$ is precisely the reverse of $L_n(u)$ [**9**].

4.2. The left special sequence. Since every Sturmian sequence u is irrational mechanical, there is a line with irrational slope α associated to u. This α can be used to determine the left special sequence of X_u^+.

Let $(d_1, d_2, ..., d_n, ...)$ be a sequence of integers with $d_1 \geq 0$ and $d_n > 0$ for $n > 1$. We associate a sequence $(s_n)_{n\geq -1}$ of blocks to this sequence by

$$s_{-1} = 1, \qquad s_0 = 0, \qquad s_n = s_{n-1}^{d_n} s_{n-2}.$$

The sequence $(s_n)_{n\geq -1}$ is a *standard sequence*, and $(d_1, d_2, ..., d_n, ...)$ is its *directive sequence*. We can then determine the left special sequence of X_u^+ with the following proposition stated in [**18**].

PROPOSITION 4.3. *Let $\alpha = [0, 1+d_1, d_2, ...]$ be the continued fraction expansion of an irrational α with $0 < \alpha < 1$, and let (s_n) be the standard sequence associated to $(d_1, d_2, ...)$. Then every s_n, $n \geq 1$, is a prefix of the left special sequence l of X_u^+ and*

$$l = \lim_{n\to\infty} s_n.$$

This is illustrated in the following two examples.

EXAMPLE 4.4. Let $\alpha = 1/\tau^2$, where $\tau = (1+\sqrt{5})/2$ is the golden mean. The continued fraction expansion of $1/\tau^2$ is $[0, 2, \overline{1}]$ where $\overline{1} = 1^\infty = 111\ldots$. By the above proposition $d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 1, ...$ The standard sequence associated to $(d_1, d_2, ...)$ is constructed as follows:

$$\begin{aligned} s_1 =& s_0^{d_1} s_{-1} = 01 \\ s_2 =& s_1^{d_2} s_0 = 010 \\ s_3 =& s_2^{d_3} s_1 = 01001 \\ s_4 =& s_3^{d_4} s_2 = 01001010 \\ &\vdots \end{aligned}$$

Continuing this process, the left special sequence of X_u^+, where u is a coding of a line with slope $1/\tau^2$, is

$$l = 010010100100101001... = f.$$

It follows that the left special sequence of X_f^+ is f.

EXAMPLE 4.5. Let $\alpha = \pi/4$. The continued fraction expansion of $\pi/4$ is

$$[0, 1, 3, 1, 1, 1, 15, 2, 72, ...].$$

By Proposition 4.3 $d_1 = 0, d_2 = 3, d_3 = 1, d_4 = 1, ...$ Then,

$$\begin{aligned} s_1 =& s_0^{d_1} s_{-1} = 1 \\ s_2 =& s_1^{d_2} s_0 = 1110 \\ s_3 =& s_2^{d_3} s_1 = 11101 \\ s_4 =& s_3^{d_4} s_2 = 111011110 \\ &\vdots \end{aligned}$$

Continuing this process, the left special sequence of X_u^+, where u is a coding of a line with slope $\pi/4$, is

$$l = 11101111011101111011110....$$

4.3. Significant blocks of a one-sided Sturmian system. In order to construct the HB diagram of a Sturmian system, it is necessary to identify the significant blocks of the system. We first note that if X is any subshift of Σ_d, then $0, 1, ..., d-1$, and d are significant blocks provided $0, 1, ..., d-1$, and d are all in $\mathcal{L}(X^{+-})$. Hence, 0 and 1 are significant blocks of any Sturmian system. Let (X_u^+, σ) be a Sturmian system with $l = l_1 l_2 l_3 ...$ the left special sequence of u. In the next two propositions we prove that given $n \geq 1$, there are exactly two significant blocks of $\tilde{X}_u$ with length n.

The first proposition applies to any subshift of Σ_2.

Proposition 4.6. *Let $X \subset \Sigma_2$ be a subshift. Suppose $a_{-n+1}a_{-n+2}...a_{-1}a_0$ is a block of length n in $\mathcal{L}(X^{+-})$. If $a_{-n+1}a_{-n+2}...a_{-1}a_0$ is significant, then $0a_{-n+2}...a_{-1}a_0$ and $1a_{-n+2}...a_{-1}a_0$ are in $\mathcal{L}(X^{+-})$.*

Proof. Assume that the block $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ is significant, so that there is a ray $b_0 b_1 \cdots \in \text{fol}(a_{-n+1}\dots a_0) \subsetneq \text{fol}(a_{-n+2}\dots a_0)$. Then there is $b \in X^{+-}$ such that $b_{-n+2}\dots b_0 = a_{-n+2}\dots a_0$, and we must have $b_{-n+1} \neq a_{-n+1}$. □

Before we state the second proposition, recall that since u is recurrent, every block is extendable to the left. Hence, $\mathcal{L}(u) = \mathcal{L}(X_u^+) = \mathcal{L}(X_u^{+-})$. We abbreviate $L_k = L_k(u)$, the left special block of length k in u.

Proposition 4.7. *Let u be a Sturmian sequence, with (X_u^+, σ) its associated Sturmian system. Then for each $n \geq 2$ there are exactly two significant blocks of $\tilde{X}_u$ of length n. The two significant blocks of length n are $0L_{n-1} = 0l_1...l_{n-1}$ and $1L_{n-1} = 1l_1...l_{n-1}$.*

Proof. By Proposition 4.6 we know that if the block $a_{-n+1}a_{-n+3}...a_{-2}a_{-1}a_0$ is significant then $0a_{-n+2}...a_{-2}a_{-1}a_0$ and $1a_{-n+2}...a_{-2}a_{-1}a_0$ are in $\mathcal{L}(X_u^{+-})$. Thus the only possible significant blocks of length n are those blocks $a_{-n+1}...a_{-2}a_{-1}a_0$ such that $a_{-n+2}...a_{-2}a_{-1}a_0$ can be extended to the left in two ways. That is, the possible significant blocks are $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ with

$$a_{-n+2}...a_{-2}a_{-1}a_0 = L_{n-1}.$$

Let $L_{n-1} = a_{-n+2}...a_0$ and $\nu \in \mathcal{L}(u)$. Since there is exactly one right special block of each length $n \in \mathbb{N}$, $0L_{n-1}\nu$ and $1L_{n-1}\nu$ cannot both be right special blocks. We first prove that $1L_{n-1}$ is significant by considering the following two cases.

Case 1: There exists $\nu \in \mathcal{L}(u)$ such that $0L_{n-1}\nu$ is right special. Then $1L_{n-1}\nu$ is not right special. This implies that $1L_{n-1}\nu 1$ is not in $\mathcal{L}(u)$, since

$$||0L_{n-1}\nu 0|_1 - |1L_{n-1}\nu 1|_1| = 2,$$

which is not permitted as u is balanced. Thus there exists a ray $a_0\nu 1...$ in $\text{fol}(L_{n-1})$ that is not in $\text{fol}(1L_{n-1})$. Hence, $1L_{n-1}$ is significant.

Case 2: There does not exist $\nu \in \mathcal{L}(u)$ such that $0L_{n-1}\nu$ is right special. This implies that there exists exactly one ray $b_1 b_2...$ that can follow $0L_{n-1}$. We claim that because u is Sturmian such a case cannot occur.

Let $u = c_0c_1c_2c_3....$ Since $0L_{n-1} \in \mathcal{L}(u)$, we know that $0L_{n-1}$ appears in u infinitely many times. Suppose $0L_{n-1}$ appears for the first time starting at position c_{m+1}. Letting $0L_{n-1} = a_{-n+2}...a_0$, we have

$$u = c_0c_1c_2c_3...c_m0L_{n-1}b_1b_2... = c_0c_1c_2c_3...c_m0a_{-n+2}...a_0b_1b_2...$$

Furthermore, there exists $r \in \mathbb{N}$ such that $0L_{n-1}$ appears again starting at b_{r+1}. As $0L_{n-1}$ can be followed only by $b_1b_2...$, this implies that

$$u = c_0c_1c_2c_3...c_m0a_{-n+2}...a_0b_1b_2...b_r0a_{-n+2}...a_0b_1b_2...b_r0a_{-n+2}...a_0b_1b_2...b_r...$$

Letting $B = 0a_{-n+2}...a_0b_1b_2...b_r$, we have that

$$u = c_0c_1c_2c_3...c_mBBBB....$$

Thus, u is eventually periodic. This, however, is a contradiction as Sturmian sequences are not eventually periodic. Hence $1L_{n-1}$ is a significant block of X_u^{+-}.

By the same argument, it can be shown that $0L_{n-1}$ is also a significant block of X_u^{+-}. □

4.4. Construction of the diagram. Recall that the HB diagram of a one-sided Sturmian system is defined to be an oriented graph whose vertices are the significant blocks of $\tilde{X}_u$ and whose arrows are defined by

$$a_{-n}...a_0 \to b_{-m}...b_0$$

if and only if $a_{-n}...a_0b_0 \in \mathcal{L}(\tilde{X}_u)$ and

$$b_{-m}...b_0 = \text{sig}(a_{-n}...a_0b_0).$$

Having determined the significant blocks of $\tilde{X}_u$, it remains only to determine the arrows. This will give us a complete description of the HB diagram of an arbitrary one-sided Sturmian system.

Let $l = l_1l_2l_3...$ be the left special sequence of X_u^+. We first consider the arrows leaving the significant blocks of length 1.

LEMMA 4.8. *If* $l_1 = 0$, *then* $0 \to 1$, $0 \to 00$ *and* $1 \to 10$. *If* $l_1 = 1$, *then* $1 \to 0$, $1 \to 11$ *and* $0 \to 01$.

PROOF. Suppose $l_1 = 0$. By definition

$$0 \to b_{-m}...b_0$$

if and only if $0b_0 \in \mathcal{L}(\tilde{X}_u)$ and

$$b_{-m}...b_0 = \text{sig}(0b_0).$$

As $b_0 \in \{0, 1\}$, we consider sig(00) and sig(01). Proposition 4.7 implies that the significant blocks of length two are 00 and 10, since $l_1 = 0$. Thus sig(00) = 00 and sig(01) = 1. Hence, $0 \to 1$ and $0 \to 00$. Additionally, consider sig($1b_0$). Since 0 is the unique right special block of length one, the balance property implies that $11 \notin \mathcal{L}(\tilde{X}_u)$. Thus there is exactly one arrow leaving the block 1, $1 \to$ sig(10), where sig(10) = 10 by Proposition 4.7.

Similarly, if $l_1 = 1$ we consider sig(10) and sig(11). In this case, Proposition 4.7 implies that the significant blocks of length two are 10 and 11. Thus sig(10) = 0 and sig(11) = 11 and $1 \to 0$ and $1 \to 11$. Furthermore, the only arrow leaving 0 is given by $0 \to$ sig(01), where sig(01) = 01 by Proposition 4.7.

