# ATTRACTORS AND ATTRACTING MEASURES

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## 1. Introduction

These are notes from a graduate course on symbolic dynamics given at the University of North Carolina, Chapel Hill, in the spring semester of 1997. The course began with some background on smooth dynamics and then mainly worked through R. Bowen's *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, dealing with the construction of Markov partitions, entropy, pressure, equilibrium states, and equilibrium (SRB) measures on attractors. The aim was to see one important source of symbolic dynamics, which was studied in its own right in the following course the next spring. The author thanks all the students who took notes, wrote them up, and typed them, and Kennan Shelton for managing the entire project.

## 2. Plan of the course. January 8, 1997 (*Notes by LK*)

In this course, we intend to study the dynamical aspects of Axiom A attractors; specifically, we want to identify such attractors and any corresponding attracting measures. *Introduction*

The required texts for this course are out of print but can be purchased in a course pack available at the bookstore. They consist of Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer-Verlag LNM 470, 1975 and sections of Denker, Grillenberger, and Sigmund, *Ergodic Theory on Compact Spaces*, Springer-Verlag LNM 527, 1976.

The following sources provide a more detailed background to ergodic theory and are on reserve in the library.

(1) Katok and Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.
(2) Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1983.
(3) Petersen, *Ergodic Theory*, Cambridge Univ. Press, 1983.
(4) Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.
(5) Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1995.
(6) Smale, *Differentiable Dynamical Systems*, Bulletin of the AMS, vol. 73 (1967), 747-817 (an older summary of the theory of dynamical systems).

As an illustration of how we can find attracting behavior in seemingly chaotic systems, we begin with an example from H. Abarbanel's *On the Analysis of Chaotic Dynamical Systems*. The example on pp 3-12 describes a cutting tool, as in machining some type of part with a lathe. This type of work needs to be precisely controlled to obtain a finished product within the specified parameters. Even so, there is still some variation in the accuracy of the cuts. A graph of the the displacement of the lathe versus the time can be found on page 3. Since this is a time series signal, we might try to use the Fourier transform to study it. However, as the graph on page 4 demonstrates, harmonic analysis does not provide much insight into the behavior of this system.

Nevertheless, we can examine the pseudo-phase space (called pseudo because it is obtained from a numerical approximation) by plotting the vector $(x(t), x(t+\tau), x(t+2\tau))$, where $x(t)$ is the displacement at time $t$ and $\tau$ is chosen in some appropriate manner. As we see from the graph on page 5, we obtain an object with some structure. Although it is not yet clear to us what this new graph signifies, it may help us to understand qualitative aspects

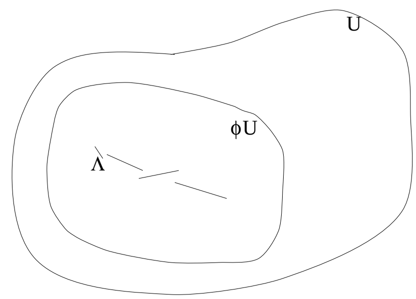



FIGURE 1. The attractor $\Lambda$

of our original system and therefore possibly make predictions about it or even control it. In fact, this procedure works with many systems and is being used currently in many parts of science. Generally, we take one-dimensional data, space it out nicely and plot it in a convenient dimension to obtain a simpler object with some obvious structure. Hopefully, that new structure will tell us something about the original system.

A *dynamical system* is a space $X$ and a family $\mathcal{S}$ of maps $\phi\colon X \to X$, where we usually assume that $\mathcal{S}$ is at least a semi-group (if $\phi, \psi \in \mathcal{S}$ then $\phi \circ \psi \in \mathcal{S}$). The space $X$ can be a topological space, a measure space, or a (compact) manifold.

Dynamical systems arise from systems of differential equations that describe some types of physical, biological or abstract systems. We want the system of equations to be *autonomous*, meaning that the laws of the system do not change over time. The system gives a *flow* in phase space described by (position, momentum) or $(y, y')$. We let $X$ be some closed invariant set (for example, the constant energy manifold) and we let $\phi\colon X \to X$ denote the time-one map of the flow. Then we have a family of maps $\mathcal{S} = \{\phi^n\}$ where $n$ can be an integer or $n$ can be restricted to the positive integers.

We pause to note that the closed invariant set $X$ will turn out to be a manifold in a natural way and thus will have a natural measure coming from Lebesgue measure. However, this measure may not see the dynamics, as the dynamically interesting part of the space may be a null set with respect to this measure class. In part, this course will focus on finding measures that are dynamically interesting.

If $X$ is a topological space (manifold) and $\phi\colon X \to X$ is a homeomorphism (diffeomorphism) then we call a set $\Lambda \subset X$ an *attractor* if there is an open set $U$ containing $\Lambda$ with $\phi U \subset U$ and $\Lambda = \bigcap_{n\geq 0} \phi^n U$, as in Figure 1.

If $x \in U$ then the set of limit points of the iterates $\phi^n x$ is contained in $\Lambda$. The *basin of attraction of* $\Lambda$ is the set $\bigcup_{n\geq 0} \phi^{-n} U$. If $\Lambda$ is anything more complicated than a periodic orbit, we call $\Lambda$ a *strange attractor*.

Sinai, Ruelle, and Bowen proved that (hyperbolic) attractors exist for Axiom A systems. We will define these terms later in the course and for now just give a few examples. *Anosov systems*, where $\Lambda$ is the entire manifold, are Axiom A systems. For example, there are one-to-one and onto maps of the torus which are Anosov because the action is hyperbolic and complicated everywhere. Solenoids are examples of Axiom A attractors which are not Anosov. On the basis of numerical studies, attractors are suspected to exist in many other cases, such as for Hénon maps and Lorentz systems.

Further, we will study the proof of the existence of attracting measures in Bowen's book. We will examine the *SRB measure* (Sinai - Ruelle - Bowen), a probability measure $\mu$ on $\Lambda$ such that for all continuous functions $f\colon U \to \mathbb{R}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\phi^k x) \to \int_\Lambda f \, d\mu$$

for a.e. $x \in U$ with respect to the Lebesgue measure class. We note that this is true even though the attractor probably has measure 0 with respect to Lebesgue measure, the measure that includes experimental observations in a laboratory. In other words, there is a set of Lebesgue measure zero that is determining the long-term behavior of what we are trying to observe. In the proof, we will see that the SRB measure is found as a Gibbs measure using equilibrium states.

## 3. Plan of the course, continued. January 10 (*Notes by LK*)

*Introduction cont.*

There are at least two reasons for studying mathematical models of the kind we discussed in the last class. First, we may satisfy our intellectual curiosity, as these examples are interesting from a purely mathematical point of view. Second, these models mirror in some respect what is going on in the real world. The exact nature of this connection with physical reality (if any), however, is in dispute.

Last class, we defined an attractor $\Lambda$ and noted that such attractors have been proved to exist in Axiom A systems. We also want to study attracting measures such as the SRB-measure defined last class. We will begin with an outline of our coverage of this topic and postpone definitions until later.

Supposing we have $\Lambda$, we will proceed through the following steps to find attracting measures. First, we will find a *Markov partition* of $\Lambda$ into sets of arbitrarily small diameter. Roughly, a Markov partition is a way to cut $\Lambda$ into sets that are mapped by $\phi$ in a nice manner; $\phi$ maps a set of the partition to a finite union of other sets of the partition.

Second, we will use the Markov partition to obtain *symbolic dynamics*. Namely, we code the *orbit* of $x$ , $\mathcal{O}(x) = \{\phi^n x\colon n \in \mathbb{Z}\}$ according to which atom of the partition $\phi^n x$ is in. Define

$$\xi\colon x \to \xi(x) \in \{0, 1, 2, \ldots, r-1\}^{\mathbb{Z}} = \Sigma_r$$

by $(\xi(x))_j = m$ if and only if $\phi^j x$ is in the $m$-th cell of the partition. Applying $\phi$ or $\phi^{-1}$ will shift the sequence in one direction or the other. Therefore, if we define $\sigma\colon \Sigma_r \to \Sigma_r$ by $(\sigma(\omega))_j = \omega_{j+1}$ where $\omega \in \Sigma_r$, then clearly $\xi(\phi x) = \sigma(\xi x)$.

We note that since the mapping on $\Lambda$ usually has some restrictions, we often don't use the full shift. Instead, we use a *shift of finite type.* We find a $\Sigma_A \subset \Sigma_r$ such that there is a map $\pi\colon \Sigma_A \to \Lambda$ which is a "tight coding", meaning that $\pi$ is one-to-one (except on a first category set) and the following diagram is commutative:

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \pi\downarrow & & \downarrow\pi \\ \Lambda & \xrightarrow{\phi} & \Lambda \end{array}$$

In fact, what we get is almost a dynamical isomorphism, so studying $\Sigma_A$ is almost the same as studying $\Lambda$. Both will have the same *topological entropy* and both will be *intrinsically ergodic*, meaning they have a unique measure of maximal entropy. Also, we will see that under the correct conditions they both will be *topologically mixing* as well.

The process described above demonstrates one of the major justifications for studying symbolic dynamics. Information theory constitutes another important application of this field.

Third, we use the *Shannon-Parry measure* on $\Sigma_A$ and map it down to the *SRB-measure* $\mu$ on $\Lambda$. The first two chapters of Bowen's book deal with finding the unique measure of maximal entropy on $\Sigma_A$, showing it is an *equilibrium state* for the constant function, and demonstrating that it is a *Gibbs measure*.

Finally, we will see that for other *Hölder continuous functions* $f\colon \Lambda \to \mathbb{R}$, we obtain that $f \circ \pi\colon \Sigma_A \to \mathbb{R}$ is a Hölder continuous function and has a unique equilibrium state which projects to one on $\Lambda$.

In our attempts to understand these steps we will learn about *expansiveness*, *specification*, *pseudo-orbit shadowing*, *stable and unstable manifolds*, and *canonical coordinates*. To study the dynamical aspects of $(\Lambda, \phi, \mu)$, we will discuss *Lyapunov exponents*, the *multiplicative* and *subadditive ergodic theorems*, *topological* and *measure-theoretic entropy*, *Hausdorff dimension*, and various formulas (Pesin, Young) relating these topics.

We note that there are approaches to the SRB-measure that do not use Markov partitions. Often there is numerical evidence (and sometimes a proof) that SRB-measures exist even when Markov partitions don't. In any case, Markov partitions are an extremely useful tool, as well as being historically important.

*Smale's horseshoe*

3.1. **Smale's horseshoe.** We are now ready for our first concrete example, the *Smale horseshoe map*. This example is useful for illustrating how chaotic dynamics can arise in a deterministic dynamical system. In this example we will observe that the attractor $\Lambda$ looks like a 2-shift. Further, we will see that this map is also found in more complicated dynamical systems.

We begin with a rectangle $R \subset \mathbb{R}^2$ and define a map $\phi\colon R \to \mathbb{R}^2$ as in Figure 2.

We can see the *hyperbolicity* as a strong uniform contraction in one direction and a strong uniform expansion in the other direction. As defined, this is not a map of manifolds. However, we could put this map onto a 2-sphere and extend it to a mapping of the entire 2-sphere (except one point) and study it there; this was the approach that Smale used.

We will use $R_0$ and $R_1$ as our basic partition; these labels also provide a coding.

We are looking for the largest $\phi$-invariant set $\Lambda = \bigcap_{n\in\mathbb{Z}} \phi^n R$. We include sketches of $R \cap \phi R$ and $R \cap \phi R \cap \phi^2 R$ in Figures 3 and 4 respectively.

If we continue this process, we see that $R \cap \phi R \cap \phi^2 R \ldots = C \times I$, a Cantor set cross an interval. Similarly, Figure 5 shows an illustration of $R \cap \phi^{-1} R$.

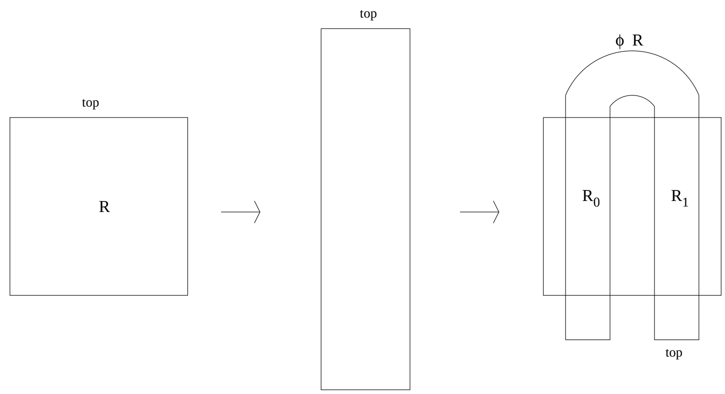



FIGURE 2. Smale's horseshoe

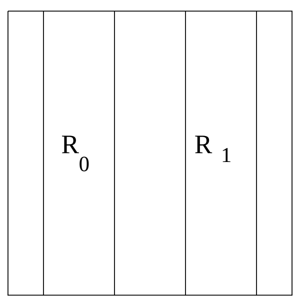



FIGURE 3. $R \cap \phi R$

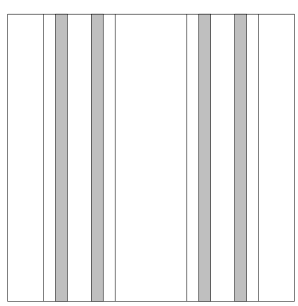


FIGURE 4. $R \cap \phi R \cap \phi^2 R$

We also obtain that $R \cap \phi^{-1}R \cap \phi^{-2}R \ldots = I \times C'$, an interval cross a Cantor set. Therefore,

$$\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^n R = (C \times I) \cap (I \times C') = C \times C'.$$

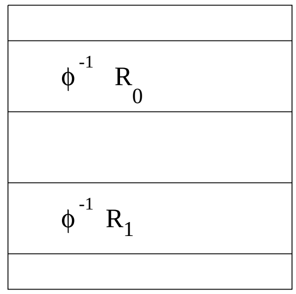



FIGURE 5. $R \cap \phi^{-1} R$

Thus, $\Lambda$ is a compact, totally disconnected, perfect set and is topologically a Cantor set.

In the next lecture, we will show that $(\Lambda, \phi|_\Lambda)$ is topologically conjugate to $(\Sigma_2, \sigma)$; that is, there exists a homeomorphism $h\colon \Sigma_2 \to \Lambda$ such that $h\sigma = \phi h$.

## 4. Smale's horseshoe. January 13 (*Notes by MA*)

*Smale's horseshoe cont.*

We continue our discussion of Smale's horseshoe and show its connection with symbolic dynamics. See Section 3.1 for a description of how Smale's horseshoe is constructed.

Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ be the space of bi-infinite sequences of zeros and ones. We can make $\Sigma_2$ into a metric space with the distance function defined by

$$d(\omega, \eta) = \frac{1}{1+k}, \text{ where } k = \inf\{|j| : \omega_j \neq \eta_j\}.$$

**Remarks 4.1.**

(1) In this metric, two points (sequences) of $\Sigma_2$ are *close* if they agree on a long central block. Basic open sets are *cylinder sets* centered about 0; i.e., sets of the form

$$U = [u_{-j} \dots u_0 \dots u_j] = \{\omega \in \Sigma_2 : \omega_i = u_i \text{ for } |i| \leq j\}.$$

(2) $\Sigma_2$ is a compact, totally disconnected metric space.

(3) The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ is a homeomorphism.

Define the map $\pi : \Sigma_2 \to \Lambda$ by

$$\pi(\omega) = \bigcap_{n \in \mathbb{Z}} \phi^{-n} R_{\omega_n}.$$

**Theorem 4.1.** *The map $\pi$ is a well-defined, one-to-one, onto, continuous map which conjugates $\sigma$ and $\phi$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \Lambda & \xrightarrow{\phi} & \Lambda \end{array}$$

**Remark 4.1.** We say that $(\Sigma_2, \sigma)$ and $(\Lambda, \phi)$ are *topologically conjugate* and write $(\Sigma_2, \sigma) \approx (\Lambda, \phi)$.

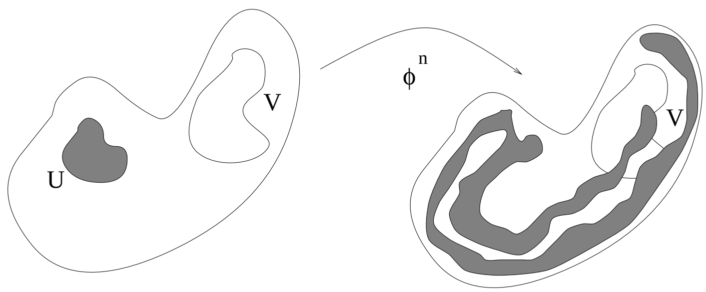



FIGURE 6. Topological Mixing

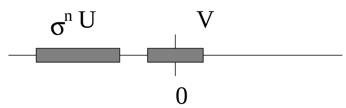



FIGURE 7. $\sigma^n U$ and $V$

The proof of this theorem is left to the reader.

Because $(\Sigma_2, \sigma)$ and $(\Lambda, \phi)$ are topologically conjugate, we can use the dynamics of the shift on $\Sigma_2$ to tell us about the behavior of $\phi$ on $\Lambda$. In particular, we will show that $\phi$ is *topologically mixing* and *topologically transitive*.

**Definition 4.1.** A system $(X, \phi)$ is called *topologically mixing* if for any nonempty open sets $U$ and $V$ in $X$, there exists an $N$ such that $\phi^n U \cap V \neq \emptyset$ for all $n \geq N$.

A system $(X, \phi)$ is called *topologically transitive* (or *topologically ergodic*) if there exists a dense orbit. Equivalently, if every closed invariant proper subset of $X$ is nowhere dense (has empty interior). See Exercise 1 (Section 5).

If $(X, \phi)$ is topologically mixing, then the images of $U$ will eventually 'fill' the entire space $X$. See Figure 6.

**Corollary 4.2.** *The map $\phi$ restricted to $\Lambda$ ($\phi|_\Lambda$) is topologically mixing, and hence topologically transitive. Further, the periodic points of $\phi$ are dense in $\Lambda$.*

*Proof.* We will actually show that $(\Sigma_2, \sigma)$ is topologically mixing (and hence topologically transitive) with dense periodic points. Since $\pi$ conjugates $\sigma$ and $\phi$, we will then have the corollary.

Let $U$ and $V$ be open subsets of $\Sigma_2$. Without loss of generality, we may consider $U$ and $V$ to be basic open sets: $U = [u_{-j} \dots u_j]$ and $V = [v_{-k} \dots v_k]$. Take $N > j + k$. Then for $n > N$, $\sigma^n U$ will be centered at $-n$, and $\sigma^n U$ will not specify any coordinates also specified by $V$ (see Figure 7).

Thus there is some $\omega$ in $\Sigma_2$ such that $\omega_{-j-n} \dots \omega_{j-n} = u_{-j} \dots u_j$, and $\omega_{-k} \dots \omega_k = v_{-k} \dots v_k$, and so $\sigma^n U \cap V \neq \emptyset$. Therefore $(\Sigma_2, \sigma)$ is topologically mixing.

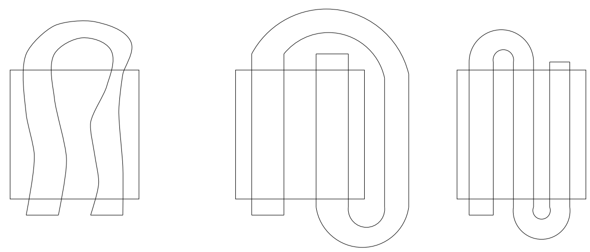


FIGURE 8. Variations of Smale's horseshoe

That the periodic points of $\sigma$ are dense in $\Sigma_2$ is easy to see: given $\omega \in \Sigma_2$, let $n$ be a large integer and set $\omega^n$ to be the finite sequence $(\omega_{-n} \dots \omega_n)$. Then define $\omega' = (\dots \omega^n \omega^n \omega^n \dots)$. $\omega'$ will be a periodic sequence (and thus a periodic point for $\sigma$) that is close to $\omega$. $\square$

Note that once we have topological mixing (and thus topological transitivity), we have some sort of *nontrivial recurrence.* This is the first sign of complicated dynamics.

**Remark 4.2.** The horseshoes will persist, at least under $C^1$ perturbations. Thus we will have the same topological dynamics even if the map $\phi$ is "wiggled" a bit. Besides perturbing $\phi$, other variations are also possible. For instance, we could have the image of $R$ intersect $R$ in several places. Examples of possible variations are given in Figure 8. Common properties of the variations include strict hyperbolicity and complete strips for $\phi(R)$ and $\phi^{-1}(R)$.

**Remark 4.3.** Horseshoes are found in actual systems. For example:

(1) Hénon map. This is a map $h$ (and $h^{-1}$) defined on $\mathbb{R}^2$ by

$$h(x, y) = (a - by - x^2, x)$$
$$h^{-1}(x, y) = \left(y, \frac{a - x - y^2}{b}\right)$$

where $a$ and $b$ are parameters. For $b = -0.3, a = 1.4$, experimental evidence seems to indicate the existence of a strange attractor.

One can show (Devaney-Nitecki) that for $b \neq 0$ and $a$ large enough (say, $a = 5, b = -0.3$), there exists a square $R_{ab}$ which has a horseshoe attractor $\Lambda$. The action of $h$ is indicated in Figure 9.

Benedicks-Carleson showed there are strange attractors for some (even many) $a, b$. Other recent work on the Hénon map by John Smillie and Zhongguo Yang (UNC Ph.D.) explores the complicated dynamics of this system for various values of the parameters.

(2) Smale's homoclinic point theorem. We will cover this next time.

## 5. SOME TOPOLOGICAL AND DIFFERENTIABLE DYNAMICS. JANUARY 15 (*Notes by MA*)

We start with a list of definitions, then an exercise.

**Definition 5.1.** Let $(X, \phi)$ be a dynamical system. Then

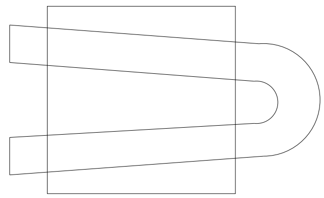



FIGURE 9. Hénon's horseshoe

(1) $(X, \phi)$ is *topologically mixing* if for $U, V$ nonempty open subsets of $X$, there exists $N$ such that for $n \geq N$, $\phi^n U \cap V \neq \emptyset$;
(2) $(X, \phi)$ is *regionally transitive* if for $U, V$ nonempty open subsets of $X$, there exists at least one $n$ such that $\phi^n U \cap V \neq \emptyset$;
(3) $(X, \phi)$ is *topologically ergodic* if every proper closed invariant subset of $X$ is nowhere dense;
(4) $(X, \phi)$ is *topologically transitive* if there exists a dense orbit.

We note that if $(X, \phi)$ is topologically transitive, then the set of points with dense orbits will be residual (the complement of a union of countably many nowhere dense sets).

**Exercise 1.** Let $X$ be a second countable, compact Hausdorff space, $\phi : X \to X$ a homeomorphism. Show that

$$\begin{aligned}(X, \phi) \text{ is topologically mixing} &\Rightarrow (X, \phi) \text{ is regionally transitive}\\ &\Leftrightarrow (X, \phi) \text{ is topologically ergodic}\\ &\Leftrightarrow (X, \phi) \text{ is topologically transitive.}\end{aligned}$$

The main thrust of the next set of lectures will be to state and explain Smale's *Homoclinic Point Theorem*. First we state the theorem, then give the background and definitions required to understand the statement. Finally, we will give an idea of the proof.

*Homoclinic Point Thm.*

**Theorem 5.1** (Smale's Homoclinic Point Theorem)**.** *Let $M$ be a compact $C^\infty$ manifold and $\phi : M \to M$ a $C^1$ diffeomorphism. Suppose that $p \in M$ is a hyperbolic periodic point for $\phi$ which has a transverse homoclinic point $x$. Then there is an $r > 0$ such that $\phi^r$ has a closed invariant hyperbolic set $\Lambda$ which contains $p$ and $x$ and such that $(\Lambda, \phi|_\Lambda)$ is topologically conjugate to the two-shift $(\Sigma_2, \sigma)$. In fact, homoclinic points $y$ and corresponding sets $\Lambda$ can be found in every neighborhood of $p$.*

### 5.1. **Some terminology from differential geometry.**

**Definition 5.2.** A *topological manifold* $M$ is a connected second countable Hausdorff space such that for each $x$ in $M$, there exists a neighborhood $U$ of $x$ and homeomorphism $h$ from $U$ to an open ball in $\mathbb{R}^d$. We call $(U, h)$ a *chart* (or system of *local coordinates*) about $x$, and we say that $M$ has *dimension $d$*. The set of charts $(U, h)$ on $M$ is called an *atlas*.

*Differential geometry background*

A $C^k$ *manifold* is a manifold such that for any two charts $(U_1, h_1)$ and $(U_2, h_2)$, the map $h_2 \circ h_1^{-1}$ on $h_1(U_1 \cap U_2)$ is a $C^k$ map (see Figure 10). In this case, a maximal atlas is called a $C^k$ *differentiable structure*.

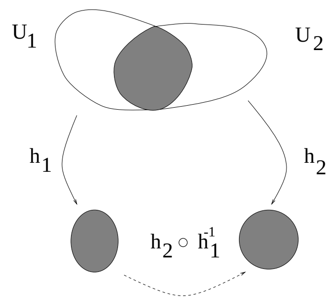



FIGURE 10

**Definition 5.3.** Let $M$ and $N$ be $C^k$ manifolds and $f : M \to N$. We say that $f$ is *differentiable* (or $C^k$) if for every pair of charts $(U,h)$ for $M$ and $(V,g)$ for $N$, the map $g \circ f \circ h^{-1}$ is differentiable (or $C^k$) on $h(U \cap f^{-1}V)$.

**Definition 5.4.** A $C^k$ *diffeomorphism* of manifolds $M$ and $N$ is a $1:1$, onto, $C^k$ map from $M$ to $N$ whose inverse is also $C^k$. We denote the space of all $C^k$ diffeomorphisms from $M$ to $N$ by $\mathrm{Diff}^k(M,N)$.

We can define a topology on $\mathrm{Diff}^k(M,N)$ — two maps $f$ and $g$ are $C^k$*-close* if for some coordinate charts $(U_1,h_1)$ on $M$ and $(U_2,h_2)$ on $N$, the maps

$$F = h_2 \circ f \circ h_1^{-1} \text{ and } G = h_2 \circ g \circ h_1^{-1}$$

are close in the $C^k$ topology on $\mathbb{R}^d$, i.e., the norms of the derivatives of $F$ and $G$ up to order $k$ are close.

Let $p$ be a point in the manifold $M$ and $\gamma$ any $(C^\infty)$ curve from $(-1,1)$ to $M$ with $\gamma(0)=p$. Then $\gamma$ acts on $C^\infty(M)$ by $f \mapsto (f\circ\gamma)'(0)$. We denote this map by $\gamma'(0)$. Note that $f\circ\gamma$ will be a map from $(-1,1)$ to $\mathbb{R}$, so the regular notion of the derivative at 0 makes sense.

We will say that two curves $\gamma$ and $\alpha$ are *equivalent* if $\gamma(0)=\alpha(0)=p$ and $\gamma'(0)=\alpha'(0)$ as maps on $C^\infty(M)$.

**Definition 5.5.** If $M$ is a manifold and $p$ is a point in $M$, then we define the *tangent space to* $M$ *at* $p$ (denoted $T_pM$) to be the set of all equivalence classes of curves $\gamma:(-1,1)\to M$ with $\gamma(0)=p$. Each equivalence class of curves is called a *tangent vector* at $p$.

**Proposition 5.2.** *Let* $M$ *be a* $d$*-dimensional manifold. Then for each* $p$ *in* $M$*,* $T_pM \cong \mathbb{R}^d$*.*

We won't prove this proposition, but will give some indication of its proof by defining the *standard basis* for $T_pM$. For $p \in M$ and chart $(U,h)$ about $p$, set $\gamma_i(t) = h^{-1}(h(p)+te_i)$, where $e_i$ is the $i^{\text{th}}$ basis element in the standard basis for $\mathbb{R}^d$ (note that we may have to rescale $t$ to keep $h(p)+te_i$ in the image of $h$ for $t$ between $-1$ and 1, but that's ok).

Then the standard basis for $T_pM$ is the set of vectors $\left\{\gamma_i'(0) := \dfrac{\partial}{\partial x_i}\right\}$.

**Definition 5.6.** The *tangent bundle* of $M$, denoted $TM$, is defined to be the disjoint union over $p \in M$ of $T_pM$; i.e., $TM = \dot{\bigcup}_{p\in M} T_pM$. We think of $TM$ as the set of ordered pairs $(p, v)$ where $p \in M$ and $v \in T_pM$.

The tangent bundle $TM$ is in fact a manifold itself. Let $(U, h)$ be any chart on $M$ and define the map $H : TM \to \mathbb{R}^d \times \mathbb{R}^d$ by

$$H(p, v) = (h(p), (v_1, v_2, \ldots, v_d))$$

where the $v_i$ are the coefficients of $v \in T_pM$ with respect to the standard basis $\left\{ \frac{\partial}{\partial x_j} \right\}$.