□

Now consider an arbitrary significant block $xl_1l_2...l_{n-1}$, where x is either 0 or 1. Again,

$$xl_1l_2...l_{n-1} \to b_{-m}...b_0$$

if and only if $xl_1l_2...l_{n-1}b_0 \in \mathcal{L}(\tilde{X}_u)$ and

$$b_{-m}...b_0 = \text{sig}(xl_1l_2...l_{n-1}b_0).$$

We consider what may follow $xl_1l_2...l_{n-1}$. There can be at most two arrows out of $xl_1l_2...l_{n-1}$, as $b_0 \in \{0,1\}$. It is always the case that $xl_1l_2...l_{n-1}l_n \in \mathcal{L}(\tilde{X}_u)$. Letting $b_0 = l_n$, we get

$$xl_1l_2...l_{n-1} \to b_{-m}...b_{-1}l_n$$

if and only if

$$b_{-m}...b_{-1}l_n = \text{sig}(xl_1l_2...l_{n-1}l_n).$$

However, $xl_1l_2...l_{n-1}l_n$ is significant; thus it must be that $b_{-m}...b_0 = xl_1l_2...l_{n-1}l_n$. Hence, we are guaranteed the arrow

$$xl_1l_2...l_{n-1} \to xl_1l_2...l_n.$$

This is stated below.

LEMMA 4.9. *Let* $xL_{n-1} = xl_1l_2...l_{n-1}$, $n > 1$, $x \in \{0,1\}$, *be a significant block of* $\tilde{X}_u$. *Then*

$$xl_1l_2...l_{n-1} \to xl_1l_2...l_n.$$

It follows from Lemma 4.9 that the left special sequence of X_u^+ is seen in the diagram by reading off the last symbol in the paths $0l_1 \to 0l_1l_2 \to 0l_1l_2l_3 \to \cdots$ and $1l_1 \to 1l_1l_2 \to 1l_1l_2l_3 \to \cdots$.

Now suppose $xl_1l_2...l_{n-1}y \in \mathcal{L}(\tilde{X}_u)$ and $y \neq l_n$. This occurs if and only if $xl_1l_2...l_{n-1}$ is a right special block. Since $R_n(u)$ is the reverse of $L_n(u) = l_1l_2...l_n$, we get the following lemma.

LEMMA 4.10. *Let* $xl_1l_2...l_{n-1}$ *be a significant block of* $\tilde{X}_u$. *In the HB diagram of* X_u^+, *two arrows leave* $xl_1l_2...l_{n-1}$ *if and only if* $xl_1l_2...l_{n-1}$ *is a right special block, equivalently if and only if* $xl_1l_2...l_{n-1} = l_nl_{n-1}...l_2l_1$.

Suppose $xl_1l_2...l_{n-1}$ is a right special significant block, where $n > 1$. Let $wl_1...l_{m-1}$, $1 \leq m < n$ be the *previous* right special significant block of $\tilde{X}_u$. In other words, there is no right special significant block of length greater than m and less than n. By definition $xl_1l_2...l_{n-1} = l_nl_{n-1}...l_2l_1$ and $wl_1...l_{m-1} = l_ml_{m-1}...l_2l_1$. We claim the following.

LEMMA 4.11. *Let* $xl_1l_2...l_{n-1}$ *and* $wl_1...l_{m-1}$ *be consecutive right special significant blocks as described and suppose* $y \neq l_n$. *Then*

$$\text{sig}(xl_1l_2...l_{n-1}y) = \text{sig}(wl_1...l_{m-1}y).$$

PROOF. Since $y \neq l_n$ it follows that $\text{sig}(xl_1l_2...l_{n-1}y) \neq xl_1l_2...l_{n-1}y$. Suppose to the contrary that

$$\text{sig}(xl_1l_2...l_{n-1}y) = \text{sig}(l_nl_{n-1}...l_2l_1y) = l_{m+i}...l_ml_{m-1}...l_2l_1y,$$

for some $i \geq 1$ with $m + i < n$.

Then $l_{m+i}...l_m l_{m-1}...l_2 l_1 y = z l_1 ... l_{m+i}$ for some $z \in \{0,1\}$, since $l_{m+i}...l_m l_{m-1}...l_2 l_1 y$ is significant. However, this implies that

$$l_{m+i}...l_m l_{m-1}...l_2 l_1 = z l_1 ... l_{m+i-1}$$

is right special. This is a contradiction, since $l_{m+i}...l_m l_{m-1}...l_2 l_1$ is a right special block of length $m+i$, with $m < m+i < n$, and $w l_1 ... l_{m-1} = l_m l_{m-1} ... l_2 l_1$ is the previous right special significant block. Hence,

$$sig(x l_1 l_2 ... l_{n-1} y) = sig(w l_1 ... l_{m-1} y).$$

□

We use the following lemma to determine the remaining arrows.

LEMMA 4.12. *Let $x l_1 l_2 ... l_{n-1}$ and $w l_1 l_2 ... l_{m-1}$, $1 \leq m < n$ be consecutive right special significant blocks as described and suppose $y \neq l_n$. If $x \neq w$, then*

$$x l_1 l_2 ... l_{n-1} \to w l_1 l_2 ... l_{m-1} l_m.$$

If $x = w$ then,

$$x l_1 l_2 ... l_{n-1} \to \mathrm{sig}(w l_1 l_2 ... l_{m-1} y).$$

PROOF. By Lemma 4.11,

$$\mathrm{sig}(x l_1 l_2 ... l_{n-1} y) = \mathrm{sig}(w l_1 ... l_{m-1} y) = \mathrm{sig}(l_m l_{m-1} ... l_2 l_1 l_m).$$

By Proposition 4.7

$$\mathrm{sig}(w l_1 ... l_{m-1} l_m) = w l_1 ... l_{m-1} l_m.$$

This gives us the arrow

$$x l_1 l_2 ... l_{n-1} \to w l_1 ... l_{m-1} l_m.$$

Suppose $x \neq w$. We know that $x l_1 l_2 ... l_{n-1} = l_n l_{n-1} ... l_2 l_1$ and $w l_1 ... l_{m-1} = l_m l_{m-1} ... l_2 l_1$, so $x \neq w$ implies that $l_n \neq l_m$. Additionally, $y \neq l_n$ implies $y = l_m$.

Now suppose $x = w$. Then $l_n = l_m$, and thus $y \neq l_m$. Therefore

$$x l_1 l_2 ... l_{n-1} \to \mathrm{sig}(w l_1 ... l_{m-1} y).$$

That is, there is an arrow leaving $x l_1 l_2 ... l_{n-1}$ that points to the same significant block as one of the arrows leaving $w L_{m-1}$. □

We summarize the construction of the HB diagram of an arbitrary one-sided Sturmian system in the following theorem.

THEOREM 4.13. *Let X_u^+ be a one-sided Sturmian system, with $l = l_1 l_2 l_3 ...$ the left special sequence of u, and $L_n = l_1 \ldots l_n$ for each $n \geq 1$. The HB diagram of X_u^+ is the directed graph with vertices $0, 1, 0L_n$, and $1L_n$, $n \geq 1$, and whose arrows are defined by*

- *(a) $0 \to 1$, $0 \to 00$, and $1 \to 10$ if $l_1 = 0$, and $1 \to 0$, $1 \to 11$, and $0 \to 01$ if $l_1 = 1$,*
- *(b) $0L_n \to 0L_{n+1}$, $1L_n \to 1L_{n+1}$,*
- *(c) If xL_n and wL_m, $n \geq m$, are consecutive right special blocks*
  - *(i) $xL_n \to wL_{m+1}$ if $x \neq w$*
  - *(ii) $xL_n \to \mathrm{sig}(wL_m y)$, $y \neq l_{m+1}$, if $x = w$.*

We describe the construction of the HB diagrams of two Sturmian systems. Recall from Example 4.1 that the Fibonacci Sturmian system is (X_f^+, σ), where $f = 0100101001001010010100....$ In Example 4.4 it is shown that the left special sequence of X_f^+ is f. By Proposition 4.7, the significant blocks of $\tilde{X}_f$ are

$$0, 1, 00, 10, 001, 101, 0010, 1010, 00100, 10100, ...$$

Furthermore, the first few right special significant blocks are $0, 10$, 0010, and 1010010. Following Theorem 4.13, we construct a portion of the HB diagram of X_f^+, as depicted in Figure 1.

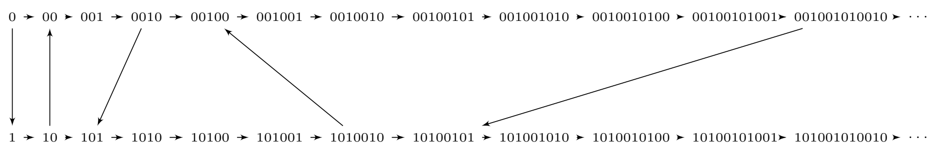

FIGURE 1. The HB diagram of X_f.

Next consider the sequence u, where u is the upper or lower mechanical sequence with slope $\alpha = \pi/4$ and intercept β, $\beta \leq 1$. Let (X_u^+, σ) be the Sturmian system associated with the sequence u. Earlier we found that the left special sequence of X_u^+ is

$$l = 1110111101110111101110....$$

Applying Proposition 4.7, the significant blocks of $\tilde{X_u}$ are

$$0, 1, 01, 11, 011, 111, 0111, 1111, 01110, 11110, ...,$$

and the first few right special significant blocks are $1, 11, 111$, 0111 and 11110111. Following Theorem 4.13, we begin construction of the HB diagram of X_u^+, as depicted in Figure 2.

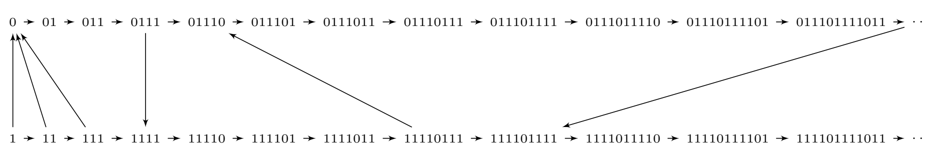

FIGURE 2. The HB diagram of X_u^+ where u is the upper or lower mechanical sequence with slope $\alpha = \pi/4$.

Unlike the situation with Markovian systems, in these diagrams there are no closed paths (since there are no periodic sequences in the subshifts).

5. General properties of HB diagrams

We consider next what HB diagrams can tell us about their associated systems. We first consider the properties of the HB diagram that hold for any subshift.

Let X^+ be a one-sided subshift with natural extension X^{+-} as previously defined.

LEMMA 5.1. *Suppose* $c_k c_{k-1} ... c_1 c_0$ *is a block in* $\mathcal{L}(X^{+-})$. *Then*

$$\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0) = \text{sig}(c_k c_{k-1} ... c_1 c_0).$$

PROOF. Let $|\text{sig}(c_k c_{k-1} ... c_1)|$ denote the length of $\text{sig}(c_k c_{k-1} ... c_1)$. Then

$$|\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0)| \leq |\text{sig}(c_k c_{k-1} ... c_1 c_0)|,$$

since $|\text{sig}(c_k c_{k-1} ... c_1)| \leq |c_k c_{k-1} ... c_1|$.

Suppose on the contrary that

$$|\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0)| < |\text{sig}(c_k c_{k-1} ... c_1 c_0)|.$$

Furthermore, suppose $\text{sig}(c_k c_{k-1} ... c_1) = c_j ... c_1$, where $1 \leq j \leq k$.