This gives us the chart $(U \times \cup_{p\in U} T_pU, H)$ on $TM$, and we see that $TM$ is a manifold (of dimension $2d$).

## 6. Some more smooth dynamics. January 17 (*Notes by MA*)

*Differential geometry background, cont.*

Now that we know what it means for a function $\phi : M \to N$ to be differentiable (see Definition 5.3), we define the *derivative* of $\phi$.

**Definition 6.1.** If $\phi : M \to N$ is differentiable, then the *derivative* of $\phi$ is the map $D\phi : TM \to TN$ (sometimes also denoted $T\phi$) given by

$$(D\phi)(p, \gamma'(0)) = (\phi(p), (\phi \circ \gamma)'(0)) \quad (1)$$

where $\gamma'(0)$ is a tangent vector in $T_p(M)$, $p \in M$, and $(\phi \circ \gamma)'(0) \in T_{\phi(p)}M$. We can speak of the derivative of $\phi$ *at a point* $p$, $D_p\phi$, which acts on $T_pM$ by restricting $D\phi$ to $T_pM$. $D_p\phi$ is a linear transformation from $T_pM$ to $T_{\phi(p)}N$. (The fact that $D\phi$ is well-defined follows from the differentiability of $\phi$). See Figure 11.

**Definition 6.2.** A *smooth vector field* $X$ on $M$ is a map from $M$ to $TM$ such that $X(p) = (p, v)$, where $v \in T_pM$, and which is smooth as a map between manifolds.

**Remark 6.1.** Recall that we can think of $X(p) \in T_pM$ as operating on $C^\infty(M)$ (Definition 5.5). Then the vector field $X$ is smooth if for each $p$, $X(p)$ sends smooth functions to smooth functions.

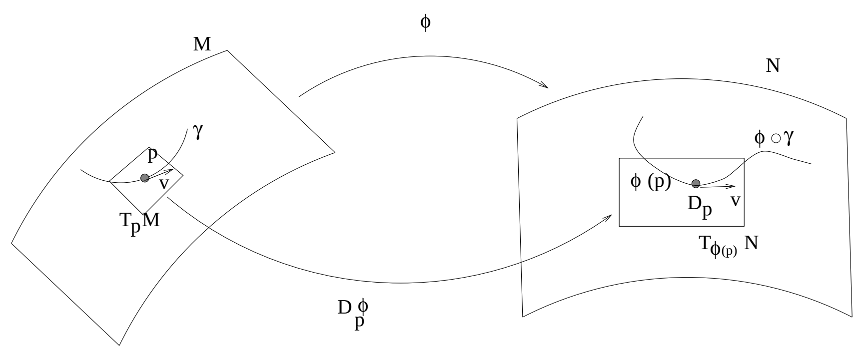



FIGURE 11. $(D_p\phi)(v)$ for $v = \gamma'(0)$

**Definition 6.3.** Let $M$ be a differentiable manifold. We say that $M$ is a *Riemannian manifold* if there is an inner product $g_p(\cdot,\cdot) =< \cdot,\cdot >_p$ defined on each tangent space $T_pM$ for $p \in M$ such that for any smooth vector fields $X$ and $Y$ on $M$, the function $p \mapsto g_p(X(p), Y(p))$ is a smooth function of $p$.

Equivalently, we could require that when we express the function $g_p$ in terms of local coordinates, each coordinate function will be a smooth function of $p$.

On a manifold $M$, there is a natural *measure class* (in the sense of absolute continuity) called the *Lebesgue measure class*: Given $(U, h)$ a chart on $M$, we say that $A \subset U$ has *measure* 0 if and only if $h(A) \subset h(U)$ has $d$-dimensional Lebesgue measure 0.

If $M$ is orientable, then we can construct elements in the Lebesgue measure class by using nondegenerate $d$-forms, called *volume forms*.

**Definition 6.4.** A *volume form* $\omega$ is a map that assigns to each $p$ an alternating $d$-tensor on the vector space $T_pM$. That is, $\omega_p$ is an alternating multilinear map from $\underbrace{T_pM \times \cdots \times T_pM}_{d \text{ times}}$ to $\mathbb{R}$. We again require that this assignment depend smoothly on $p$.

It is a basic result that there exists a nondegenerate volume form on $M$ iff oriented charts can consistently be chosen for $M$ iff the space $\bigwedge^d T^*M$ is one-dimensional ($T_p^*M$ is the space of linear maps from $T_pM$ to $\mathbb{R}$).

Let $M$ be a Riemannian (orientable) manifold. For each $p$, let $u_1, \dots, u_d$ be the standard orthonormal, positively oriented basis for $T_pM$. Then define the volume form $\omega$ by $\omega_p(u_1, \dots, u_d) = 1$.

We will use $\omega$ to define a measure $\mu_\omega$ on $M$ in the following way: If $(U, h)$ is a chart and $A \subset U$, define $\mu_\omega(A)$ by

$$\mu_\omega(A) = \int_{h(A)} \left|\omega_{h^{-1}x}\left(D_xh^{-1}(e_1), \dots, D_xh^{-1}(e_d)\right)\right| dx_1 \dots dx_d, \tag{2}$$

where $(e_1, \dots, e_d)$ is the standard basis for $\mathbb{R}^d$, $h^{-1}$ is a map from (a subset of) $\mathbb{R}^d$ to $M$, and so $D_xh^{-1}$ is a map from $T_x\mathbb{R}^d$ to $T_{h^{-1}x}M$.

The idea is to define the volume of a small box-like subset of $M$ which approximates the volume of the box spanned by $(\epsilon u_1, \dots, \epsilon u_d)$ in $T_{h^{-1}x}M$, for small $\epsilon$, to be about $\omega_{h^{-1}x}(\epsilon u_1, \dots, \epsilon u_d)$. Then, instead of integrating over $A$ in $M$, we integrate over $h(A) \subset \mathbb{R}^d$, where integration makes sense (compare (2) with the usual change of variables formula). See Figure 12.

*Dynamics background*

**Definition 6.5.** A *hyperbolic periodic point* of $\phi : M \to M$ is a point $p \in M$ such that

(1) $\phi^m(p) = p$ for some $m > 0$ (when $m$ is the smallest possible, we say that $p$ has *period* $m$);
(2) $D_p(\phi^m) : T_pM \to T_pM$ is a *hyperbolic linear map* (no eigenvalues with modulus equal to 1).

When $p$ is a hyperbolic periodic point, we can decompose $T_pM$ into a *stable* subspace $E_p^s$ (the eigenvalues of the restriction of $D_p(\phi^m)$ to $E_p^s$ have modulus less than 1) and an *unstable* subspace $E_p^u$ (the eigenvalues of the restriction of $D_p(\phi^m)$ have modulus greater than 1). Thus $T_pM = E_p^s \oplus E_p^u$.

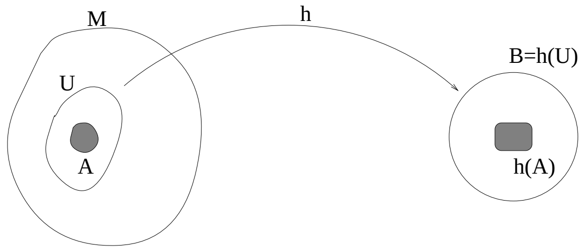



FIGURE 12. The map $h$ takes $U \subset M$ to $B \subset \mathbb{R}^d$

## 7. Hartman-Grobman and Stable Manifold theorems. January 22 (*Notes by NP*)

We will start off today's notes with some topological definitions which lead up to another condition equivalent to topological ergodicity. Assume that $X$ is a second countable compact Hausdorff space.

**Definition 7.1.** A set $G \subset X$ is *residual* if it contains the countable intersection of dense open sets. A subset of the complement of a dense open set is called *nowhere dense*. A set $F \subset M$ is said to be *first category* if it is the countable union of nowhere dense sets. A set $E \subset M$ is said to have the *property of Baire* if $E = (G \cup M_1)\backslash M_2$ with $G$ open and $M_1, M_2$ first category.

**Exercise 2.** Show that the following condition can be added to the list of conditions equivalent to topological ergodicity studied in Exercise 1:

> If $E$ is an invariant set for $\phi$ (i.e. $\phi(E) = E$) with the property of Baire, then either $E$ or $E^c$ is first category.

This condition should be compared to the measure-theoretic definition of ergodicity: $\phi$ is metrically (measure-theoretically) ergodic for an invariant measure $\mu$ if and only if $X$ is indecomposable, i.e., $\phi(E) = E \Rightarrow \mu(E) = 0$ or $\mu(E^c) = 0$. The equivalence of these various forms of topological ergodicity is stated in an article by John Oxtoby in the Proceedings of the National Academy of Sciences, 1937.

7.1. **Stable and unstable sets.** Let's get back to defining the terms we need in order to understand the Smale Homoclinic Point Theorem. To this point we have defined the topological space everything is taking place on (a $C^\infty$ Riemannian manifold $M$), what kind of action is taking place (a $C^1$ diffeomorphism), and what it means for $p \in M$ to be a hyperbolic periodic point for $\phi$. In order to study the ideas in a notationally and conceptually simpler fashion we will restrict our attention to hyperbolic fixed points of $\phi$; this makes sense because if $p$ is a periodic point with $\phi^m(p) = p$ then it is a fixed point of the diffeomorphism $\phi^m$.

*Dynamics background cont.*

Given any $p \in M$ we can consider the behavior under $\phi$ of other points in $M$ through comparison to the behavior of $p$ under $\phi$. The *stable or forward asymptotic set* of $p$ is the set of all points whose forward iterates approach the forward iterates of $p$; formally:

$$W^s(p) = \{x \in M : d(\phi^n x, \phi^n p) \to 0 \text{ as } n \to \infty\}.$$

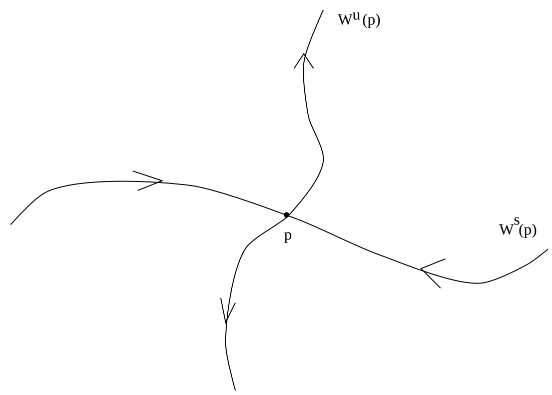



FIGURE 13. The stable and unstable sets of $p$

Similarly, we can look at the backwards iterates of points to define the *unstable or backward asymptotic set* of $p$; it is the set of all points whose iterates under $\phi^{-1}$ approach those of $p$:

$$W^u(p) = \{x \in M : d(\phi^{-n}x, \phi^{-n}p) \to 0 \text{ as } n \to \infty\}.$$

Note that since we are on a $C^\infty$ manifold it is possible to define a metric on $M$ which is consistent with the ambient differentiable structure and this is why it makes sense to look at the distance between the iterates of points in $M$. Figure 13 shows the movement under $\phi$ of points in the stable and unstable sets of $p$.

7.2. **Properties of hyperbolic fixed points.** The following very important theorem shows that near its hyperbolic fixed points a function is locally linearizable; it acts just like its derivative tells you it should. That is, there is a topological conjugacy between $\phi$ and $D_p\phi$ on a neighborhood $U$ of $p$.

**Theorem 7.1** (Hartman-Grobman)**.** *Let $p$ be a hyperbolic fixed point of a $C^k$ diffeomorphism $\phi : M \to M$. Then locally $\phi$ is the same as $D_p\phi$ up to a continuous change of coordinates, i.e. there are a neighborhood $U$ of $p$ and a homeomorphism $h : U \to T_pM$ such that*

$$(D_p\phi) \circ h = h \circ \phi$$

*where both are defined.*

The situation is depicted in this commutative diagram.

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \phi(U) \\ h\downarrow & & \downarrow h \\ T_pM & \xrightarrow{D_p\phi} & T_pM \end{array}$$

At hyperbolic fixed points the tangent space splits into contracting and expanding subspaces; in fact these subspaces are copies of the stable and unstable sets of the fixed point. It is also true that the stable and unstable sets are copies of Euclidean spaces inside $M$,

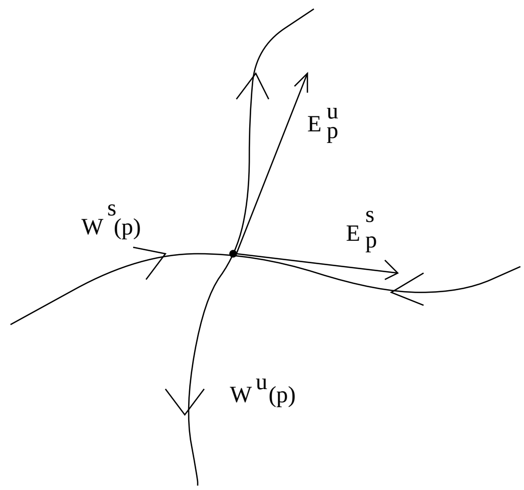



FIGURE 14. The tangent space to $M$ at $p$.

although they are not actually submanifolds of $M$. (By abuse of terminology, $W^s(p)$ and $W^u(p)$ are sometimes referred to as the stable and unstable manifolds of p.) The topological conjugacy is now made precise in the Stable Manifold Theorem.

**Definition 7.2.** Given a manifold $M$ and a set $N \subset M$ we say that $N$ is *injectively immersed* in $M$ if there is a manifold $\hat{N}$ and a 1:1 smooth map $\psi : \hat{N} \to M$ with $D\phi$ injective and $\psi(\hat{N}) = N$. We say that $N$ is *tangent at* $p = \psi(x)$ to the subspace $E$ of $T_pM$ if $D_x\psi : T_x\hat{N} \to T_pM$ has image $E$.

**Theorem 7.2** (Stable Manifold Theorem)**.** *Let $p$ be a hyperbolic fixed point of a $C^k$ diffeomorphism $\phi : M \to M$ with splitting $T_pM = E_p^s \oplus E_p^u$ into $D_p\phi$-invariant contracting and expanding subspaces. Then $W^s(p)$ and $W^u(p)$ are injectively immersed images of Euclidean spaces in $M$ which are tangent to $E_p^s$ and $E_p^u$ and thus have dimensions* $\dim(E_p^s)$ *and dim$(E_p^u)$ respectively.*

Elements of the statement and proof of this theorem go back to Poincaré, Hadamard, Perron, and others, but the modern statement presented here was given by Hirsch and Pugh.