Then

$$\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0) = \text{sig}(c_j ... c_1 c_0) = c_m ... c_1 c_0$$

with $0 \leq m \leq j$, and

$$\text{sig}(c_k c_{k-1} ... c_1 c_0) = c_r ... c_1 c_0$$

with $m < r \leq k$.

Since $c_r ... c_1 c_0$ is significant, we know that $\text{fol}(c_r ... c_1 c_0) \subsetneq \text{fol}(c_{r-1} ... c_1 c_0)$. Thus there exists a one-sided ray $b_0 b_1 b_2 ...$ in $\text{fol}(c_{r-1} ... c_1 c_0)$ that is not in $\text{fol}(c_r ... c_1 c_0)$. Furthermore, $b_0 b_1 b_2 ...$ is such that there exists a two-sided ray b in the natural extension of X^+, with $b_{-r+1} ... b_0 = c_{r-1} ... c_0$. However, this implies that $b_{-1} b_0 b_1 ...$ is a one-sided ray in $\text{fol}(c_{r-1} ... c_1)$ that is not in $\text{fol}(c_r ... c_1)$. Hence, $c_r ... c_1$ is significant.

It follows that $r \leq j$. If not $\text{sig}(c_k c_{k-1} ... c_1) = c_r ... c_1$. However, we have assumed $r > m$. This contradicts $\text{sig}(c_j ... c_1 c_0) = c_m ... c_1 c_0$, since $c_r ... c_1 c_0$ is a longer significant block than $c_m ... c_1 c_0$ that is also a suffix of $c_j ... c_1 c_0$. Thus,

$$|\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0)| = |\text{sig}(c_k c_{k-1} ... c_1 c_0)|,$$

and so

$$\text{sig}(\text{sig}(c_k c_{k-1} ... c_1) c_0) = \text{sig}(c_k c_{k-1} ... c_1 c_0).$$

□

The following corollary, which is stated as an exercise in [**2**], follows from Lemma 5.1 by induction.

COROLLARY 5.2. *Let* $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$ *be a finite path on a HB diagram* $\mathcal{D}$. *Suppose* $\alpha_0 = b_{-k} ... b_0$ *and let* a_i *be the last letter of* α_i. *Then for all* $n \in \mathbb{N}$

$$\alpha_n = \text{sig}(b_{-k} ... b_0 a_1 a_2 ... a_n).$$

Shifting our attention to the paths on $\mathcal{D}$, we define the *length* of a path to be the number of vertices in the path. From Corollary 5.2 we get the following result.

THEOREM 5.3. *Let* X *be a subshift with HB diagram* $\mathcal{D}$. *If* $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{n-1}$ *and* $\beta_0 \to \beta_1 \to \cdots \beta_{n-1}$ *are two paths of length* n *on* $\mathcal{D}$ *with* α_0 *and* β_0 *blocks of length 1, then* $\hat{\pi}(\alpha_0 \alpha_1 ... \alpha_{n-1}) = \hat{\pi}(\beta_0 \beta_1 ... \beta_{n-1})$ *if and only if* $\alpha_i = \beta_i$ *for all* $0 \leq i \leq n-1$.

PROOF. Let a_i and b_i be the last letters of α_i and β_i respectively.

Suppose $\hat{\pi}(\alpha_0\alpha_1...\alpha_{n-1}) = \hat{\pi}(\beta_0\beta_1...\beta_{n-1})$. By Corollary 5.2,

$$\alpha_i = \text{sig}(\alpha_0 a_1 ... a_i) = \text{sig}(a_0 a_1 ... a_i)$$

and

$$\beta_i = \text{sig}(\beta_0 b_1 ... b_i) = \text{sig}(b_0 b_1 ... b_i).$$

Furthermore, $a_0 a_1 ... a_i = \hat{\pi}(\alpha_0\alpha_1...\alpha_i)$ and $b_0 b_1 ... b_i = \hat{\pi}(\beta_0\beta_1...\beta_i)$. As $\hat{\pi}(\alpha_0\alpha_1...\alpha_i)$ and $\hat{\pi}(\beta_0\beta_1...\beta_i)$ are prefixes of $\hat{\pi}(\alpha_0\alpha_1...\alpha_{n-1})$ and $\hat{\pi}(\beta_0\beta_1...\beta_{n-1})$ respectively, it follows that

$$\hat{\pi}(\alpha_0\alpha_1...\alpha_i) = \hat{\pi}(\beta_0\beta_1...\beta_i).$$

Hence,

$$\alpha_i = \beta_i = \text{sig}(\hat{\pi}(\alpha_0\alpha_1...\alpha_i)) = \text{sig}(\hat{\pi}(\beta_0\beta_1...\beta_i)).$$

Now suppose $\alpha_i = \beta_i$ for all $0 \le i \le n-1$. Then $a_i = b_i$ for all $0 \le i \le n-1$. Thus $a_0...a_{n-1} = b_0...b_{n-1}$. By definition, $\hat{\pi}(\alpha_0\alpha_1...\alpha_{n-1}) = a_0...a_{n-1}$ and $\hat{\pi}(\beta_0\beta_1...\beta_{n-1}) = b_0...b_{n-1}$. Hence $\hat{\pi}(\alpha_0\alpha_1...\alpha_{n-1}) = \hat{\pi}(\beta_0\beta_1...\beta_{n-1})$. □

It follows from Theorem 5.3 that on the HB diagram $\mathcal{D}$ of a subshift X each distinct path of length n starting at a block of length one projects to a distinct block of length n in $\mathcal{L}(X)$.

Now suppose B is a block in $\mathcal{L}(X)$. We ask, does there exist a path on $\mathcal{D}$ starting with a block of length one that projects to the block B? In general, this is not the case.

EXAMPLE 5.4. Let $u = 1\bar{0}$ and consider the system (X^+, σ) where X^+ is the orbit closure of $\{\sigma^n u | n \in \mathbb{N}\}$. We claim that $1 \notin \mathcal{L}(X^{+-})$. For 1 to be in $\mathcal{L}(X^{+-})$, 1 must appear in a two-sided sequence b such that $b_p b_{p+1} ... \in X^+$ for all $p \in \mathbb{Z}$. Suppose

$$b = ...b_{-n-2} 1 b_{-n} ... b_{-2} b_{-1} b_0 b_1 b_1 ....$$

Then b_{-n-2} cannot equal 0 since $01 \notin \mathcal{L}(X^+)$, and b_{-n-2} cannot equal 1 since $11 \notin \mathcal{L}(X^+)$. Thus, 1 does not appear in any two-sided sequence b with the property that $b_p b_{p+1} ... \in X^+$ for all $p \in \mathbb{Z}$. Hence, $1 \notin \mathcal{L}(X^{+-})$. This implies that 1 is not a vertex in the HB diagram $\mathcal{D}$ for X^+. As a result, there is no path on $\mathcal{D}$ starting with a block of length one that projects to any block in $\mathcal{L}(X^+)$ that begins with a 1.

In this example, we see that the relationship between $\mathcal{L}(X)$ and $\mathcal{L}(X^{+-})$ depends on the paths that appear on $\mathcal{D}$. If X is a two-sided subshift, obviously $X^{+-} = X$ and $\mathcal{L}(X) = \mathcal{L}(X^{+-})$. In contrast, if X^+ is a one-sided subshift it is not as easy to determine whether $\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$. Recall that

$$X^{+-} = \{x \in \mathcal{A}^{\mathbb{Z}} | \text{ for all } p \in \mathbb{Z}, \ \ x_p x_{p+1} ... \in X^+\}.$$

Hence, $\mathcal{L}(X^{+-}) \subset \mathcal{L}(X^+)$. We provide a construction of X^{+-} and thus a necessary and sufficient condition on X^+ such that $\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$.

Let $a^{(n)} = a_0^{(n)} a_1^{(n)} a_2^{(n)} a_3^{(n)} ...$ be points in X^+. We construct a sequence $(x_n(a^{(n)}))$ of two-sided sequences as follows. Let $b^{(n)} = 0^\infty . a^{(n)}$ and set $x_n(a^{(n)}) = \sigma^n b^{(n)}$.

PROPOSITION 5.5. *Let X^+ and $(x_n(a^{(n)}))$ be as described above. Then X^{+-} is the set of limit points of all $(x_n(a^{(n)}))$, $a^{(n)} \in X^+$ for all $n \ge 0$.*

PROOF. By compactness, the sequence $(x_n(a^{(n)}))$ has a limit point x. That is, there exists a subsequence $(x_{n_k}(a^{(n_k)}))$ of $(x_n(a^{(n)}))$ that converges to x. We claim that any such limit point is in X^{+-}. Suppose on the contrary that $x \notin X^{+-}$. Then there exists $p \in \mathbb{Z}$ such that $x_p x_{p+1} x_{p+2} \ldots \notin X^+$. This, however, is impossible since the initial blocks of any right ray in x can be found as the initial blocks of a point in X^+ by construction. Hence, $x_p x_{p+1} x_{p+2} \ldots$ is in the closure of X^+ and thus is in X^+.

Conversely, let $b = \ldots b_{-3} b_{-2} b_{-1} . b_0 b_1 b_2 b_3 \ldots$ be an arbitrary bisequence in X^{+-}. We show that b is a limit point of a subsequence of $(x_n(a^{(n)}))$, for some $a^{(n)} \in X^+$. Let
$a^{(n)} = \pi_{X^+}(\sigma^{-n} b)$, where π_{X^+} is as defined earlier. That is

$$\begin{aligned} a^{(1)} &= \pi_{X^+}(\sigma^{-1} b) = b_{-1} b_0 b_1 \ldots \\ a^{(2)} &= \pi_{X^+}(\sigma^{-2} b) = b_{-2} b_{-1} b_0 \ldots \\ &\vdots \\ a^{(n)} &= \pi_{X^+}(\sigma^{-n} b) = b_{-n} b_{-n+1} b_{-n+2} \ldots . \end{aligned}$$

It follows that,

$$b = \lim_{n \to \infty} x_n(a^{(n)}).$$

Thus any point in X^{+-} is a limit point of a subsequence of $(x_n(a^{(n)}))$. □

COROLLARY 5.6. *$\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$ if and only if for every block B in $\mathcal{L}(X^+)$ and for all $n \geq 0$ there exists $a^{(n)} \in X^+$ such that B appears in $a^{(n)}$ starting at position n.*

PROOF. Suppose that $B \in \mathcal{L}(X^+)$ and for all $n \geq 0$ there exists $a^{(n)} = a_0^{(n)} a_1^{(n)} a_2^{(n)} \ldots \in X^+$ such that B appears in $a^{(n)}$ starting at position n. Construct the sequence $(x_n(a^{(n)})$ as defined previously. For all $n \geq 0$,

$$x_n(a^{(n)}) = 0^\infty a_0^{(n)} \ldots a_{n-1}^{(n)} . B a_{|B|+n}^{(n)} \ldots .$$

Let x be any limit point of the sequence $(x_n(a^{(n)}))$. Then $x \in X^{+-}$ by Proposition 5.5 and $x = \ldots x_{-1} x_0 . B x_{|B|} \ldots$. That is, B appears in x starting at position 0. Hence $B \in \mathcal{L}(X^{+-})$.