Figure 14 is a diagram intended to depict the stable and unstable manifolds of $p$ along with its tangent space decomposition. Remember that the manifolds and subspaces involved here need not be just one-dimensional.

Putting the Stable Manifold Theorem together with the Hartman-Grobman Theorem we see that a neighborhood of $p$ in $M$ is compressed in the direction of the stable set and stretched in the direction of the unstable set. Points near the stable set move closer to $p$ while those near the unstable set move away from it. If there is a point $x$ which is in both sets, how do points near $x$ behave under $\phi$? Such a point $x$ is called a *homoclinic point* and the existence of such a point causes interesting dynamics. The big idea is:

$\boxed{\textit{hyperbolicity} + \textit{recurrence} \Rightarrow \textit{very complicated behavior.}}$

## 8. Homoclinic points. January 24 (*Notes by NP*)

*Dynamics background cont.*

8.1. **Homoclinic points and the Smale Homoclinic Point Theorem.** Assume that $p$ is a hyperbolic fixed point for $\phi$ and let $x \in W^s(p) \cap W^u(p)$, so that $x$ is a homoclinic point for $p$. We say that $x$ is a *transverse* homoclinic point for $p$ if the tangent spaces to $W^s(p)$ and $W^u(p)$ at $x$ span $T_xM$. See Figure 15, where $x$ is a transverse homoclinic point of $p$ but the point $y$, while homoclinic, has only a partial tangency and is therefore not transverse.

In the presence of a transverse homoclinic point $x$ for a hyperbolic fixed point $p$ we find $\phi$ to be very unpredictable. In fact, it is just as unpredictable as Smale's horseshoe mapping (a.k.a. the full 2-shift!). In fact, if one examines the behavior of points in a neighborhood $U$ of $p$ (such as the neighborhood of $p$ containing $x$ in Figure 16,) one sees that $\phi$ stretches out $U$ along the unstable set according to the linear map $D_p(\phi)$.

For $k$ sufficiently large $\phi^k(U)$ will stretch around to contain $x$ again. In fact, this is how the horseshoe mapping and coding onto the full 2-shift is seen in this system at the hyperbolic point $p$, and it is the idea of the proof of Smale's Homoclinic Point Theorem (Theorem 5.1), which we will now outline.

*Proof of Smale's Homoclinic Point Thm.*

8.2. **A sketch of the proof of Smale's Homoclinic Point Theorem.** A reference for this proof is found in S. Newhouse's Lectures on Dynamical Systems (CIME lectures, Bressanone, 1978, Birkhauser, 1980, pages 1 - 114).

Without loss of generality we can assume that the hyperbolic periodic point $p$ is actually a fixed point for $\phi$ (if not, replace $\phi$ with $\phi^m$ where $m$ is the period of $p$). Construct a neighborhood $U$ of $p$ as (in local coordinates) the product of a small disk in $W^u(p)$ and a small open ball in $W^s(p)$. This neighborhood contains a homoclinic point $x$ since all

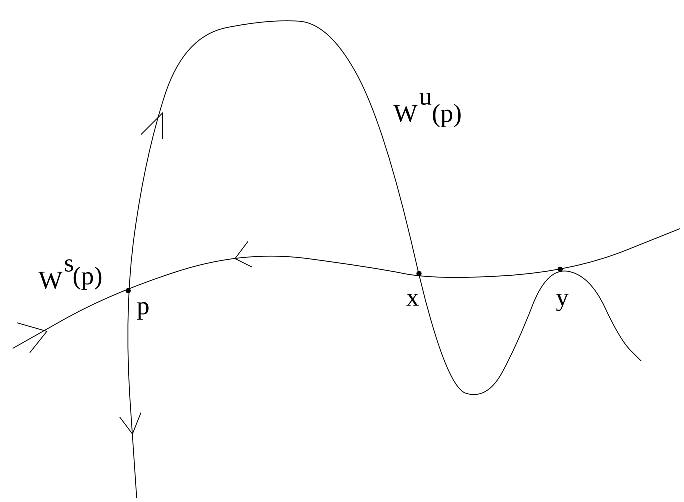



Figure 15. A hyperbolic point $p$ and a transverse homoclinic point for $p$.

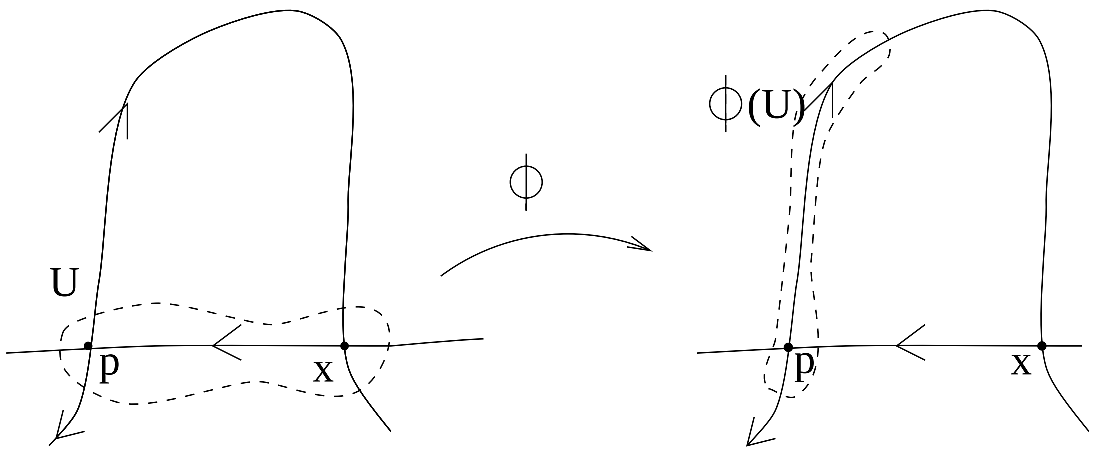



FIGURE 16. The action of $\phi$ on a neighborhood of a hyperbolic point.

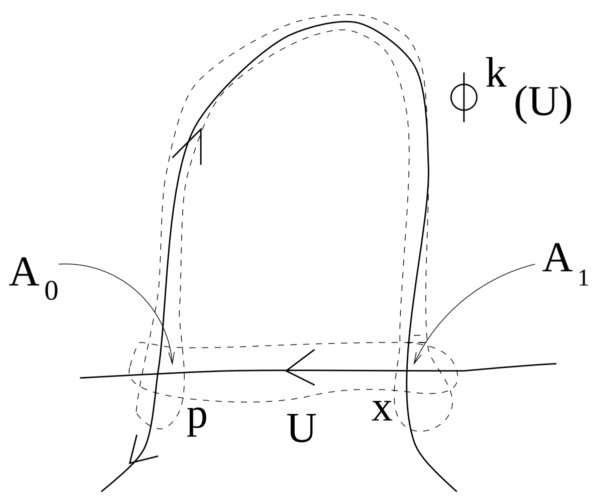



FIGURE 17. Under repeated iteration of $\phi$ there is recurrence.

homoclinic points are in $W^s(p)$ and therefore have forward iterates arbitrarily close to $p$; any image under $\phi$ of a homoclinic point is also homoclinic.

Since $p$ is a hyperbolic point for $\phi$ we know that the neighborhood $U$ is being stretched along $W^u(p)$ and will eventually stretch enough to contain $x$ again. Choose the first $k$ for which $\phi^k(U) \cap U$ is nonempty and for which $x \in \phi^k(U)$. Looking at Figure 17, one can see the Smale horseshoe mapping by restricting attention to the behavior of $\phi$ in the neighborhood $U$. (The disk in $W^u(p)$ is taken small enough that what these pictures show actually happens, except perhaps with other intersections of $\phi^k(U)$ and $U$ also "between" $p$ and $x$.)

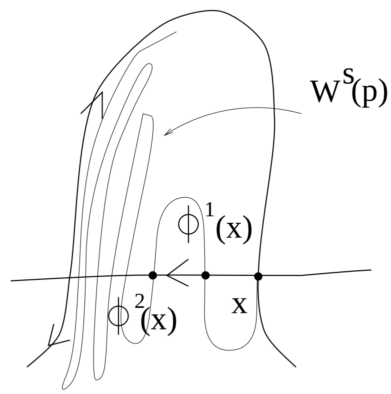



FIGURE 18. $W^s(p)$ accumulates back on itself.

Label $A_0$ the connected component of $\phi^k(U) \cap U$ containing $p$, and $A_1$ the one which contains $x$, and call $\Lambda$ the largest $\phi$-invariant set in $A_0 \cup A_1$. That is,

$$\Lambda = \bigcap_{j=-\infty}^{\infty} (\phi^k)^j (A_0 \cup A_1).$$

Now we can code $\Lambda$ over to the full 2-shift via a map $h$; for any $\omega \in \Sigma_2$ let

$$h(\omega) = \bigcap_{j=-\infty}^{\infty} (\phi^k)^{-j} A_{\omega_j}$$

This gives the commutative diagram:

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \\ h\downarrow & & \downarrow h \\ \Lambda & \xrightarrow{\phi} & \Lambda \end{array}$$

The mapping treats a point $\omega \in \Sigma_2$ as the itinerary of some point $u \in U$ as it travels via $\phi$ through the sets $A_0$ and $A_1$. The map $h$ is one to one since the sets $A_0$ and $A_1$ are disjoint, and it is clear that $h$ intertwines $\sigma$ and $\phi$. The surjectivity and bicontinuity of the mapping should also be checked, as well as the fact that $\Lambda$ is a hyperbolic set, a notion to be defined in the next lecture.

8.3. **The homoclinic mesh.** Since it is true that the stable and unstable sets of a hyperbolic fixed point are preserved under $\phi$, we can investigate their structure by looking at the forward and backward iterates of a transverse homoclinic point $x$ (if there is one). It is easy to see that each $\phi^j(x)$ is also a homoclinic point for $p$; it is also transverse since $\phi$ is a diffeomorphism. So, as we watch the iterates of $x$ under $\phi$ march through the stable set of $p$ we see that $W^s(p)$ must behave as in Figure 18.

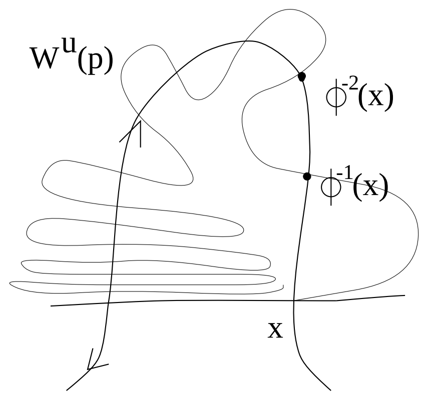



FIGURE 19. $W^u(p)$ accumulates back on itself.

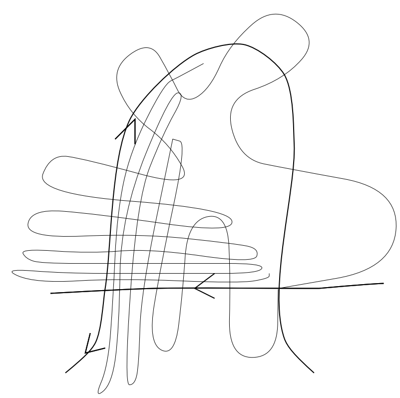


FIGURE 20. The homoclinic mesh.

A similar phenomenon occurs when we examine the behavior of the backwards iterates of $x$, and look at the unstable set (see Figure 19).

When we combine the information from these pictures we get a feeling for how chaotic the set of homoclinic points must actually be (see Figure 20).

## 9. Axiom A systems. January 27 (*Notes by MA*)

*Axiom A systems*

We now start our study of *Axiom A diffeomorphisms*. These form a large class of diffeomorphisms that contain many interesting examples, such as hyperbolic toral automorphisms and time one maps of gradient flows. More generally, Axiom A diffeomorphisms include two important classes of diffeomorphisms: *Anosov* and *Morse-Smale* (definitions below). For related material from the coursepack, see Bowen, Chap. 3 (p. 68) and DGS, Chap. 23 (p. 224).

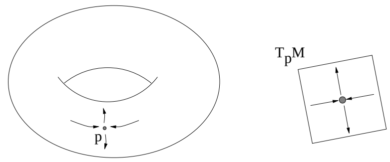



FIGURE 21. Automorphism of a torus, showing the splitting of $T_pM$

**Definition 9.1.** A closed $\phi$-invariant subset $\Lambda$ of a (compact, connected, $C^\infty$, Riemannian) manifold is a *hyperbolic set* for $\phi$ if there is a (continuous) splitting of the tangent bundle $T\Lambda = E^u \oplus E^s$ ($T_pM = E^u_p \oplus E^s_p$ for $p \in \Lambda$) such that

(1) $(D_p\phi)E^u_p = E^u_{\phi(p)}$ and $(D_p\phi)E^s_p = E^s_{\phi(p)}$

(2) there are constants $c > 0$ and $\lambda < 1$ (independent of $p$) such that

$$||D_p(\phi^n)v|| \leq c\lambda^n\, ||v|| \text{ for } v \in E^s_p, n \geq 0$$
$$||D_p(\phi^n)v|| \leq c\lambda^{-n}\, ||v|| \text{ for } v \in E^u_p, n \leq 0.$$

Note that $c$ and $\lambda$ may depend on the Riemannian norm. If the Riemannian metric is such that $c = 1$, then the metric is called *adapted*. It is always possible to find an adapted Riemannian metric, so we really only need consider $n = \pm 1$. Also, that the splitting varies continuously (in the topology on the tangent bundle) is actually a consequence of conditions (1) and (2).

**Remarks 9.1.**

(1) The attractor $\Lambda$ in the Smale horseshoe is an example of a hyperbolic set. In this case, we have uniform hyperbolicity.

(2) The horseshoe we found in Smale's Homoclinic Point Theorem (5.1) is a hyperbolic set. Figure 20 indicates hyperbolicity at each point, and in fact estimates can be made precise using the hyperbolicity at the fixed (periodic) point $p$.

(3) You can prove hyperbolicity by finding a *field of cones* $C_p \subset T_p\Lambda$ such that $(D_p\phi)C_p = C_{\phi(p)}$, and there is an $m$ such that $D_p\phi^m$ expands on $C_p$ and $D_p\phi^{-m}$ expands on $T_pM \backslash C_p$. This is sometimes easier than finding an exact splitting of the tangent spaces.

An important class of diffeomorphisms are *Anosov* diffeomorphisms:

**Definition 9.2.** The dynamical system $(M, \phi)$ is *Anosov* if all of $M$ is a hyperbolic set for $\phi$.

**Example 9.1** (Hyperbolic automorphisms of the torus)**.** Let $\phi$ be the map on the torus $\mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then it is easy to see that each point of the torus is in the hyperbolic set for $\phi$. See Figure 21.