Now assume $\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$ and $B \in \mathcal{L}(X^+)$ and $n \geq 0$ are given. Since $B \in \mathcal{L}(X^{+-})$, B appears in some $x \in X^{+-}$. Futhermore, there exists $m \in \mathbb{Z}$ such that $\sigma^m x$ has B appearing in position n. By definition, the ray $\pi_{X^+}(\sigma^m x)$ is in X^+. Setting $a^{(n)} = \pi_{X^+}(\sigma^m x)$, we have the desired result. □

The following Corollary is an immediate consequence of Corollary 5.6.

COROLLARY 5.7. *$\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$ if and only if every block in $\mathcal{L}(X^+)$ is left extendable. In particular, if X^+ is minimal, then $\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$.*

We now focus our attention on subshifts X such that $\mathcal{L}(X) = \mathcal{L}(X^{+-})$.

THEOREM 5.8. *Let X be a one or two-sided subshift with $\mathcal{L}(X) = \mathcal{L}(X^{+-})$. Let
$w = w_0 w_1 \ldots w_n$ be a block in $\mathcal{L}(X)$. Then there exists a unique path $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$ in the HB diagram of X with $\alpha_0 = w_0$ and $\hat{\pi}(\alpha_0 \alpha_1 \ldots \alpha_n) = w$.*

PROOF. Let $w = w_0w_1...w_n$ be a block in $\mathcal{L}(X)$. Since $w_0...w_i$ appears in w for $0 \le i \le n$, it follows that $w_0...w_i \in \mathcal{L}(X) = \mathcal{L}(X^{+-})$ for all $0 \le i \le n$. Set $\alpha_i = \mathrm{sig}(w_0w_1...w_i)$. By definition $\mathrm{sig}(w_0w_1...w_i)$ is a significant block in X^{+-} that ends with the letter w_i. It follows that if $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$ is a path on $\mathcal{D}$ then $\hat{\pi}(\alpha_0\alpha_1...\alpha_n) = w_0w_1...w_n$.

It remains to show that for $0 \le i \le n-1$ there exist arrows from $\alpha_i \to \alpha_{i+1}$ in the HB diagram of X. By definition $\alpha_i \to \alpha_{i+1}$ if and only if $\alpha_{i+1} = \mathrm{sig}(\alpha_i w_{i+1})$. Since $\alpha_i = \mathrm{sig}(w_0w_1...w_i)$, it suffices to show that $\alpha_{i+1} = \mathrm{sig}(\mathrm{sig}(w_0w_1...w_i)w_{i+1})$. Lemma 5.2 implies that

$$\mathrm{sig}(\mathrm{sig}(w_0w_1...w_i)w_{i+1}) = \mathrm{sig}(w_0w_1...w_iw_{i+1}),$$

where

$$\mathrm{sig}(w_0w_1...w_iw_{i+1}) = \alpha_{i+1}.$$

Thus, the desired path exists. Furthermore, this path is unique by Theorem 5.3. □

Here we have shown that if B is a block in $\mathcal{L}(X)$ then there exists a path on $\mathcal{D}$ starting with a block of length one that projects to the block B, provided $\mathcal{L}(X) = \mathcal{L}(X^{+-})$ (this condition is necessary—see Example 5.4). It immediately follows that if X is a subshift with $\mathcal{L}(X) = \mathcal{L}(X^{+-})$, then for any point x in the one-sided subshift X^+ there exists a unique path α in the HB diagram of X starting with a block of length one such that $\hat{\pi}^+(\alpha) = x$. Furthermore, from Theorems 5.3 and 5.8 we get the following corollary.

COROLLARY 5.9. *Let X be a one or two-sided subshift with $\mathcal{L}(X) = \mathcal{L}(X^{+-})$ and let p_X be the complexity function of X. Then the number of distinct paths of length n that occur on $\mathcal{D}$, the HB diagram of X, that begin with a block of length one is equal to $p_X(n)$.*

We provide an alternate statement and proof of Corollary 5.9 that is specific to Sturmian systems.

THEOREM 5.10. *Let (X_u^+, σ) be a Sturmian system. In the HB diagram of X_u^+, for $n \ge 1$ there are $p_u(n) = n + 1$ paths of length n starting from either the vertex labeled 0 or the vertex labeled 1.*

PROOF. We denote the number of paths of length n by P_n. Let $n = 1$. As any path can begin with 0 or 1, $P_1 = 2$. Next let $n = 2$. If 0 is right special, the paths of length two are $0 \to 00, 0 \to 1$, and $1 \to 10$. If 1 is right special, the paths of length two are $1 \to 11, 1 \to 0$, and $0 \to 01$. In either case, $P_2 = 3$.

We proceed by induction. Fix $n \ge 2$ and assume $P_{n-1} = p_u(n-1)$. Then $P_{n-1} = n$. We wish to show that $P_n = n + 1$. Consider the n distinct paths of length $n-1$. Because each of these paths can be continued, there are at least n paths of length n. Suppose there are $n+2$ paths of length n. From Theorem 5.3, each distinct path of length n starting with either 0 or 1 yields a distinct block of length n by reading off the last symbol of every vertex encountered. Then, that there are $n+2$ paths of length n implies that there are $n+2$ distinct blocks of length n in $\mathcal{L}(X_u^+)$. This contradicts $p_u(n) = n + 1$, so $n \le P_n < n + 2$.

To prove that $P_n = n + 1$, we show that exactly one of the paths of length $n-1$ can be continued in two ways. Consider the n paths of length $n-1$ with initial vertex 0 or 1. Each of these paths projects to a distinct block of length $n-1$, hence there is a path corresponding to every block in $\mathcal{L}_n(u)$. It follows that exactly one of these blocks is right special. Call this block $w = l_{n-1}l_{n-2}...l_1$, where $l = l_1l_2l_3...$ is the left special sequence of u. Let $\alpha_0 \to \cdots \to \alpha_{n-2}$ be the path that projects to w. That is $w = \hat{\pi}(\alpha_0...\alpha_{n-2}) = l_{n-1}l_{n-2}...l_1$. By Corollary 5.2, $\alpha_{n-2} = \text{sig}(l_{n-1}l_{n-2}...l_0)$. However, $\text{sig}(l_{n-1}l_{n-2}...l_0) = l_ml_{m-1}...l_0$, $0 \le m \le n-1$. Thus α_{n-2} is a right special significant block. This implies that the path $\alpha_0 \to \cdots \to \alpha_{n-2}$ can be continued in two ways. Thus, there are exactly $n+1$ paths of length n, as desired. □

In this proof, we not only show that there are $p_u(n)$ paths of length n with initial vertex 0 or 1, but we identify the path of length n that extends in two ways.

It follows from Corollary 5.9 that given a subshift X with the property that $\mathcal{L}(X) = \mathcal{L}(X^{+-})$, we can recover the complexity function for X by counting paths in the HB diagram of X.

Let X be any subshift and $\mathcal{D}$ its HB diagram. Recall that $X(\mathcal{D})$ is the set of all bi-infinite paths that occur on $\mathcal{D}$.

Definition 5.11. A sequence $a \in X^{+-}$ is *eventually Markov* at time $p \in \mathbb{Z}$ if there exists $N = N(x,p)$ such that for all $n \ge N$

$$\text{fol}(a_{p-n}...a_p) = \text{fol}(a_{p-N}...a_p).$$

The *eventually Markov part* $X^{+-}_M \subset X^{+-}$ is the set of $a \in X^{+-}$ which are eventually Markov at all times $p \in \mathbb{Z}$.

The following theorem, due to Hofbauer and Buzzi, shows that $\hat{\pi} : X(\mathcal{D}) \to X^{+-}_M$ is an isomorphism [**1**, **2**, **12**].

Theorem 5.12. *The natural projection $\hat{\pi}$ from the Hofbauer shift $X(\mathcal{D})$ to the subshift X^{+-} defined by*

$$\hat{\pi} : \alpha \in X(\mathcal{D}) \mapsto a \in X^{+-}$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{Z}$ is well defined and is a Borel isomorphism from $X(\mathcal{D})$ to X^{+-}_M.

Hence, one could say that $X(\mathcal{D})$ is "partially isomorphic" to X^{+-}.

It is apparent that the HB diagram of a Sturmian system X^+_u does not contain any bi-infinite paths, thus $X^+_u(\mathcal{D})$ is the empty set. This may seem alarming, but it turns out that the eventually Markov part of X^{+-}_u is also empty. In fact, we show that if the natural extension of a subshift is infinite and minimal, then the eventually Markov part of the natural extension is empty. Thus, it will follow that if X^+ is minimal the isomorphism in Theorem 5.12 is between two copies of the empty set. Nevertheless, Theorem 5.8 gives an isomorphism between $X(\mathcal{D})^+$ (paths in the HB diagram that start with blocks of length one, see Definition 3.7) and X^+.

Proposition 5.13. *If X^+ is a subshift such that X^{+-} is infinite and minimal, then the eventually Markov part of X^{+-} is empty.*

Proof. Suppose on the contrary that there exists $x \in X^{+-}$ that is eventually Markov at time $p \in \mathbb{Z}$. Then there exists $N = N(x,p)$ such that for all $n \ge N$,

$$\text{fol}(x_{p-n}...x_p) = \text{fol}(x_{p-N}...x_p).$$

Let $B = x_{p-N}...x_p$. Since x is left recurrent there exists $n > 2N+1$ such that $x_{p-n}...x_{p-n+N} = B$. Let $A = x_{p-n+N+1}...x_{p-N-1}$.

By the definition of X^{+-}, $x_{p-n}x_{p-n+1}... = BABx_{p+1}x_{p+2}... \in X^+$. Thus there exists a ray r_1 in fol(B) that has $x_{p-n+N}AB = x_pAB$ as a prefix. Since

$$\text{fol}(B) = \text{fol}(x_{p-N}...x_p) = \text{fol}(x_{p-n}...x_p) = \text{fol}(BAB),$$

it follows that $r_1 \in \text{fol}(BAB)$. This implies that there exists an $a^{(1)} \in X^+$ with prefix $BABAB$.

Since $a^{(1)} \in X^+$, there exists a ray r_2 in fol(B) that has a_pABAB as a prefix. Then fol$(B) = $ fol(BAB) implies that $r_2 \in \text{fol}(BAB)$. Hence there exists $a^{(2)} \in X^+$ with prefix $BABABAB$. Continuing in this manner, we construct a sequence $(a^{(n)}) \subset X^+$ with

$$\lim_{n\to\infty}(a^{(n)}) = BABABABABA....$$

Let $b^{(n)} = 0^\infty.a^{(n)}$ and $x_n(a^{(n)}) = \sigma^n b^{(n)}$ as in Proposition 5.5. Then any limit point y of $(x_n(a^{(n)}))$ is a periodic bisequence in X^{+-}. This is a contradiction, since X^{+-} does not contain any periodic points. Thus x is not eventually Markov at any time $p \in \mathbb{Z}$ and X^{+-}_M, the eventually Markov part of X^{+-}, is empty. □

As previously discussed, a Sturmian sequence u is syndetically recurrent and is not periodic. Since X_u^+ is the orbit closure of the almost periodic sequence u it follows that X_u^+ is minimal. Furthermore, since u is not periodic, X_u^+ is infinite. However, a priori, we don't know that X_u^{+-} is minimal.

PROPOSITION 5.14. *If X^+ is minimal, then the natural extension X^{+-} of X^+ is minimal. In fact, for any $x \in X^{+-}$ both the forward orbit $\mathcal{O}^+(x) = \{\sigma^n x | n \geq 0\}$ and the backward orbit $\mathcal{O}^-(x) = \{\sigma^{-n} x | n \geq 0\}$ are dense in X^{+-}.*

PROOF. Since X^+ is minimal, every block $B \in \mathcal{L}(X^+)$ appears with bounded gap in each $a \in X^+$. By Corollary 5.7, $\mathcal{L}(X^+) = \mathcal{L}(X^{+-})$. Therefore each block $B \in \mathcal{L}(X^{+-})$ appears in each long-enough block in $\mathcal{L}(X^{+-})$. Hence for all $x \in X^{+-}$, the block B appears with bounded gap to the left and the right in x. Thus $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are dense in X^{+-}. □

REMARK 5.15. Since any infinite minimal subshift contains no periodic points, it follows from Proposition 5.14 that if X^+ is both minimal and infinite, then X^{+-} is minimal and infinite.

6. The Morse minimal subshift

To describe the construction of the HB diagram of one particular substitution system, the Morse minimal subshift, we have to recall some well-known properties of the Prouhet-Thue-Morse sequence,

$$\omega = .\omega_0\omega_1\omega_2... = .0110100110010110....$$

For the sake of simplicity, we shall refer to this sequence as the Morse sequence. The one-sided subshift associated with ω is the *Morse minimal subshift*. It is defined by the pair (X_ω^+, σ), where X_ω^+ is the closure of $\{\sigma^n\omega | n \in \mathbb{N}\}$.

This sequence has many interesting properties. Axel Thue, concerned with constructing bi-infinite sequences on two symbols with controlled repetitions, constructed the two-sided Morse sequence

$$M = ...0110100110010110.0110100110010110...,$$

which he defined as having the property that the sequence contains no blocks of the form BBb where B is a block and b is the first letter of B [**5**, **7**]. Thue's results were published in 1912. In 1917, Marston Morse, not knowing of Thue's results, constructed the Morse sequence in his dissertation. In [**19**] Morse and Hedlund proved that every element in the Morse minimal set, the closure of $\{\sigma^n(M)|n \in \mathbb{Z}\}$, has the no BBb property. It was later shown by Gottshalk and Hedlund that the elements of the Morse minimal set are the only bi-infinite sequences with the no BBb property [**11**]. While the Gottshalk and Hedlund result does not carry over to the one-sided Morse sequence [**7**], it is still the case that the one-sided Morse sequence ω has the no BBb property.

The Morse sequence is also generated by iterating a substitution. Following Chapter 5 of [**9**], we recall how this is done and how the construction allows us to deduce important properties of the sequence. Let ζ be the substitution map defined by $\zeta(0) = 01$ and $\zeta(1) = 10$. The Morse sequence is the infinite sequence which begins with $\zeta^n(0)$ for every $n \in \mathbb{N}$. It follows from this construction that the Morse sequence is syndetically recurrent and neither periodic nor eventually periodic.

6.1. Recognizability of the Morse substitution. Since the Morse sequence arises from a substitution map, it is natural to consider how to "decompose" or "desubstitute" a block that occurs in $\mathcal{L}(X_\omega^+)$. The notion of recognizability deals with this problem [**23**].

Definition 6.1. A substitution γ over the alphabet $\mathcal{A}$ is *primitive* if there exists $k \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$ the letter a occurs in $\gamma^k(b)$.

In the context of recognizability we consider only primitive substitutions. Note that the Morse substitution ζ is primitive since 0 and 1 both appear in $\zeta(0)$ and $\zeta(1)$.

Let $u = u_0u_1...$ be any fixed point of an aribitrary primitive substitution γ.

Definition 6.2. For every $k \geq 1$, $E_k = \{0\} \cup \{|\gamma^k(u_0u_1...u_{p-1})| \mid p > 0\}$ is the set of *cutting bars of order k*.

Definition 6.3. The substitution γ is said to be *recognizable* if there exists an integer $K > 0$ such that

$$n \in E_1 \text{ and } u_nu_{n+1}...u_{n+K} = u_mu_{m+1}...u_{m+K} \text{ implies } m \in E_1.$$

The smallest integer K satisfying this is the *recognizability index* of γ.

In other words, a substitution is recognizable if it is possible to determine if u_m is the first letter of a substituted block by examining the K terms that follow it. The Morse substitution ζ is recognizable with recognizability index 3. This means that it is possible to determine if 0 (or 1) is the first letter of $\zeta(0)$ (or $\zeta(1)$) by examining the three letters which follow it.

Note that this definition of recognizability does not satisfactorily guarantee desubstitution in the general setting. Even very simple primitive, aperiodic substitutions may fail to have the recognizability property. For example, the substitution γ on the alphabet $\{0,1\}$ defined by $\gamma(0) = 010$ and $\gamma(1) = 10$ fails to be recognizable. Brigitte Mossé introduces another notion of recognizability, bilateral recognizability, in [**20**].

Definition 6.4. A substitution γ is said to be *bilaterally recognizable* if there exists an integer $L > 0$ such that

$$n \in E_1 \text{ and } u_{n-L}...u_{n+L} = u_{m-L}...u_{m+L} \text{ implies } m \in E_1.$$

One advantage of Mossé's definition is that every primitive aperiodic substitution is bilaterally recognizable. Furthermore, if u is a fixed point of a primitive aperiodic substitution γ and $X_u^+ = \text{cl}\{\sigma^k u | k \in \mathbb{N}\}$ (cl denotes closure), then any block in $\mathcal{L}(X_u^+)$ can be "desubstituted" up to some prefix and some suffix at the ends of the block [**9**, **20**, **23**]. Since the Morse substitution is recognizable, we do not rely on bilateral recognizability. However, the consequences of bilateral recognizability could be useful in extending the results for the Morse minimal subshift to general substitution systems.

We now consider the decomposition of blocks appearing in the fixed point of a substitution. Let $b = u_i ... u_{i+|w|-1}$ be a block appearing in u. Since $\gamma(u) = u$ there exists an index j, a length l, a suffix S of $\gamma(u_j)$ and a prefix P of $\gamma(u_{j+l+1})$ such that

$$b = S\gamma(u_j + 1)...\gamma(u_{j+l})P.$$

Definition 6.5. Let b be as above. The *1-cutting at the index i* of b is

$$S \dagger \gamma(u_{j+1}) \dagger ... \dagger \gamma(u_{j+l}) \dagger P,$$

and we say that b comes from the block $u_j ... u_{j+l+1}$. The block $u_j ... u_{j+l+1}$ is the *ancestor block* of b [**9**].

Note that S and P are not necessarily proper suffixes and prefixes, respectively. Furthermore, the 1-cutting yields a string on an enlarged alphabet. For the Morse sequence this alphabet is $\{0, 1, \dagger\}$.

To illustrate this, consider the block $\omega_4 ... \omega_9 = 1001100$ appearing in the Morse sequence. Let $S = 10 = \zeta(\omega_2)$, and $P = 0$, the one letter prefix of $\zeta(\omega_5)$. Then

$$1001100 = 10 \dagger 01 \dagger 10 \dagger 0 = 10 \dagger \zeta(\omega_3) \dagger \zeta(\omega_4) \dagger 0,$$

and has $\omega_2 ... \omega_5 = 1010$ as an ancestor block.

In this example, it is apparent that the 1-cutting of the block 1001100 partitions the block into a concatenation of the subblocks 10 and 01 with daggers in between. We define the *1-blocks* of the Morse sequence to be the blocks 01 and 10. By partitioning a block into its 1-blocks, it is possible to determine its ancestor block. The following lemma, found in [**9**], is a result of the recognizability of the Morse sequence.

Lemma 6.6. *In the Morse sequence, every block of length at least five has a unique 1-cutting, or decomposition into 1-blocks, possibly beginning with the last letter of a 1-block and possibly ending with the first letter of a 1-block.*

Remark 6.7. If a block has a unique 1-cutting, then the block has a unique ancestor block. The only blocks of length less than five appearing in the Morse sequence which do not have a unique partition into 1-blocks are 010, 101, 0101, and 1010, each of which has two possible ancestor blocks. Furthermore, a block has a unique 1-cutting if and only if that block has either 00 or 11 as a subblock.

We denote the *dual* of a letter $a \in \{0, 1\}$ by $\overline{a}$. If $a = 0$ then $\overline{a} = 1$ and vice versa. Note that each 1-block consists of a pair of dual letters.

Lemma 6.8. *Let* $a_{-n}a_{-n+1}a_{-n+2}...a_0$ *be a block of length* n *in* $\mathcal{L}(X_\omega^+)$ *that has a unique 1-cutting. If the unique 1-cutting has a dagger immediately to the right of* a_{-n+1}, *then* $a_{-n} = \overline{a}_{-n+1}$. *That is,* a_{-n} *is uniquely determined by* a_{-n+1}.

Proof. Since $a_{-n+1}a_{-n+2}...a_0$ has a unique decomposition into 1-blocks, $a_{-n+1}a_{-n+2}...a_0$ has 00 or 11 as a subblock. Without loss of generality, suppose 00 is a subblock of $a_{-n+1}a_{-n+2}...a_0$. Since 00 is not a 1-block, it follows that in the 1-cutting of $a_{-n+1}a_{-n+2}...a_0$ there is a dagger in the middle of 00. Furthermore, since there is a dagger immediately to the right of a_{-n+1}, it follows that there exists k, $1 \le k \le n/2$, such that $a_{-n-1+2k}a_{-n+2k} = 00$.

Now consider $a_{-n}a_{-n+1}...a_0 \in \mathcal{L}(X_\omega^+)$. Since 00 is a subblock of $a_{-n}a_{-n+1}...a_0$, there exists a unique 1-cutting of $a_{-n}a_{-n+1}a_{-n+2}...a_0$. Additionally, there must be a dagger in between $00 = a_{-n-1+2k}a_{-n+2k}$ in the unique 1-cutting. Thus there is a dagger between $a_{-n-1+2i}a_{-n+2i}$ for all $1 \le i \le n/2$. Letting $i = 1$, this implies that there is a dagger between $a_{-n+1}a_{-n+2}$. Hence, $a_{-n}a_{-n+1}$ must be a 1-block and $a_{-n} = \overline{a_{-n+1}}$. □

6.2. Significant blocks of the Morse minimal subshift. Using the properties of the Morse sequence detailed above, we determine the significant blocks of the Morse minimal subshift. Note that it follows from the minimality of the Morse sequence that $\mathcal{L}(X_\omega^+) = \mathcal{L}(X_\omega^{+-})$.

PROPOSITION 6.9. *Let $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ be a block of length $n \in \mathbb{N}$, $n \ge 2$, in $\mathcal{L}(X_\omega^+)$. Then $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ is significant if and only if $0a_{-n+2}...a_{-2}a_{-1}a_0$ and $1a_{-n+2}...a_{-2}a_{-1}a_0$ are in $\mathcal{L}(X_\omega^+)$.*

PROOF. One direction is proved in Proposition 4.6.