**Definition 9.3.** The *nonwandering set* for $\phi$, denoted $\Omega(\phi)$, is the set of all $x \in M$ for which given any neighborhood $U$ of $x$, there exists an $n > 0$ such that $\phi^n U \cap U \neq \emptyset$.

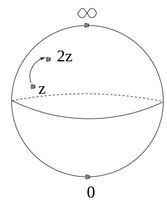



FIGURE 22. The map $z \mapsto 2z$ on $S^2$

Near a point in the nonwandering set, the dynamics of $p$ exhibits a weak form of recurrence.

**Exercise 3.** Show that $\Omega(\phi)$ is closed and $\phi$-invariant.

At the other 'extreme' of diffeomorphisms are *Morse-Smale* diffeomorphisms:

**Definition 9.4.** The dynamical system $(M, \phi)$ is *Morse-Smale* if

(1) $\Omega(\phi)$ is finite (hence there are finitely many periodic points);
(2) each periodic point is hyperbolic;
(3) if $x, y \in \Omega(\phi)$, then $W^s(x)$ and $W^u(y)$ intersect transversally, i.e., at each point $z$ in the intersection, the tangent spaces (from the immersions) span $T_z M$.

**Example 9.2.** Time one maps of gradient flows, such as the North-South map on $S^2 =$ Riemann sphere ($z \mapsto 2z$, see Figure 22) are Morse-Smale. In this case, $\Omega(\phi) = \{0, \infty\}$. The point 0 is a repelling fixed point (a *source*) while the point $\infty$ is an attracting fixed point (a *sink*), so we have that

$$W^s(0) = \{0\} \qquad W^u(0) = S^2 - \{\infty\}$$
$$W^s(\infty) = S^2 - \{0\} \qquad W^u(\infty) = \{\infty\}.$$

It's clear that the points 0 and $\infty$ are hyperbolic, and for any point $z$ in the intersection of $W^u(0)$ and $W^s(\infty)$, $T_z(W^u(0)) + T_z(W^s(\infty)) = T_z S^2$, i.e., the intersection is transverse.

## 10. STRUCTURAL STABILITY. JANUARY 29 (*Notes by MA*)

To continue the discussion of Morse-Smale systems, we note that Condition 3 of Definition 9.4 is sometimes referred to as *strong transversality*, but the definition may vary. *Axiom A systems cont.*

More examples of Morse-Smale systems include time one maps of gradient flows on a torus with $n$ 'holes'. In this case, there will be $2(n+1)$ points in the nonwandering set $\Omega(\phi)$ ($n+1$ sources and $n+1$ sinks). See Figure 23.

Anosov and Morse-Smale systems represent two 'extremes' of classes of diffeomorphisms. In an effort to combine these (and other) classes, Smale introduced the notion of *Axiom A*:

**Definition 10.1.** The dynamical system $(M, \phi)$ is called *Axiom A* if

(1) $\Omega(\phi)$ is a hyperbolic set (note: $\Omega(\phi)$ will be a *potential attractor*);
(2) the periodic points for $\phi$ are dense in $\Omega(\phi)$.

**Remarks 10.1.**

(1) It is clear that Morse-Smale systems are Axiom A.
(2) It is also true that Anosov systems are Axiom A — this follows from the *Anosov Closing Lemma* (Theorem 22.3).
(3) In dimension 2, condition 1 of the definition implies 2 (Newhouse-Palis).
(4) In dimensions greater than 2, condition 1 need not imply 2 (Dankner).

In Axiom A systems, the hyperbolicity and recurrence combine to give us complicated dynamics (on $\Omega(\phi)$). There are three things to discuss: existence of complicated dynamics, the persistence of qualitative dynamic behavior under perturbation, and typicality (genericity).

To make the notion of *persistence* more precise, we have

**Definition 10.2.** The dynamical system $(M, \phi)$ is $(C^1)$ *structurally stable* if there exists a neighborhood $N$ of $\phi$ in the $C^1$ topology such that each $\psi \in N$ is topologically conjugate to $\phi$ (i.e., there exists a homeomorphism $h : M \to M$ such that $h \circ \phi = \psi \circ h$).

Physically, structurally stable systems are the ones that are useful. Since we are not able to make precise measurements, it is necessary to know that even when our numbers (say, the coefficients in a system of differential equations) are a bit off, the overall behavior of our observed system is the same as for one with nearby values of the parameters.

In particular, the topological dynamics of $(M, \phi)$ and $(M, \psi)$ will be the same: Periodic points of one correspond to periodic points of the other; both are either topologically transitive (or mixing) or not; invariant Borel measures of one correspond to invariant Borel measures of the other; closed invariant sets of one correspond to closed invariant sets of the other; and, their topological entropies will be equal.

**Theorem 10.1** (Robbin, Robinson)**.** *Axiom A with strong transversality implies structural stability.*

Converses to this theorem have been conjectured and proved for some cases. The following two corollaries were proved before Theorem 10.1

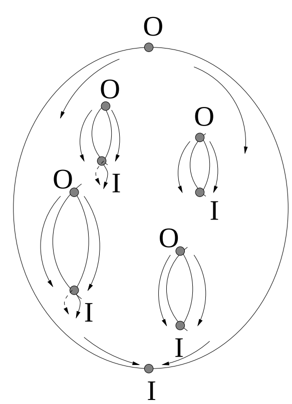



FIGURE 23. The gradient flow on a torus of genus 4. Points marked $O$ are sources; points marked $I$ are sinks.

**Corollary 10.2** (Palis-Smale). *Morse-Smale systems are structurally stable.*

For example, in the time one map of the gradient flow $\phi$ indicated by Figure 23, any small perturbation of $\phi$ would simply move the 'holes' of $M$ slightly. No new holes would be created, or old ones removed.

**Corollary 10.3** (Anosov). *Anosov systems are structurally stable.*

To discuss how 'typical' Axiom A systems are, we need the notion of *generic*:

**Definition 10.3.** A subset $E$ of a complete metric space is called *generic* if it is the complement of a first category set. Equivalently, $E$ is generic if it contains a dense $G_\delta$ set (a $G_\delta$ set is a countable intersection of open sets).

**Theorem 10.4** (Kupka-Smale). *Conditions (2) and (3) of Definition 9.4 (hyperbolic periodic points and strong transversality) are generic in the $C^1$ topology on $\mathcal{D}^1(M)$ (the space of $C^1$ diffeomorphisms from $M$ to $M$).*

**Corollary 10.5.** *The existence of homoclinic points and hence horseshoes is generic.*

**Remark 10.1.** For an explicit method for finding homoclinic points, refer to the Poincare-Melnikov-Arnold method detailed in Robinson.

**Remark 10.2.** Are there Axiom A systems that do *not* have strong transversality? Probably not — there could be some tangencies that do not persist under perturbations.

As an example of an Axiom A system satisfying strong transversality, consider the Smale's horseshoe map extended to $S^2$ (see Newhouse, p. 43). In this case, the nonwandering set is the attractor, plus a few fixed points needed to fit it onto $S^2$. Hyperbolicity on the nonwandering set is easy to see, as is the fact that periodic points are dense (since they are dense in the 2-shift). And, while it is not so easy to see the stable and unstable manifolds for individual points in the nonwandering set, it is true that they satisfy strong transversality. We conclude then that this map is structurally stable.

## 11. Smale's solenoid. January 31 (*Notes by PS*)

Recall the definition of *Axiom A* (Definition 10.1):

**Definition.** A dynamical system $(M, \phi)$ is called *Axiom A* if

(1) the nonwandering set $\Omega(\phi)$ is hyperbolic, and
(2) $\Omega(\phi)$ is the closure of the periodic points for $\phi$, i.e. the periodic points for $\phi$ are dense in $\Omega(\phi)$.

Axiom A systems include Anosov and Morse-Smale systems. Different pieces of the nonwandering set $\Omega(\phi)$ can be sources, sinks or attracting/repelling in different tangent directions.

One noninvertible example is the map $f(z) = z^2$ on the complex sphere. The nonwandering set consists of the attracting fixed points 0 and $\infty$, as well as the equator $|z| = 1$ , where the dynamics are complicated.

11.1. **Smale's solenoid, an Axiom A attractor.** Let $\overline{M} = S^1 \times D^1 = \{(\theta, z) : 0 \le \theta < 2\pi, |z| \le 1\}$ be a solid torus with boundary. *Smale's solenoid*

Define the map $\phi : \overline{M} \to \overline{M}$ by wrapping around twice in the $\theta$ direction and shrinking by a factor of $\frac{1}{4}$ in the $z$ direction, with a twist:

$$\phi(\theta, z) = \left(2\theta, \frac{1}{4}z + \frac{1}{2}e^{i\theta}\right).$$

See Figure 26.

**Remarks 11.1.**

(1) $\phi(\overline{M})$ wraps around twice.
(2) $z$ is shrunk by $\frac{1}{4}$ and translated by at most $\frac{1}{2}$, so $\phi(\overline{M}) \subset \overline{M}$.
(3) $\phi$ is injective.

To see 3, consider the cross-section of $\phi(\overline{M})$ for a fixed $\theta$:

$$\left\{\frac{z}{4} + \frac{1}{2}e^{i(\theta/2)} : |z| \le 1\right\} \cup \left\{\frac{z}{4} + \frac{1}{2}e^{i(\pi+\theta/2)} : |z| \le 1\right\}.$$

We have two disks of radius $\frac{1}{4}$ centered at $\frac{1}{2}e^{i(\theta/2)}$ and $\frac{1}{2}e^{i(\pi+\theta/2)}$.

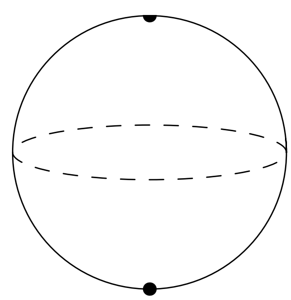


FIGURE 24. The complex sphere. The north pole ($\infty$) and the south pole (0) are attracting fixed points for $f(z) = z^2$.

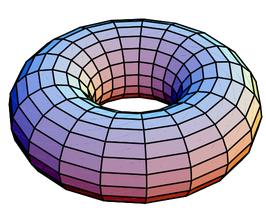


FIGURE 25. The solid torus $\overline{M}$.

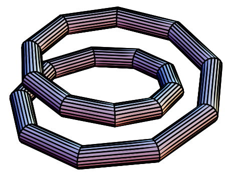


FIGURE 26. The image of $\phi : \overline{M} \to \overline{M}$.

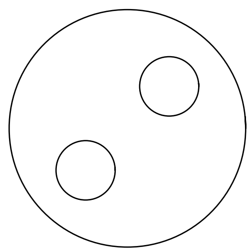


FIGURE 27. A cross-section of $\phi(\overline{M})$ for a fixed $\theta$.

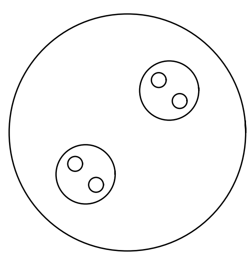


FIGURE 28. A cross-section of $\phi^2(\overline{M})$ for a fixed $\theta$.

We define $\Lambda = \cap_{n \geq 0} \phi^n(\overline{M})$, a closed, $\phi$-invariant set, on which $\phi : \Lambda \to \Lambda$ is a homeomorphism. This $\Lambda$ is an attractor, in the sense that for all $x \in \overline{M}$, the distance $\mathrm{d}(\phi^n(x), \Lambda) \to 0$ as $n \to \infty$.

Each cross-section of $\Lambda$ is a Cantor set. To see this, consider the cross-sections of $\phi^n(\overline{M})$ for fixed $\theta$. At each stage, we get a nested pair of circles, the intersection of which will be a Cantor set.

Note, however, that the whole of $\Lambda$ is a connected set, because each $\phi^n(\overline{M})$ is connected. It should be clear that the nonwandering set $\Omega(\phi)$ is contained in $\Lambda$ (in fact, it equals $\Lambda$). There is a proof in Hasselblatt-Katok that $\Lambda$ is hyperbolic. This should be no surprise; we have expansion in the $\theta$ direction and contraction in the $z$ direction.

## 12. Symbolic dynamics in action. February 3 (*Notes by PS*)

*Smale's solenoid is Axiom A*

To show that Smale's solenoid is Axiom A, we need only show that $\Lambda = \Omega(\phi)$ and that $\Lambda$ is the closure of the periodic points of $\phi$. To demonstrate this, we first prove the following:

**Theorem 12.1.** *There is a continuous map $h : \Sigma_2 \to \Lambda$ such that $\phi h = h\sigma$, i.e. $(\Lambda, \phi)$ is a topological/dynamic factor of $(\Sigma_2, \sigma)$. If this is true, the above assertions are immediate corollaries. For example, because $(\Sigma_2, \sigma)$ is topologically mixing, there are dense orbits in $\Lambda$ as well.*

*Proof.* We partition $\overline{M}$ into two sets:

$$A_0 = \{(\theta, z) : 0 \le \theta < \pi\} \text{ and } A_1 = \{(\theta, z) : \pi \le \theta < 2\pi\}$$

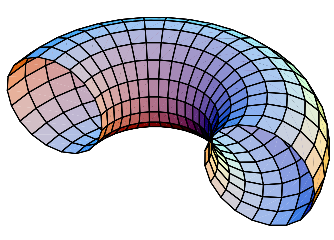
and define

$$h(\omega) = \bigcap_{i \in \mathbb{Z}} \phi^i(A_{\omega_i}).$$

Backwards intersections divide the torus $\overline{M}$, first in two, then quarters, then eighths, etc., eventually narrowing down exactly what angle $\theta$ our point lies at. Forward intersections keep track of which of the two loops, top or bottom, the point falls into.

Considering only cross-sections, forward intersections determine which of the nested circles we fall into. Together, forward and backward intersections determine a single point in $\Lambda$, so our map $h$ is well defined.

Using standard arguments, $h$ intertwines $\phi$ and $\sigma$. To see that $h$ is continuous, consider two nearby points $\omega_1$ and $\omega_2$ in $\Sigma_2$ that agree on a long central block. Iterating backwards, we see that their images in $\Lambda$ have nearly the same $\theta$, and iterating forwards, we see that their images fall into the same nested circles. Thus, the images of the two points are near each other in $\Lambda$ as well. □



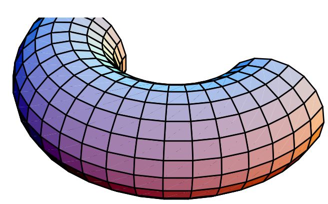


Figure 29. The partitions of $\overline{M}$: $A_0$ and $A_1$.