We first prove the converse for 01,10, 010, 101, 0101, and 1010. By examining the sequence ω it is apparent that each of these blocks satisfies the hypothesis that $0a_{-n+2}...a_{-2}a_{-1}a_0$ and $1a_{-n+2}...a_{-2}a_{-1}a_0$ are in $\mathcal{L}(X_\omega^+)$. To prove that each block is significant we construct a ray for each block that is in $\text{fol}(a_{-n+2}...a_{-2}a_{-1}a_0)$, but not in $\text{fol}(a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0)$.

Consider 010. Since the Morse sequence contains no blocks of the form BBb, $01010 \notin \mathcal{L}(X_\omega^+)$. However, $1010 \in \mathcal{L}(X_\omega^+)$. Thus the ray $\omega_3\omega_4\omega_5... = 0100110...$ is in fol(10) but not fol(010). Hence 010 is significant. Similarly, it can be shown that

$$\omega_2\omega_3\omega_4... = 101001... \in \text{fol}(1) \text{ but } 101001... \notin \text{fol}(01),$$
$$\omega_{10}\omega_{11}\omega_{12}... = 010110... \in \text{fol}(0) \text{ but } 010110... \notin \text{fol}(10),$$
$$\omega_{11}\omega_{12}\omega_{13}... = 101101... \in \text{fol}(01) \text{ but } 101101... \notin \text{fol}(101),$$
$$\omega_4\omega_5\omega_6... = 100110... \in \text{fol}(101) \text{ but } 100110... \notin \text{fol}(0101),$$
$$\omega_{12}\omega_{13}\omega_{14}... = 011010... \in \text{fol}(010) \text{ but } 011010... \notin \text{fol}(1010).$$

Therefore, 01, 10, 101, 0101, and 1010 are all significant.

We now prove the converse for the remaining blocks $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ satisfying the property that $0a_{-n+2}...a_{-2}a_{-1}a_0$ and $1a_{-n+2}...a_{-2}a_{-1}a_0$ are in $\mathcal{L}(X_\omega^+)$. To prove that each of these blocks is significant, we explicitly construct a ray that is in the follower set of $a_{-n+2}...a_{-2}a_{-1}a_0$ but not in the follower set of $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$. This is done by repeatedly desubstituting $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ and choosing a ray based on the ancestor block of $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$.

It follows from Lemma 6.6 and Remark 6.7 that each of the remaining blocks has a unique 1-block decomposition. If $n > 3$, then partition $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ into 1-blocks. Lemma 6.8 implies that if the unique partition of $a_{-n+1}a_{-n+2}...$ $a_{-2}a_{-1}a_0$ has a dagger directly after a_{-n+2}, then a_{-n+1} is uniquely determined by a_{-n+2}, and hence $0a_{-n+2}...a_{-2}a_{-1}a_0$ and $1a_{-n+2}...a_{-2}a_{-1}a_0$ are not both in $\mathcal{L}(X_\omega)$. It follows that for $n > 3$ the partition of $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$ is

$$a_{-n+1} \dagger a_{-n+2}a_{-n+3} \dagger \cdots \dagger a_{-2}a_{-1} \dagger a_0$$

if n is even, and

$$a_{-n+1} \dagger a_{-n+2}a_{-n+3} \dagger \cdots \dagger a_{-3}a_{-2} \dagger a_{-1}a_0$$

if n is odd.

Next we map each 1-block in the partition to its preimage under ζ, or ancestor block. Let $s_{1,i/2}$ denote the preimage of the 1-block $a_{-n+i}a_{-n+i+1}$, where i is even and $2 \leq i \leq n-1$. Since there is a dagger placed directly after a_{-n+1}, a_{-n+1} uniquely determines the letter that can precede it. Let a_{-n} be this letter and let $s_{1,0}$ denote the preimage of $a_{-n}a_{-n+1}$. Similarly, when n is even a_0 uniquely determines a_1. In this case let $s_{1,n/2}$ denote the preimage of a_0a_1. The resulting block is $s_{1,0}s_{1,1}...s_{1,(n-1)/2}$ if n is odd, and $s_{1,0}s_{1,1}...s_{1,n/2}$ if n is even. For ease of notation, denote $s_{1,0}s_{1,1}...s_{1,(n-1)/2}$ or $s_{1,0}s_{1,1}...s_{1,n/2}$ by S_1. Note that S_1 is the ancestor block of $a_{-n+1}a_{-n+2}...a_{-2}a_{-1}a_0$.

At this stage, consider the length of S_1. If $|S_1| \geq 4$ and S_1 has a unique 1-block partition, then map each 1-block to its preimage under ζ. That is, map S_1 to its ancestor block. If $|S_1| < 4$ or $S_1 = 0101$ or 1010, then do nothing. We continue this process so that in general, if $|S_j| \geq 4$ and S_j can be uniquely partitioned into 1-blocks, we map S_j to its ancestor block, and otherwise do nothing. During this process, if the 1-block decomposition of S_j has a dagger between $s_{j,0}$ and $s_{j,1}$, then there is only one letter that can precede $s_{j,0}$. Let $s_{j,-1}$ be this letter. Then $s_{j+1,0}$ is defined to be the preimage of the 1-block $s_{j,-1}s_{j,0}$.

Example 6.10. We illustrate this process for the block $00110100 \in \mathcal{L}(X_\omega^+)$. The unique 1-block decomposition of 00110100 is

$$0 \dagger 01 \dagger 10 \dagger 10 \dagger 0.$$

Mapping each 1-block to its preimage under ζ, we get $S_1 = 10110$. Since $|10110| = 5$, we partition 10110 into 1-blocks as follows:

$$1 \dagger 01 \dagger 10.$$

Again, map each 1-block to its preimage under ζ to get $S_2 = 001$. As $|001| < 4$, the decomposition process is complete.

We claim that there exists an $m \in \mathbb{N}$ such that $S_m = s_{m,0}s_{m,1}s_{m,2}$, where $s_{m,1}s_{m,2} \in \{01, 10\}$; or $S_m = s_{m,0}s_{m,1}s_{m,2}s_{m,3}$, where $s_{m,1}s_{m,2}s_{m,3} \in \{010, 101\}$. Since the decomposition process is repeated until $3 \leq |S_m| \leq 4$ we need only show that

$$S_m \in \{001, 101, 010, 110, 0010, 1010, 0101, 1101\}.$$

Suppose on the contrary that S_m is not one of the above blocks. Since $S_m \in \mathcal{L}(X_\omega^+)$, it follows that

$$S_m \in \{100, 011, 1001, 0110\}.$$

Each of these blocks has a unique partitioning in which there is a dagger directly to the right of $s_{m,1}$. Hence $s_{m,0}$ is uniquely determined by $s_{m,1}$, and $s_{m,0} = \overline{s_{m,1}}$, by Lemma 6.8. We prove that if $s_{m,0}$ is uniquely determined by $s_{m,1}$ then there is only one possible value for a_{-n+1}.

First suppose that for all $1 \leq k < m$ the partition of S_k has a dagger between $s_{k,0}$ and $s_{k,1}$. That is,

$$s_{k,0} \dagger s_{k,1}s_{k,2} \dagger \cdots \dagger s_{k,j} \qquad \text{and} \qquad s_{k,0} \dagger s_{k,1}s_{k,2} \dagger \cdots \dagger s_{k,j}s_{k,j+1}$$

are the 1-cuttings for $1 \leq k < m$ when $|S_k|$ is even and odd respectively, but

$$s_{m,0}s_{m,1} \dagger s_{m,2} \qquad \text{and} \qquad s_{m,0}s_{m,1} \dagger s_{m,2}s_{m,3}$$

are the 1-cuttings for $s_{m,0}$ when S_m has length 3 and 4 respectively.

If this is the case, then $\zeta(s_{k+1,0}) = s_{k,-1}s_{k,0}$ for all $1 \leq k < m$. Then

$$\zeta^2(s_{k+1,0}) = \zeta(s_{k,-1})\zeta(s_{k,0}) = \zeta(s_{k,-1})s_{k-1,-1}s_{k-1,0},$$

and in general

$$\zeta^j(s_{k+1,0}) = \zeta^{j-1}(s_{k,-1})\zeta^{j-2}(s_{k-1,-1})...\zeta(s_{k-j+2,-1})s_{k-j+1,-1}s_{k-j+1,0}$$

for $j \leq k$. Hence

$$\begin{aligned}\zeta^m(s_{m,0}) &= \zeta^{m-1}(s_{m-1,-1})\zeta^{m-2}(s_{m-2,-1})...\zeta^2(s_{2,-1})\zeta(s_{1,-1})\zeta(s_1,0)\\ &= \zeta^{m-1}(s_{m-1,-1})\zeta^{m-2}(s_{m-2,-1})...\zeta^2(s_{2,-1})\zeta(s_{1,-1})\overline{a_{-n+1}}a_{-n+1}.\end{aligned}$$

Thus the last letter of $\zeta^m(s_{m,0})$ is a_{-n+1}. Since each desubstitution is unique, this implies that a_{-n+1} is uniquely determined by $s_{m,0}$.

Let $S_m = s_{m,0}...s_{m,q}$, where $q \in \{2,3\}$. By a similar argument to that used above, it can be shown that the block $a_{-n+2}a_{-n+3}..a_1a_0$ is a prefix of $\zeta^m(s_{m,1}...s_{m,q})$. That is,

$$\zeta^m(S_m) = \zeta^m(s_{m,1}...s_{m,q}) = a_{-n+2}a_{-n+3}...a_1a_0C,$$

where $C \in \mathcal{L}(X_\omega^+)$. Since each desubstitution of $a_{-n+2}a_{-n+3}...a_1a_0$ is unique, this implies that $s_{m,1}$ is uniquely determined by the block $a_{-n+2}a_{-n+3}...a_1a_0$.

Furthermore, since we have assumed that $s_{m,0}$ is uniquely determined by $s_{m,1}$, it follows that $\zeta^m(s_{m,0}) = \zeta^m(\overline{s_{m,1}})$. Hence, a_{-n+1}, the last letter of $\zeta^m(s_{m,0})$, is uniquely determined by $s_{m,1}$. Thus, there is only one possible value for a_{-n+1}. Therefore, $0a_{-n+2}a_{-n+3}...a_1a_0$ and $1a_{-n+2}a_{-n+3}...a_1a_0$ cannot both be in $\mathcal{L}(X_\omega^+)$, contradicting $a_{-n+1}a_{-n+2}a_{-n+3}...a_1a_0$ being a significant block.

Now suppose that there exists $1 \leq k < m$ such that the 1-block decomposition of S_k has a dagger after $s_{k,1}$. Then there exists an r, $1 \leq r \leq k$, such that for all $1 \leq j < r$ the partition of S_j has a dagger between $s_{j,0}$ and $s_{j,1}$, but the partition of S_r has a dagger after $s_{r,1}$. By the previous argument, it follows that a_{-n+1} is uniquely determined by $s_{r,0}$ and thus by $s_{r,1}$. Hence, $0a_{-n+2}a_{-n+3}..a_1a_0$ and $1a_{-n+2}a_{-n+3}..a_1a_0$ cannot both be in $\mathcal{L}(X_\omega^+)$. Therefore,

$$S_m \notin \{100, 011, 1001, 0110\}.$$

REMARK 6.11. Let $S_m = s_{m,0}...s_{m,q}$, where $q \in \{2,3\}$. By the above argument it must be the case that for all $1 \leq k < m$ the partition of S_k has a dagger between $s_{k,0}$ and $s_{k,1}$. This implies that a_{-n+1} is the last letter of $\zeta^m(s_{m,0})$ and $a_{-n+2}a_{-n+3}..a_1a_0$ is a prefix of $\zeta^m(s_{m,1}...s_{m,q})$, as discussed.

Having established the existence of an $m \in \mathbb{N}$ such that $S_m = s_{m,0}s_{m,1}s_{m,2}$, where $s_{m,1}s_{m,2} \in \{01, 10\}$, or $S_m = s_{m,0}s_{m,1}s_{m,2}s_{m,3}$, where $s_{m,1}s_{m,2}s_{m,3} \in \{010, 101\}$, define the ray ν as follows.
If $s_{m,0} = 0$ then,

$$\nu = \begin{cases} \omega_5\omega_6\omega_7... = 0011001..., & \text{if } s_{m,1}s_{m,2} = 01 \\ \omega_6\omega_7\omega_8... = 0110010..., & \text{if } s_{m,1}s_{m,2s_{m,3}} = 010 \\ \omega_4\omega_5\omega_6... = 1001100..., & \text{if } s_{m,1}s_{m,2} = 10 \\ \omega_5\omega_6\omega_7... = 0011001..., & \text{if } s_{m,1}s_{m,2s_{m,3}} = 101 \end{cases}$$

If $s_{m,0} = 1$ then,

$$\nu = \begin{cases} \omega_{13}\omega_{14}\omega_{15}... = 1101001..., & \text{if } s_{m,1}s_{m,2} = 10 \\ \omega_{14}\omega_{15}\omega_{16}... = 1010010..., & \text{if } s_{m,1}s_{m,2s_{m,3}} = 101 \\ \omega_{12}\omega_{13}\omega_{14}... = 0110100..., & \text{if } s_{m,1}s_{m,2} = 01 \\ \omega_{13}\omega_{14}\omega_{15}... = 1101001..., & \text{if } s_{m,1}s_{m,2s_{m},3} = 010 \end{cases}$$

Note that the sequence ν is defined so that $s_{m,1}s_{m,2}\nu$ (or $s_{m,1}s_{m,2}s_{m,3}\nu$) is in X_ω^+, but $s_{m,0}s_{m,1}s_{m,2}\nu$ (or $s_{m,0}s_{m,1}s_{m,2}s_{m,3}\nu$) is not in X_ω^+. We provide an example of this.

Example 6.12. Let $s_{m,0}s_{m,1}s_{m,2} = 001$. Then $s_{m,0}s_{m,1}s_{m,2}\nu = 0010011001....$ Consider the 1-block decomposition of the prefix block 00100. Since a dagger must be placed between consecutive zeros we get $0 \dagger 01 \dagger 0 \dagger 0$. This, however, is not an allowed 1-block decomposition. Thus 00100 is not in $\mathcal{L}(X_\omega^+)$ and $s_{m,0}s_{m,1}s_{m,2}\nu \notin X_\omega^+$.

Suppose $s_{m,1}s_{m,2} \in \{01, 10\}$. Let $\zeta^m(s_{m,1}s_{m,2}\nu) = d_1d_2d_3....$ Note that the first $n-1$ letters of $\zeta^m(s_{m,1}s_{m,2}\nu)$ form the block $a_{-n+2}a_{-n+3}...a_0$. We claim that $d_{n-1}d_nd_{n+1}...$ is in $\text{fol}(a_{-n+2}a_{-n+3}...a_0)$ but not in $\text{fol}(a_{-n+1}a_{-n+2}...a_0)$. Since $s_{m,1}s_{m,2}\nu \in X_\omega^+$, it follows that $\zeta^m(s_{m,1}s_{m,2}\nu) \in X_\omega^+$, as ω is fixed under the substitution ζ. Thus, $d_{n-1}d_nd_{n+1}...$ is in $\text{fol}(a_{-n+2}a_{-n+3}...a_0)$. It remains to show that $a_{-n+1}\zeta^m(s_{m,1}s_{m,2}\nu)$ is not in X_ω^+.

Let $\nu = \nu_1\nu_2\nu_3...$ and consider the block $a_{-n+1}\zeta^m(s_{m,1}s_{m,2}\nu_1\nu_2)$. Since the first $n-1$ terms of $\zeta^m(s_{m,1}s_{m,2}\nu_1\nu_2)$ form the block $a_{-n+2}a_{-n+3}...a_0$, the block $a_{-n+1}a_{-n+2}a_{-n+3}...a_0$ is a prefix of the block $a_{-n+1}\zeta^m(s_{m,1}s_{m,2}\nu_1\nu_2)$. Implementing the decomposition process m times, the resulting block is $s_{m,0}s_{m,1}s_{m,2}\nu_1\nu_2$. However, $s_{m,0}s_{m,1}s_{m,2}\nu_1\nu_2 \notin \mathcal{L}(X_\omega^+)$. As the ancestor block of any block in $\mathcal{L}(X_\omega^+)$ is in $\mathcal{L}(X_\omega^+)$ it follows that $a_{-n+1}\zeta^m(s_{m,1}s_{m,2}\nu_1\nu_2) \notin \mathcal{L}(X_\omega^+)$. Therefore, $a_{-n+1}\zeta^m(s_{m,1}s_{m,2}\nu)$ is not in X_ω^+. Therefore, $a_{-n+1}a_{-n+2}a_{-n+3}...a_0$ is significant.

Using similar arguments, it can be shown that $a_{-n+1}a_{-n+2}...a_0$ is significant in the case that

$$s_{m,1}s_{m,2}s_{m,3} \in \{010, 101\}.$$

□

Remark 6.13. We relate this result to Proposition 4.7 from the Sturmian case. Since there is not a unique left special sequence as in the Sturmian case, let m_n be the number of blocks of length n in $\mathcal{L}(X_\omega^+)$ that can be extended to the left in two ways. Denote the set of such blocks by $\{L_n^i\}$, $1 \le i \le m_n$. Then for each $n \ge 2$ the significant blocks of length n of X_ω^+ are $\{0L_{n-1}^i, 1L_{n-1}^i\}$, $1 \le i \le m_{n-1}$.

6.3. HB diagram of the Morse minimal subshift. Since there is no left special sequence to direct us to the blocks in $\mathcal{L}(X_\omega^+)$ that can be extended to the left in two ways, the process of determining the significant blocks is more tedious. Nevertheless, it was possible above to use the no BBb property and desubstitution to identify the significant blocks.

After determining the significant blocks of the Morse minimal subshift, the next step is to determine the arrows. As in the Sturmian case, only those blocks that can be extended to the right in two ways will have two arrows out. However, for the Morse minimal subshift there is no easy technique for determining the significant blocks that satisfy this property. Hence the process of constructing the HB diagram of the Morse minimal subshift is not nearly as streamlined as for the Sturmians. We construct the HB diagram in the following way.

Begin by generating a list of signficant blocks. Start with the blocks 0 and 1. Since $00, 10, 01$, and 11 are all in $\mathcal{L}(X_\omega^+)$, it follows that 0 and 1 can both be extended to the left in two ways. Hence $00, 10, 01$, and 11 are all significant. Next consider $\mathcal{L}_2(X_\omega^+) = \{00, 01, 10, 11\}$, the blocks of length 2 in $\mathcal{L}(X_\omega^+)$. As the Morse minimal subshift has no blocks of the form BBb, 000 and 111 are not in $\mathcal{L}(X_\omega^+)$. Hence 00 and 11 cannot be extended to the left in two ways. However, 10 and 01 can be extended to the left in two ways. Thus $110, 010, 001$, and 101 are the significant blocks of length 3.

Now consider $\mathcal{L}_3(X_\omega^+) = \{001, 010, 011, 100, 101, 110\}$. By the no BBb property, it follows that 0001 and 1110 are not in $\mathcal{L}(X_\omega^+)$. The remaining blocks can all be extended to the left in two ways. Thus the significant blocks of length 4 are 0010, 1010, 0011, 1011, 0100, 1100, 0101, and 1101. Continuing in this manner, we are able to generate a list of significant blocks of X_ω^+;

0,1, 00, 10, 01, 11, 110, 010, 001, 101, 0010, 1010, 0011, 1011, 0100, 1100, 0101, 1101, 00110, 01001,10110, 11010,... .

To determine the arrows in the HB diagram of X_ω^+, we consider the right extensions of each significant block. We illustrate the process of determining the arrows by considering those arrows that start at a significant block of length 4. It is easily seen that $0011, 1011, 0100$, and 1100 can only be extended to the right in one way. Thus there is exactly one arrow out of each of these blocks, and these arrows are:

$$\begin{aligned} 0011 &\to \mathrm{sig}(00110) = 00110 \\ 1011 &\to \mathrm{sig}(10110) = 10110 \\ 0100 &\to \mathrm{sig}(01001) = 01001 \\ 1100 &\to \mathrm{sig}(11001) = 11001. \end{aligned}$$

Additionally, 1010 and 0101 can be extended to the right in only one way, as the blocks 10101 and 01010 are of the form BBb. The arrows out of these blocks are:

$$\begin{aligned} 1010 &\to \mathrm{sig}(10100) = 0100 \\ 0101 &\to \mathrm{sig}(01011) = 1011. \end{aligned}$$

Lastly, consider the blocks 1101 and 0010. Instead of using the no BBb property, we consider the 1-block decomposition of each block. This gives us $1\dagger 10\dagger 1$ and $0\dagger 01\dagger 0$. Since extending each block to the right must yield a legal 1-block decomposition, it follows that 1101 can be followed only by a 0, and 0010 can be followed only by a 1. Thus, the arrows out of these blocks are:

$$1101 \to \mathrm{sig}(11010) = 1010$$
$$0010 \to \mathrm{sig}(00101) = 0101.$$

Note that although there is exactly one arrow out of each significant block of length 4, this is not the case in general. For example, using the same process it can be shown that all four significant blocks of length 5 can be extended to the right in two ways.

Figure 3 depicts a portion of the HB diagram of the Morse minimal subshift that has been constructed using the process described above.

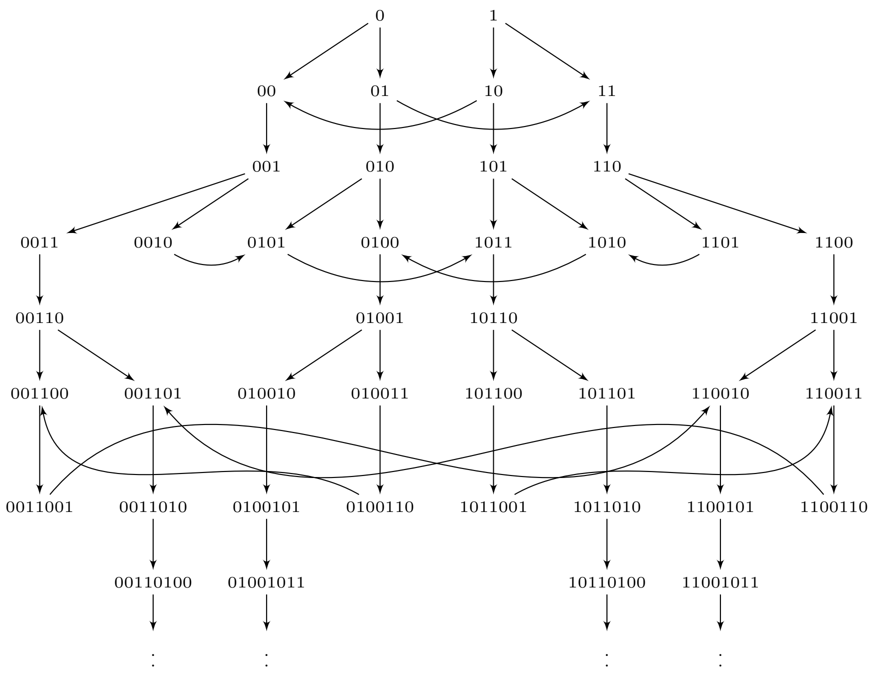

FIGURE 3. The HB diagram of the Morse minimal subshift.