Figure 30. The forward images of $A_0$ and $A_1$.

What do the stable and unstable manifolds $W^s(\theta, z)$ and $W^u(\theta, z)$ look like?

Examining $W^s(\theta, z)$ first, note that all $(\theta, y)$ in the same cross-section at $\theta$ will eventually be asymptotic with the orbit of $(\theta, z)$, since the nested circles contract under iteration. Now, suppose you have a point $(\theta', y)$ such that $2^n\theta' = 2^n\theta \pmod{2\pi}$ for some $n$. Then eventually $\phi^n(\theta', y)$ and $\phi^n(\theta, z)$ will fall into the same cross-section, and thereafter will converge. On the other hand, if $2^n\theta' \neq 2^n \pmod{2\pi}$ for any $n$, then $\theta$ and $\theta$' will never converge. Thus:

$$W^s(\theta, z) = \{(\theta', y) : 2^n\theta' = 2^n\theta, \text{ for some } n\}$$

## 13. Topological ergodicity. February 5 (*Notes by PS*)

*Solutions to Exercises 1 and 2*

**Definition.** Let $(X, \phi)$ be a dynamical system. Then

(1) $(X, \phi)$ is *topologically mixing* if for $U, V$ nonempty open subsets of $X$, there exists $N$ such that for $n \geq N$, $\phi^n(U) \cap V \neq \emptyset$.
(2) $(X, \phi)$ is *regionally transitive* if for $U, V$ nonempty open subsets of $X$, there exists at least one n such that $\phi^n(U) \cap V \neq \emptyset$.
(3) $(X, \phi)$ is *topologically ergodic* if every proper closed invariant subset of $X$ is nowhere dense.
(4) $(X, \phi)$ is *topologically transitive* if there exists a dense orbit. In fact (as we will show later), the set of all points with dense orbit is residual.

**Definition.** A subset $E$ in $X$ is *first category* if it is the union of countably many nowhere dense sets. A subset $E$ is *residual* if it contains the intersection of countably many open dense sets. The complement of a first category set is residual.

**Theorem 1.** *Topological mixing implies regional transitivity. It is also true that regionally transitive, topologically ergodic and topologically transitive are all equivalent properties.*

*Proof.*

(1) Assume $(X, \phi)$ is regionally transitive. Let $C$ be a proper closed subset of $X$ and assume $C$ is not nowhere dense. Then there is a nonempty open set $U \subset C$. In addition, $C^c$ is also open and nonempty. By regional transitivity, there exists an $n$ for which $\phi^n(U) \cap C^c \neq \emptyset$, contradicting the fact that $C$ is invariant. Thus, $C$ must be nowhere dense, and $X$ is topologically ergodic.

(2) Assume $(X, \phi)$ is topologically ergodic. Since $X$ is a 2nd countable space, there is a countable basis $\{B_i\}$ for $X$. Define the orbit of each $B_i$, $O(B_i) = \cup_{n\in\mathbf{Z}}\phi^n(B_i)$. Both $O(B_i)$ and its complement $O(B_i)^c$ are invariant. Since $O(B_i)^c$ is a closed invariant set, by topological ergodicity it is nowhere dense, and thus $O(B_i)$ is an open dense set.

Let $F = \cap_{n\geq 0}O(B_i)$, then $F$ is a residual set. We claim that every $x \in F$ has a dense orbit. Let $U$ be any open set in $X$. Then there exists a basic open set $B_i \subset U$. By the definition of $F$, $x \in O(B_i)$, and thus there is an $n$ for which $\phi^n(x) \in O(B_i) \subset U$. Since the orbit of $x$ meets every open set $U$, the orbit is dense in $X$. Thus $(X, \phi)$ is topologically transitive. As we claimed above, the set of all points whose orbit is dense ($F$) is in fact residual.

(3) Suppose $(X, \phi)$ is topologically transitive. Let $x$ be a point with dense orbit. Let $U, V$ be nonempty open sets. Since $O(x)$ is dense, there exist $n$ and $m$ such that $\phi^n(x) \in U$ and $\phi^m(x) \in V$. Therefore, $\phi^{m-n}(U) \cap V$ contains at least the point $\phi^m(x)$ and is nonempty. Thus, $(X, \phi)$ is regionally transitive.

□

**Definition.** A set $E$ *satisfies the property of Baire* if $E = (G \cup M_1) \setminus M_2$, where $M_1$ and $M_2$ are first category sets.

**Definition.** A dynamical system $(X, \phi)$ is *Baire ergodic* if any for any invariant set $E$ satisfying the property of Baire, either $E$ or $E^c$ is residual. Equivalently, either $E$ or $E^c$ is first category.

**Theorem 2.** *Baire ergodicity is equivalent to the properties in the previous theorem.*

*Proof.*

(1) Suppose $(X, \phi)$ is Baire ergodic. Let $C$ be a closed invariant subset. Trivially, $C^c$ satisfies the property of Baire, since it is open. Thus, either $C$ or $C^c$ is residual. If $C$ is residual, it is both dense and closed, and therefore all of $X$. If $C^c$ is residual, it is dense. Therefore, it is impossible for $C$ to have an open subset, so $C$ must be nowhere dense. We have therefore shown that $(X, \phi)$ must be topologically ergodic.

(2) Suppose $(X, \phi)$ is topologically transitive. Let $E$ be an invariant subset of $X$, with the property of Baire, that is $E = (G \cup M_1) \setminus M_2$, where $M_1$ and $M_2$ are first category. Let $F$ be the set of all points whose orbit is dense.

If $G \cap F$ is empty, then $E^c$ must be residual, since $F \setminus M_1 \subset E^c$. If $G \cap F$ is nonempty, then its orbit $O(G)$ is open and dense. Since $E$ is invariant, $O(G) \setminus O(M_2) \subset E$, and thus $E$ is residual.

□

## 14. Stable and unstable manifolds. February 7 (*Notes by MA*)

We will now examine more closely the structure of the nonwandering set of an Axiom A diffeomorphism. To do this, we will need the *Stable Manifold Theorem for a hyperbolic set* $\Lambda$ *in* $M$. This theorem essentially says that the splitting of the tangent bundle $T\Lambda = E^s \oplus E^u$ (i.e., for $p \in \Lambda$, $T_pM = E^s_p \oplus E^u_p$) is *integrable* — it arises as tangent spaces to injectively immersed manifolds.

For $\epsilon > 0$, and $x \in M$, we define

$$W^s_\epsilon(x) = \{y \in M : d(\phi^n x, \phi^n y) \leq \epsilon \text{ for all } n \geq 0\}$$
$$W^u_\epsilon(x) = \{y \in M : d(\phi^n x, \phi^n y) \leq \epsilon \text{ for all } n \leq 0\}.$$

**Theorem 14.1** (Hirsch-Pugh)**.** *Let $\Lambda$ be a hyperbolic set for a $C^k$ $(k \geq 1)$ diffeomorphism $\phi : M \to M$, with a splitting $T_pM = E^s_p \oplus E^u_p$ for $p \in \Lambda$, as in the definition of the hyperbolic set. We assume (as usual) that $M$ has an adapted Riemmanian metric. Then for small $\epsilon > 0$,*

*Stable Manifold Theorem*

1. *$W^s_\epsilon$ and $W^u_\epsilon$ are $C^k$ disks (injectively immersed) of dimension $\dim E^s_x$ and $\dim E^u_x$ respectively, with $T_xW^s_\epsilon = E^s_x$, $T_xW^u_\epsilon(x) = E^u_x$.*
2. *There is $0 < \lambda < 1$ such that $d(\phi^n x, \phi^n y) \leq \lambda^n d(x, y)$ for $y \in W^s_\epsilon(x)$, and $d(\phi^{-n}x, \phi^{-n}y) \leq \lambda^n d(x, y)$ for $y \in W^u_\epsilon$, for $n \geq 0$.*
3. *$W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ vary continuously with $x \in \Lambda$.*

Some consequences of Theorem 14.1:

1. Statement (2) implies that $W^s_\epsilon(x) \subset W^s(x)$ for each $x$ in $\Lambda$, since $d(\phi^n x, \phi^n y)$ is approaching 0 exponentially fast. Similarly, $W^u_\epsilon(x) \subset W^u(x)$.
2. $\phi(W^s_\epsilon(x)) \subset W^s_\epsilon(\phi x)$ and $\phi^{-1}(W^u_\epsilon(x)) \subset W^u_\epsilon(\phi^{-1}x)$.
3. $W^s(x) = \cup_{n\geq 0}\phi^{-n}W^s_\epsilon(\phi^n x)$ and $W^u(x) = \cup_{n\geq 0}\phi^n W^u_\epsilon(\phi^{-n}x)$. These are *increasing* unions.
4. So for $x \in \Lambda$, $W^s(x)$ and $W^u(x)$ are injectively immersed copies of Euclidean spaces of the right dimensions.

**Corollary 14.2.** *If $\Lambda$ is a hyperbolic set in $(M, \phi)$, then $(\Lambda, \phi|_\Lambda)$ is* expansive*, i.e., there is a $\delta > 0$ such that if $x, y \in \Lambda$, $x \neq y$, then there is some integer $n$ such that $d(\phi^n x, \phi^n y) \geq \delta$.*

*Proof.* The proof hinges on the claim that for $\epsilon$ small enough, $W^s_\epsilon(x) \cap W^u_\epsilon(x) = \{x\}$. Given this, it is easy to see that if $x$ and $y$ are such that $d(\phi^n x, \phi^n y) < \epsilon$ for all $n$, then $y \in W^s_\epsilon(x) \cap W^u_\epsilon(x)$, and so $x = y$.

Thus, we need only show that $W^s_\epsilon(x) \cap W^u_\epsilon(x) = \{x\}$. But this is clear, since $W^\epsilon_s(x)$ and $W^\epsilon_u(x)$ are small ($C^k$ injectively immersed) disks which intersect transversely at $x$, just like their tangent spaces (which have complementary dimensions and span $T_xM$). □

## 15. Expansiveness, canonical coordinates, basic sets. February 10 (*Notes by RJ*)

**Definition 15.1.** A dynamical system $(X, \phi)$ is said to be *expansive* if there exists a $\delta > 0$ such that if $x, y \in X$, $x \neq y$, then there exists an integer $n$ such that $d(\phi^n x, \phi^n y) \geq \delta$.

**Corollary 15.1.** *If $\Lambda$ is a hyperbolic set for a $C^k$ diffeomorphism $\phi\colon M \to M$, then $(\Lambda, \phi|_\Lambda)$ is expansive.*

*Proof.* Choose $\delta$ so that $\delta < \epsilon$ where $\epsilon$ is as in the Stable Manifold Theorem for Hyperbolic Sets (Theorem 14.1). If $d(\phi^n x, \phi^n y) < \delta$ for all $n \in Z$, then $y \in W^s_\delta(x) \cap W^u_\delta(x)$. But this intersection consists only of the point $x$. □

The following theorem of Smale describes the *Canonical Coordinates* or *Local Product Structure* of an Axiom A diffeomorphism.

*Canonical coordinates*

**Theorem 15.2** (Smale)**.** *Let $\phi\colon M \to M$ be a $C^k$ Axiom A diffeomorphism. Then for every small $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in \Omega(\phi)$ and $d(x, y) < \delta$ then $W^s_\delta(x) \cap W^u_\delta(y)$ consists of exactly one point $[x, y]$. Furthermore, $[x, y]$ is in $\Omega(\phi)$ and $[x, y]$ is a continuous function of $(x, y)$.*

*Proof.* For the first statement we choose $\delta$ small enough to say that the pictures won't change much. Specifically, $W^s_\delta(x)$ and $W^u_\delta(x)$ are transverse, so for $y$ near $x$, $W^s_\delta(x)$ and $W^u_\delta(y)$ remain transverse and hence have a single intersection point. This, and the continuity of $[x, y]$ in $x$ and $y$, follow from the continuity assertion in the Stable Manifold Theorem.

It remains to show that $[x, y] \in \Omega(\phi)$. We first reduce to the case in which $x$ and $y$ are fixed points. (In an Axiom A system, periodic points are dense in $\Omega(\phi)$, so choose $x$ and $y$ to be periodic. Then under an appropriate power of $\phi$, $x$ and $y$ are fixed.)

Let $U$ be an arbitrary neighborhood of $[x, y]$ in $M$. We claim that

$$W^u_\epsilon(x) \subset \overline{\cup_{n\geq 0}\phi^n U}.$$

Since $\phi$ is an expansion along $W^u(x)$, if we can find a small neighborhood $V$ containing $x$ in $W^u_\epsilon(x)$ with $V \subset \overline{\cup_{n\geq 0}\phi^n U}$ then the entire set $W^u_\epsilon(x) \subset \overline{\cup_{n\geq 0}\phi^n U}$.

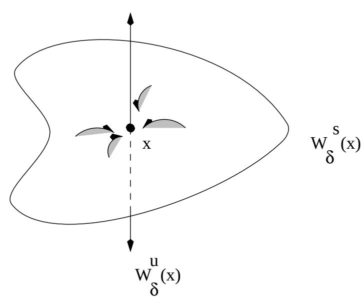



Figure 31. $W^s_\delta$ and $W^u_\delta$ are closed disks of the appropriate dimension. There is only one intersection point if we keep $\delta$ small.

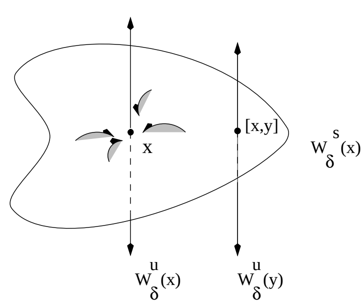



Figure 32. $[x, y]$ is unique for small enough $\delta$.

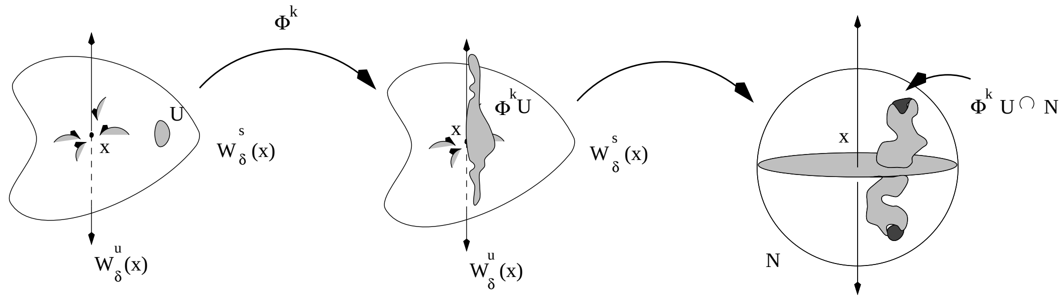



FIGURE 33. Starting with $U$ we take a high enough power $\phi^k U$ and restrict to a small neighborhood $N$ of $x$

Let $N$ be a small neighborhood around $x$ in $M$ and replace $U$ by $N \cap (\phi^k U)$ for some large $k$. Now the entire picture is local, so the Hartman-Grobman Theorem applies and says that $\phi = D_x\phi$ up to a continuous change of coordinates.