7. Conclusion and Further Directions

We have described the construction of HB diagrams for some highly non-Markovian systems, that is, systems with long-range memory. These HB diagrams provide a way to visualize all the possibilities for extending any given block and present useful information about the languages of such systems and therefore about the structures of the systems themselves. Our examples of Sturmian systems and the Morse system are among the most-studied symbolic dynamical systems, so it is unlikely that these diagrams can lead to any new results about them or new proofs of known properties (indeed the construction of the diagrams used many known facts about the sequences and systems). For example, the complexity functions of subshifts can be read off the diagrams by counting paths that begin at significant blocks of length 1 (Corollary 5.9). The complexity function of the Morse minimal subshift is given by (see [**9**, Ch. 5]) $p_\omega(1) = 2$, $p_\omega(2) = 4$ and for $n \geq 3$ if $n = 2^r + q + 1$, $r \geq 0$, $0 < q \leq 2^r$, then

$$p_\omega(n) = \begin{cases} 6(2^{r-1}) + 4q & \text{if } 0 < q \leq 2^{r-1} \\ 8(2^{r-1}) + 2q & \text{if } 2^{r-1} < q \leq 2^r. \end{cases}$$

It is apparent in examining the portion of the HB diagram of the Morse minimal subshift shown in Figure 3 that the number of paths with initial vertex 0 or 1 is equal to $p_\omega(n)$ for $n \leq 8$. But to use the diagram to prove the formula for all lengths would probably require the same properties used to prove the formula directly (in addition to the many properties used to establish the diagram), so here the diagrams seem not to provide any advantage.

K. Johnson [**14**] (see also [**10**, **22**]) observed that every one-sided subshift (X, σ) with alphabet $\{0, 1, \dots, d-1\}$ has a unique lexicographically maximal element $\omega_{\max}(X, \sigma)$, which, since it lexicographically dominates all its shifts, is the expansion of 1 base β for a unique $\beta \in (1, \infty)$ and thus determines a smallest beta-shift that contains (X, σ). In particular, there is a one-to-one correspondence between minimal subshifts on alphabets $\{0, 1, \dots, d-1\}$ and real $\beta \in (1, \infty)$ for which the expansions of 1 base β are uniformly recurrent (syndetic, also sometimes called almost periodic). HB diagrams provide an efficient way to identify the lexicographically maximal element in a subshift: start at the vertex $d-1$, and whenever two arrows leave a significant block follow the one with the larger label (i.e., the one whose target vertex has largest final entry). Johnson gave algorithms for constructing the maximal elements in constant-length and some variable-length substitution systems, showing in particular that the maximal element in the Fibonacci system is $1f$, where $f = 010\dots$ is the fixed point of the substitution, and the maximal element in the Morse system is $\sigma\omega$, the shift of the Morse sequence ω. Applying our method to the part of the HB diagram that we have constructed for the Morse system corroborates Johnson's result, but proving that the maximal path in the diagram determines the Morse sequence would probably involve arguments similar to those of [**14**]. However, for Sturmian systems (including the Fibonacci) we can use the diagrams to identify the maximal elements.

THEOREM 7.1. *Let (X_u^+, σ) be a Sturmian subshift with left special sequence $l = l_1 l_2 \dots$. Then the lexicographically maximal sequence in X_u^+ is $\omega_{\max}(X_u^+, \sigma) = 1l$.*

PROOF. Let L_0 denote the empty block and for each $n = 0, 1, 2, \dots$ let $L_n = l_1 l_2 \dots l_n$ denote the left special block of length n in u. By Proposition 4.7, the significant blocks (vertices of the diagram) are the blocks $0L_{n-1}$ and $1L_{n-1}, n = 1, 2, \dots$. So our path following edges with maximal labels begins at the block $1L_0$. By Lemma 4.9, we always have arrows $xL_{k-1} \to xL_k, x \in \{0, 1\}$, so our maximally labeled path starting at $1L_0$ will be forced to proceed, as long as there is only one exiting arrow from each of its vertices, along vertices $1L_k$ until it reaches a vertex $1L_{n-1}$ which has two exiting arrows. By Lemma 4.10, then $1L_{n-1}$ is right special, and $1L_{n-1} = 1l_1 \dots l_{n-1} = l_n l_{n-1} \dots l_1$, so that $l_n = 1$ and the edge labeled 1 (i.e., to the target vertex with final entry 1) is to $1L_n$. Continuing in this way always to follow the arrow with the largest label whenever there is a choice then produces the maximal element $\omega_{\max}(X_u^+, \sigma) = 1l$. □

A similar argument shows that the lexicographically *minimal* element in a Sturmian subshift with left special sequence l is $\omega_{\min}(X_u^+, \sigma) = 0l$.

The diagrams might be put to further use in various other ways. How can invariant measures be represented on the diagrams? Can we detect unique ergodicity or minimality from these diagrams? In Section 6 we were able to construct the HB diagram of the Morse minimal subshift because the recognizability property of the Morse substitution allowed us to say precisely which blocks are significant. Can this result be generalized to any recognizable, or bilaterally recognizable, substitution? It is known that beta-shifts, as well as their factors, have unique measures of maximal entropy [**4**, **12**, **24**]. The HB diagram of a beta-shift turns out to be just a relabeling of the well-known beta-shift graph (see, e.g., [**14**, **22**]). Is there a simple way to transform the HB diagram of a subshift to produce the HB diagram of one of its factors? A relation between the two diagrams could help to understand factor maps, and in particular to identify measures of maximal entropy or maximal relative entropy.

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References

[1] Jérôme Buzzi, *Intrinsic ergodicity of smooth interval maps*, Israel J. Math. **100** (1997), 125–161, DOI 10.1007/BF02773637. MR1469107

[2] Jérôme Buzzi, *Entropy theory on the interval* (English, with English and French summaries), École de Théorie Ergodique, Sémin. Congr., vol. 20, Soc. Math. France, Paris, 2010, pp. 39–82. MR2856509

[3] Jérôme Buzzi and Pascal Hubert, *Piecewise monotone maps without periodic points: rigidity, measures and complexity*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 383–405, DOI 10.1017/S0143385703000488. MR2054049

[4] Vaughn Climenhaga and Daniel J. Thompson, *Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors*, Israel J. Math. **192** (2012), no. 2, 785–817, DOI 10.1007/s11856-012-0052-x. MR3009742

[5] Ethan M. Coven, Michael Keane, and Michelle Lemasurier, *A characterization of the Morse minimal set up to topological conjugacy*, Ergodic Theory Dynam. Systems **28** (2008), no. 5, 1443–1451, DOI 10.1017/S0143385707000831. MR2449536

[6] Doris Fiebig and Ulf-Rainer Fiebig, *Covers for coded systems*, Symbolic dynamics and its applications (New Haven, CT, 1991), Contemp. Math., vol. 135, Amer. Math. Soc., Providence, RI, 1992, pp. 139–179, DOI 10.1090/conm/135/1185086. MR1185086

[7] Earl D. Fife, *Binary sequences which contain no BBb*, Trans. Amer. Math. Soc. **261** (1980), no. 1, 115–136, DOI 10.2307/1998321. MR576867

[8] Roland Fischer, *Sofic systems and graphs*, Monatsh. Math. **80** (1975), no. 3, 179–186. MR0407235

[9] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel. MR1970385

[10] Shaobo Gan, *Sturmian sequences and the lexicographic world*, Proc. Amer. Math. Soc. **129** (2001), no. 5, 1445–1451 (electronic), DOI 10.1090/S0002-9939-00-05950-5. MR1814171

[11] W. H. Gottschalk and G. A. Hedlund, *A characterization of the Morse minimal set*, Proc. Amer. Math. Soc. **15** (1964), 70–74. MR0158386

[12] Franz Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. **34** (1979), no. 3, 213–237 (1980), DOI 10.1007/BF02760884. MR570882

[13] Franz Hofbauer and Peter Raith, *Topologically transitive subsets of piecewise monotonic maps, which contain no periodic points*, Monatsh. Math. **107** (1989), no. 3, 217–239, DOI 10.1007/BF01300345. MR1008681

[14] Kimberly Christian Johnson, *Beta-shift dynamical systems and their associated languages*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The University of North Carolina at Chapel Hill. MR2699401

[15] Wolfgang Krieger, *On sofic systems. I*, Israel J. Math. **48** (1984), no. 4, 305–330, DOI 10.1007/BF02760631. MR776312

[16] Wolfgang Krieger, *On sofic systems. II*, Israel J. Math. **60** (1987), no. 2, 167–176, DOI 10.1007/BF02790789. MR931874

[17] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR1369092

[18] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002, A collective work by Jean Berstel, Dominique Perrin, Patrice Seebold, Julien Cassaigne, Aldo De Luca, Steffano Varricchio, Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon, Veronique Bruyere, Christiane Frougny, Filippo Mignosi, Antonio Restivo, Christophe Reutenauer, Dominique Foata, Guo-Niu Han, Jacques Desarmenien, Volker Diekert, Tero Harju, Juhani Karhumaki and Wojciech Plandowski, With a preface by Berstel and Perrin. MR1905123 (2003i:68115)

[19] Marston Morse and Gustav A. Hedlund, *Unending chess, symbolic dynamics and a problem in semigroups*, Duke Math. J. **11** (1944), 1–7. MR0009788

[20] Brigitte Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution* (French), Theoret. Comput. Sci. **99** (1992), no. 2, 327–334, DOI 10.1016/0304-3975(92)90357-L. MR1168468

[21] Karl Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, Cambridge, 1989. Corrected reprint of the 1983 original. MR1073173

[22] Karl Petersen, *Information compression and retention in dynamical processes*, Dynamics and randomness (Santiago, 2000), Nonlinear Phenom. Complex Systems, vol. 7, Kluwer Acad. Publ., Dordrecht, 2002, pp. 147–217, DOI 10.1007/978-94-010-0345-2_6. MR1975578

[23] Martine Queffélec, *Substitution dynamical systems—spectral analysis*, 2nd ed., Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 2010. MR2590264

[24] Peter Walters, *Equilibrium states for β-transformations and related transformations*, Math. Z. **159** (1978), no. 1, 65–88. MR0466492

[25] Benjamin Weiss, *Subshifts of finite type and sofic systems*, Monatsh. Math. **77** (1973), 462–474. MR0340556

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