On $N$, $D_x\phi$ acts on eigenspaces, shrinking along the contracting subspace and blowing up along the expanding subspace. In this setting it is clear that the forward images of $U$ accumulate along the unstable manifold of $x$.

We now play "hyperbolic pinball": see Figure 34.

The point $[y, x] \in W^s_\epsilon(y) \cap W^u_\epsilon(x)$, so for some $n_1$, $U' = \phi^{n_1} U \cap W^s_\epsilon(y)$ contains a point near $[y, x]$. Repeating the argument made for $U$ with $U'$ shows that the forward images of $U'$ accumulate along $W^u_\epsilon(y)$, and for some large $n_2$, $\phi^{n_2} U' \cap U$ is nonempty. So $\phi^k \phi^{n_1+n_2} U \cap U$ is nonempty and the point $[x, y]$ is nonwandering. □

Now we have the tools necessary to prove the following result, the first part of which is due to Smale and the second to Bowen:

*Spectral decomposition of the non-wandering Set*

**Theorem 15.3.** *Let $\phi\colon M \to M$ be an Axiom A diffeomorphism. Then*

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n,$$

*where the* "basic sets" *$\Omega_i$ are pairwise disjoint closed $\phi$-invariant sets and each $(\Omega_i, \phi)$ is topologically transitive.*

*Moreover, for each basic set $\Omega_i$,*

$$\Omega_i = X_i^0 \cup X_i^1 \cup \cdots \cup X_i^{n_i-1},$$

*where the* "elementary sets" *$X_i^j$ are pairwise disjoint closed sets which $\phi$ maps cyclically (i.e. $\phi X_i^j = X_i^{j+1 \pmod{n_i}}$), and, further, each $(X_i^j, \phi^{n_i})$ is topologically mixing.*

## 16. Proof of Spectral Decomposition Theorem. February 12 (*Notes by RJ*)

We begin proving the Spectral Decomposition Theorem (Theorem 15.3), which was stated last time.

*Proof.* For each periodic point $p \in \Omega$, let $X_p = \overline{W^u(p) \cap \Omega}$.

In order to show that the $X_p$ separate $\Omega$ in the manner of the theorem, we first wish to show that each such $X_p$ is both open and closed in $\Omega$. To this end, let $\delta > 0$ be as in

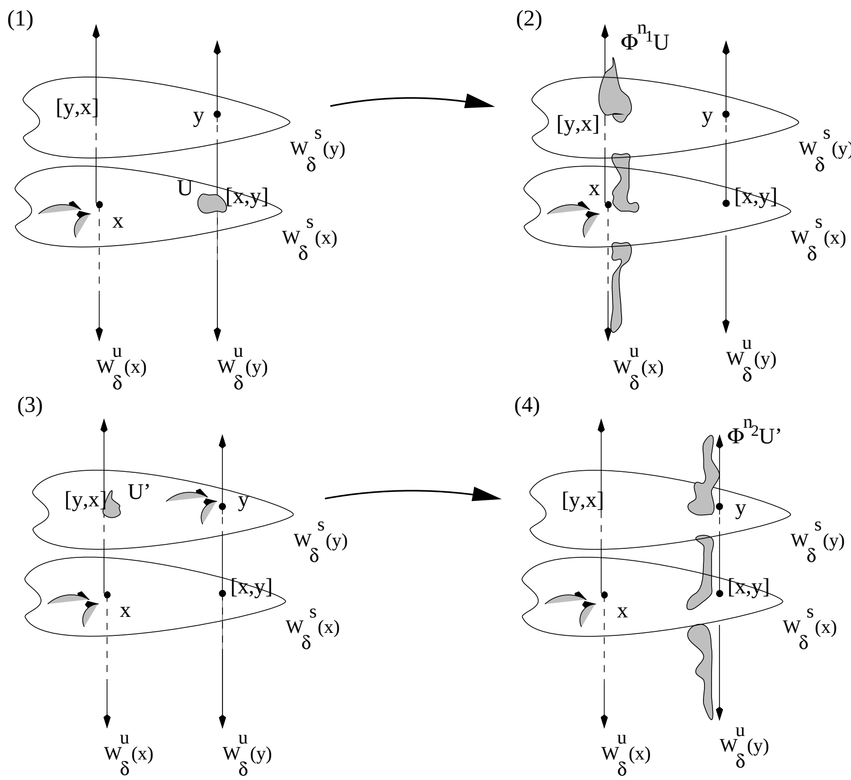



FIGURE 34. "Hyperbolic Pinball"

the Canonical Coordinates Theorem so that if $x, y \in \Omega$ and $d(x, y) < \delta$, there is a unique $[x, y] \in W^s_\epsilon(x) \cap W^u_\epsilon(x) \cap \Omega$.

**Claim 1:** *We claim that* $X_p = B_\delta(X_p) = \{y \in \Omega \colon d(y, X_p) < \delta\}$.

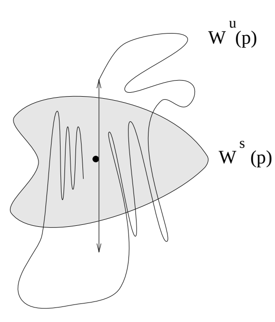



FIGURE 35. $X_p = \overline{W^u(p) \cap \Omega}$.

Since the periodic points are dense in $\Omega$, it is enough to show that for $q \in B_\delta(X_p)$ periodic, $q \in X_p$. Since $q \in \overline{W^u(p) \cap \Omega}$, we can find an $x \in W^u(p) \cap \Omega$ so that $d(x, q) < \delta$. Let $x' = [q, x]$. By definition, $[q, x] \in W^s_\epsilon(q) \cap W^u_\epsilon(x) \cap \Omega$ and since $x \in W^u(p)$, $[q, x] \in W^s(q) \cap W^u(p) \cap \Omega$. (We have both

$$d(\phi^{-n}x', \phi^{-n}x) \to 0 \text{ as } n \to \infty \text{ and } d(\phi^{-n}x, \phi^{-n}p) \to 0 \text{ as } n \to \infty,$$

so by the triangle inequality,

$$d(\phi^{-n}x', \phi^{-n}p) \to 0 \text{ as } n \to \infty.)$$

Let $\psi = \phi^{\text{per}(p)\cdot\text{per}(q)}$, so $\psi p = p$ and $\psi q = q$. Now because $x' \in W^s(q)$, $d(\psi^k x', q) = d(\psi^k x', \psi^k q) \to 0$ as $k \to \infty$.

But at the same time $\psi^k x' \in W^u(p) \cap \Omega$ for all $k$. So $q \in \overline{W^u(p) \cap \Omega} = X_p$. Thus each $X_p$ is both open and closed in $\Omega$.

Of course many periodic points may generate the same set $X_p$.

**Claim 2:** *We claim that if $p$ and $q$ are periodic and $q \in X_p$, then $X_p = X_q$.*

First note that $\phi X_p = X_{\phi p}$ and $\psi X_p = X_{\psi p} = X_p$.

If $q \in X_p$ then trivially $W^u_\delta(q) \subset B_\delta(X_p) = X_p$. And (recall that the first union is an increasing union)

$$\begin{aligned} W^u(q) &= \cup_{n\geq 0}\phi^n W^u_\delta(\phi^{-n}q) \\ &= \cup_{k\geq 0}\psi^k W^u_\delta(\psi^{-k}q) \\ &= \cup_{k\geq 0}\psi^k W^u_\delta(q) \\ &\subset \cup_{k\geq 0}\psi^k X_p \\ &= X_p. \end{aligned}$$

Hence $X_q = \overline{W^u(q) \cap \Omega} \subset X_p$.

We claim also that $p \in X_q$. This being the case, we can reverse the roles of $p$ and $q$ in the above argument to obtain $X_p \subset X_q$. So it remains to show that $p \in X_q$.

Proceeding approximately as before, let $x' = [q, p]$. Then $x' \in W^s_\epsilon(q) \cap W^u_\epsilon(p) \cap \Omega$ and $d(\psi^k x', q) = d(\psi^k x', \psi^k q) \to 0$ as $k \to \infty$. Recall from 1, however, that $X_q$ is open in $\Omega$, so for large k, $\psi^k x' \in X_q$. Since $X_q$ is $\psi$-invariant, $\psi^{-j}x' \in X_q$ for all $j$. But also $d(\psi^{-j}x', p) \to 0$ as $j \to \infty$, so $p \in \overline{X_q} = X_q$. And thus $X_p = X_q$.

**Claim 3:** *Therefore we claim $\Omega = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_n}$ where the $X_{p_i}$ are closed, pairwise disjoint sets and each $X_{p_i}$ is invariant under $\phi^{\text{per}(p_i)}$.*

The $X_p$ are clearly closed. Since the $X_p$ are also open, if there were $X_p$ and $X_q$ with $X_p \cap X_q \neq \emptyset$, then this intersection is open and contains a periodic point $q'$. But then $X_p = X_{q'} = X_q$. So the $X_p$ are pairwise disjoint. Moreover, since $\{X_p\}$ forms a cover of the compact set $\Omega$, there is a finite subcover: $\Omega = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_n}$.

Since $\phi X_{p_i} = X_{\phi p_i}$, $\phi$ acts as a permutation on the equivalence classes $X_{p_i}$. These $X_{p_i}$ are the *elementary parts*. The *basic sets* $\Omega_j$ are the unions of the cycles of this permutation, i.e. $\Omega_j = X_{p_j} \cup \phi X_{p_2} \cup \cdots \cup \phi^{n_i - 1} X_{p_n}$.

Next time we will complete the proof by showing that $(X_{p_i}, \phi^{n_i})$ is topologically mixing.

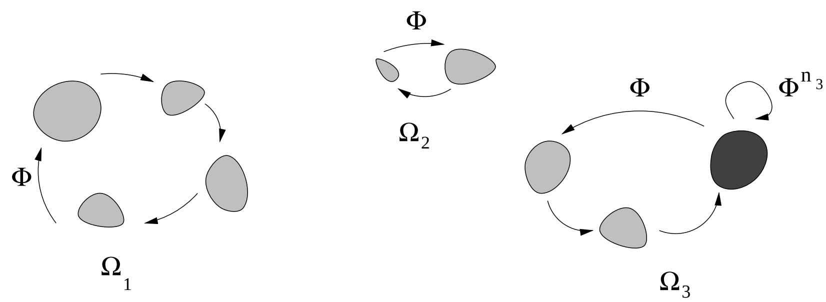



FIGURE 36. Elementary and Basic Sets

## 17. Topological mixing on basic sets. February 14 (*Notes by RJ*)

We wish to complete the proof of the Spectral Decomposition of the nonwandering set by showing that $(X_{p_i}, \phi^{n_i})$ is topologically mixing.

*Proof (cont).*

**Claim 4:** $(X_{p_i}, \phi^{n_i})$ *is topologically mixing.*

Take $U$ and $V$ to be nonempty open subsets of $X_{p_i}$. To show that $(X_{p_i}, \phi^{n_i})$ is mixing, we will produce a point which is in both $V$ and $\phi^N U$ for some value of $N$. Since the periodic points are dense , we can find periodic points $p \in U$ and $q \in V$. Call their periods $m$ and $n$, respectively.

For any (large) $t$, we can write $tn_i = kmn + j$ with $0 \leq j < mn$. Then $\phi^j p = \phi^{tn_i}\phi^{-kmn}p = \phi^{tn_i}p \in X_{p_i}$

Again letting $\delta > 0$ be small enough to ensure the existence of canonical coordinates, for each integer $j$, $0 \leq j < mn$, we can choose $x_j \in W^u(\phi^j p) \cap \Omega$ with $d(x_j, q) < \delta$. And let $x'_j = [q, x_j] \in W^s(q) \cap W^u(x_j) \cap \Omega \subset W^s(q) \cap W^u(\phi^j p) \cap \Omega$.

Now $\phi^j U$ is a neighborhood of $\phi^j p$, so for all large enough $l$, $\phi^{-lmn}x'_j \in \phi^j U$. Let $y_j = \phi^{-lmn}x'_j$. Since $x'_j \in W^s(q)$ and $W^s(q)$ is $\phi$-invariant, $y_j \in W^s(q)$. So for all large enough $k$, $\phi^{kmn}y_j \in V$. Summarizing, $y_j \in \phi^j U$ and $\phi^{kmn}y_j \in V$. But this implies $\phi^{-j}y_j \in U$, i.e. $\phi^{kmn-tn_i}y_j \in U$.

So $\phi^{kmn}y_j \in V \cap \phi^{tn_i}U$ for all large enough t, and hence $(X_{p_i}, \phi^{n_i})$ is topologically mixing.

□

**Example 17.1** (Morse-Smale Systems)**.** In a Morse-Smale System, $\Omega(\phi)$ is finite. Each periodic point is an elementary part and the basic sets are the periodic orbits.

**Example 17.2** (the $n$-torus)**.** Consider the system $(\mathbb{T}^n, A)$ where $\mathbb{T}^n$ is the $n$-torus and $A$ is an $n$-by-$n$ integer matrix with $\det A = \pm 1$. If no eigenvalue has modulus 1, then $(\mathbb{T}^n, A)$ is hyperbolic. Since the periodic points (the points with rational coordinates) are dense in $\mathbb{T}^n$, the nonwandering set $\Omega(\phi)$ is all of $\mathbb{T}^n$. Therefore, since $\Omega(\phi)$ is connected, $\Omega(\phi)$ consists of a single elementary part. Hence, $(\mathbb{T}^n, A)$ is topologically mixing.

Similarly, every Anosov diffeomorphism on $\mathbb{T}^n$ is topologically mixing.

**Definition 17.1.** Let $I$ be a (possibly infinite) interval of integers, and let $\alpha > 0$. Then an *$\alpha$-pseudo-orbit* for $\phi$ is a sequence of points $(x_i)_{i\in I}$ in $X$ such that $d(\phi x_i, x_{I+1}) < \alpha$ for all $i$, $i+1 \in I$

*$\alpha$-pseudo-orbits and $\beta$-shadowing*

**Remark 17.1.** An easy way to produce an $\alpha$-pseudo-orbit for $\phi$ is to pick points $(x_i)_{i\in I}$ near an actual orbit $\{y, \phi y, \phi^2 y, \ldots\}$. By the continuity of $\phi$, if $d(x_1, \phi^n y)$ is small enough, then $d(\phi x_1, \phi^{n+1} y)$ will also be small, and by the triangle inequality, if $d(x_2, \phi^{n+1} y)$ is small and $d(\phi x_1, \phi^{n+1} y)$ is small, then $d(\phi x_1, x_2)$ is small.

**Definition 17.2.** Given a sequence of points $(x_i)_{i\in I}$ in $X$ and a point $x \in X$, we say that *$x$ $\beta$-shadows* $(x_i)_{i\in I}$ if $d(\phi^i x, x_i) < \beta$ for all $i \in I$.

Next time we will prove the following theorem of Bowen:

**Theorem 17.1.** *Let $(X, \phi)$ be an Axiom A system. Then given $\beta > 0$, there is an $\alpha > 0$ such that every $\alpha$-pseudo-orbit in $\Omega(\phi)$ (even of infinite length) is $\beta$-shadowed by some point in $\Omega(\phi)$.*

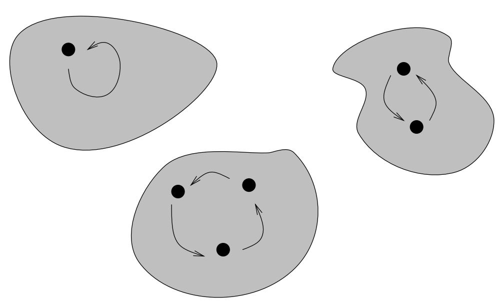


FIGURE 37. Basic Sets and Elementary Parts of a Morse-Smale system

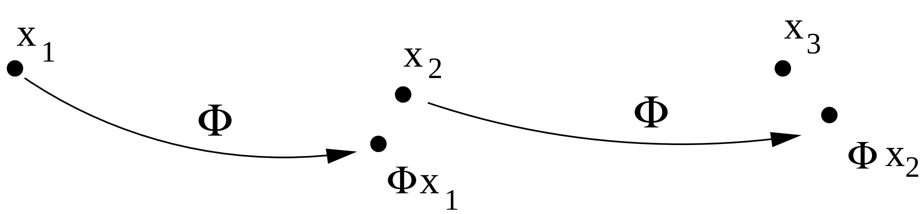



FIGURE 38. $(x_i)_{i\in I}$ is an $\alpha$-pseudo-orbit for $\phi$

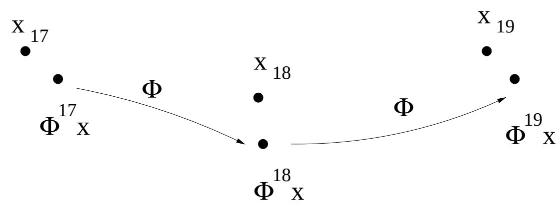



FIGURE 39. $x$ $\beta$-shadows $(x_i)_{i\in I}$

## 18. SHADOWING. FEBRUARY 17 (*Notes by SS*)

**Example 18.1.** (Basic sets and elementary parts of Axiom A diffeomorphisms)

(1) For a Morse-Smale system, the basic sets are the periodic orbits and the elementary parts are the individual periodic points.
(2) For an Axiom A automorphism $\phi$ of the $n$-torus $\mathbb{R}^n/\mathbb{Z}^n$, which is given by an $n\times n$ integer matrix with det $=\pm 1$, no eigenvalue has modulus 1, since it is hyperbolic. Then the periodic points $=\pi(\mathbb{Q}^n)$, where $\pi\colon \mathbb{R}^n\to\mathbb{R}^n/\mathbb{Z}^n$ is the natural projection. For, if $x\in\pi(\mathbb{Q}^n)$, then $x=(p_1/q_1,\dots,p_n/q_n)$, $p_i,q_i\in\mathbb{Z}, 0\le p_i<q_i$, for all $i$. Let $q=\operatorname{lcm}\,(q_1,\dots,q_n)$. Then $x=(r_1/q,\dots,r_n/q)$, where $r_i\in\{0,1,\dots,q-1\}\cong\mathbb{Z}_q$, for all $i$. Now $\phi$ induces a one to one, onto map $\phi_q\colon(\mathbb{Z}_q)^n\to(\mathbb{Z}_q)^n$ of a finite set and for this map every orbit is finite, i.e., every point is a periodic point. Conversely, since $\phi$ has no eigenvalue with modulus 1, $\det(\phi^k-I)\neq 0$ for all $k\ge 1$. So, if $x\in\mathbb{R}^n/\mathbb{Z}^n$ and $\phi^k x=x$, i.e., $(\phi^k-I)x\in\mathbb{Z}^n$, then by Cramer's rule, $x$ has rational coordinates, i.e., $x\in\pi(\mathbb{Q}^n)$. Thus the nonwandering set $\Omega(\phi)$ is $\mathbb{R}^n/\mathbb{Z}^n$, since the periodic points are dense in $\mathbb{R}^n/\mathbb{Z}^n$. Then $\Omega(\phi)=\mathbb{R}^n/\mathbb{Z}^n$ consists of a single elementary part, since it is connected. Hence $(\mathbb{R}^n/\mathbb{Z}^n,\phi)$ is topologically mixing.

*Pseudo-orbit Shadowing Theorem*

**Theorem 18.1.** *The nonwandering set of every Axiom A diffeomorphism has the pseudo-orbit shadowing property: Given $\beta>0$, there is $\alpha>0$ such that every $\alpha$-pseudo-orbit in $\Omega$ is $\beta$-shadowed by the orbit of some point in $\Omega$.*

*Proof.* Fix a very small $\epsilon>0$. Choose $\delta$, $0<\delta<\epsilon$, small enough for the existence of canonical coordinates on $\Omega$, i.e., if $x,y\in\Omega$ with $d(x,y)<\delta$, then there exists $[x,y]\in W^s_\epsilon(x)\cap W^u_\epsilon(y)\cap\Omega$. Pick $K$ with $\lambda^K\epsilon<\delta/2$, where $\lambda$, $0<\lambda<1$, is the hyperbolicity constant. Choose $\alpha>0$ so that a tight enough $\alpha$-pseudo-orbit of length $(K+1)$ in $\Omega$ is $\delta/2$-shadowed by iterates of its starting point:

$$(y_i)_{0\le i\le K} \text{ is an } \alpha\text{-pseudo-orbit} \Rightarrow d(\phi^i y_0, y_i)<\delta/2 \text{ for } i=0,1,\dots,K.$$

It is possible. For, choose $\alpha_0>0$ such that if $d(x,y)<\alpha_0$, then $d(\phi^i x,\phi^i y)<\delta/4K$ for all $i=1,\dots,K-1$. Let $\alpha=\min\,\{\alpha_0,\frac{\delta}{4}\}$. If $(y_j)_{0\le j\le K}$ is an $\alpha$-pseudo orbit in $\Omega$, then $d(\phi^i(\phi y_j),\phi^i(y_{j+1}))<\delta/4K$, for all $i=1,\dots,K-1$, for all $j=0,1,\dots,K-1$, and so for

$i = 0, 1, \ldots, K$,

$$d(\phi^i y_0, y_i) \le \sum_{l=0}^{i-2} d(\phi^{i-(l+1)}(\phi y_l), \phi^{i-(l+1)} y_{l+1}) + d(\phi y_{i-1}, y_i)$$
$$\le \sum_{l=0}^{i-2} \frac{\delta}{4K} + \alpha < \frac{\delta}{4K} K + \frac{\delta}{4} = \frac{\delta}{2}.$$

We break the proof into steps:

(1) Try to $\beta$-shadow a finite $\alpha$-pseudo-orbit of the form $(x_i)_{0 \le i \le rK}$.

Define $x'_0 = x_0$ and $x'_K = [x_K, \phi^K x'_0] \in W^s_\epsilon(x_K) \cap W^u_\epsilon(\phi^K x'_0) \cap \Omega$ (it is possible because $d(x_K, \phi^K x'_0) = d(x_K, \phi^K x_0) < \delta/2$ by choice of $\alpha$).

Inductively, define

$$x'_{iK} = [x_{iK}, \phi^K x'_{(i-1)K}] \in W^s_\epsilon(x_{iK}) \cap W^u_\epsilon(\phi^K x'_{(i-1)K}) \cap \Omega$$

for $i = 2, \ldots, r$. (It is possible. For, by choice of $\alpha$, $d(x_{iK}, \phi^K x_{(i-1)K}) < \delta/2$, and since $x'_{(i-1)K} \in W^s_\epsilon(x_{(i-1)K})$, we have $d(\phi^K x_{(i-1)K}, \phi^K x'_{(i-1)K}) < \lambda^K \epsilon < \delta/2$. So $d(x_{iK}, \phi^K x'_{(i-1)K}) < \delta$.) Now $x'_{rK} = [x_{rK}, \phi^K x'_{(r-1)K}] \in W^s_\epsilon(x_{rK}) \cap W^u_\epsilon(\phi^K x'_{(r-1)K}) \cap \Omega$.

Then $x = \phi^{-rK} x'_{rK}$ $\beta$-shadows $(x_i)_{0 \le i \le rK}$, i.e., $d(\phi^i x, x_i) < \beta$ for $i = 0, 1, \ldots, rK$. For suppose $sK \le i < (s+1)K$, $0 \le s \le r$. Then,

$$d(\phi^i x, \phi^{i-sK} x'_{sK}) = d(\phi^{i-rK} x'_{rK}, \phi^{i-sK} x'_{sK})$$
$$\le \sum_{l=s+1}^{r} d(\phi^{i-lK} x'_{lK}, \phi^{i-(l-1)K} x'_{(l-1)K})$$
$$= \sum_{l=s+1}^{r} d(\phi^{i-lK} x'_{lK}, \phi^{i-lK}(\phi^K x'_{(l-1)K}))$$
$$\le \sum_{l=s+1}^{r} \lambda^{-(i-lK)} \epsilon$$
$$\le \frac{\epsilon\lambda}{1-\lambda},$$

since $x'_{lK} \in W^u_\epsilon(\phi^K x'_{(l-1)K})$ and $i - lK \le 0$ for all $l = s+1, \ldots, r$.

Also, $d(\phi^{i-sK} x'_{sK}, \phi^{i-sK} x_{sK}) \le \epsilon$, since $x'_{sK} \in W^s_\epsilon(x_{sK})$ and $i - sK \ge 0$, and $d(\phi^{i-sK} x_{sK}, x_i) < \delta/2$ by choice of $\alpha$.

Thus

$$d(\phi^i x, x_i) \le d(\phi^i x, \phi^{i-sK} x'_{sK}) + d(\phi^{i-sK} x'_{sK}, \phi^{i-sK} x_{sK}) + d(\phi^{i-sK} x_{sK}, x_i)$$
$$< \frac{\epsilon\lambda}{1-\lambda} + \epsilon + \delta/2.$$

Set $\epsilon$ at the beginning such that the right side is less than $\beta$.

(2) To shadow a finite $\alpha$-pseudo-orbit of the form $(x_i)_{0\le i\le n}$ ($n$ may be not a multiple of $K$), just extend it to a new $\alpha$-pseudo-orbit $(x_i)_{o\le i\le rK}$, where $x_i = \phi^{i-n}x_n$, for $i = n+1, \dots, rK$ and $rK \le n < (r+1)K$. Then, an $x$ that $\beta$-shadows $(x_i)_{0\le i\le rK}$ will also $\beta$-shadow $(x_i)_{0\le i\le n}$.

(3) For an arbitrary finite $\alpha$-pseudo-orbit $(x_i)_{s\le i\le s+l}$, set $x'_i = x_{i+s}$ for $i = 0, 1, \dots, l$. We can $\beta$-shadow it with some point $x \in \Omega$, i.e., $d(\phi^i x, x'_i) = d(\phi^i x, x_{i+s}) < \beta$ for $i = 0, 1, \dots, l$. Then, $\phi^{-s}x$ $\beta$-shadows $(x_i)_{s\le i\le s+l}$ with the times synchronized with the indices as required, since $d(\phi^i(\phi^{-s}x), x_i) = d(\phi^{i-s}x, x'_{i-s}) < \beta$ for $i = s, s+1, \dots, s+l$.

(4) Finally, suppose we have an infinite $\alpha$-pseudo orbit $(x_i)_{-\infty<i<\infty}$.

For $m \ge 1$, choose $x^{(m)}$ which $\beta/2$-shadows $(x_i)_{-m\le i\le m}$ and let $x$ be a limit point of $(x^{(m)})_{m\ge 1}$ (recall that $M$ is compact). Then, $x$ $\beta$-shadows $(x_i)_{-\infty<i<\infty}$. Indeed, check it on a range, i.e., $[-w, w]$ for an arbitrary number $w$. Pick a large $m \ge w$ so that $d(\phi^i x, \phi^i(x^{(m)})) < \beta/2$ for all $i \in [-w, w]$. Also $d(\phi^i(x^{(m)}), x_i) < \beta/2$ for all $i \in [-w, w]$, since $x^{(m)}$ $\beta/2$-shadows $(x_i)_{-m\le i\le m}$ and so $(x_i)_{-w\le i\le w}$. Thus $d(\phi^i x, x_i) < \beta$ for all $i \in [-w, w]$.

□

## 19. Specification. February 19 (*Notes by SS*)

**Corollary 19.1.** *Given $\beta > 0$, there is $\alpha > 0$ such that if $x \in \Omega$ and $d(\phi^p x, x) < \alpha$, then there exists $y \in \Omega$ with $\phi^p y = y$ which $\beta$-shadows $x, \phi x, \dots, \phi^{p-1}x$.*

*Proof.* We may assume that $\beta < \epsilon/2$, where $\epsilon$ is the expansive constant. Choose $\alpha$ as in the Pseudo-Orbit Shadowing Theorem (Theorem 18.1). Then $(x_i)_{-\infty<i<\infty}$, where $x_i = \phi^k x$, $i \equiv k \bmod p$, $k \in \{0, 1, \dots, p-1\}$, i.e.,

$$\cdots, \phi^{p-1}x, x, \phi x, \cdots, \phi^{p-1}x, x, \phi x, \cdots$$

is an $\alpha$-pseudo orbit, since $x, \phi x, \dots, \phi^{p-1}x$ is an $\alpha$-pseudo orbit and by hypothesis,

$$d(\phi(\phi^{p-1}x), x) < \alpha.$$

So take $y \in \Omega$ which $\beta$-shadows it. Then $\phi^p y$ also $\beta$-shadows it and so for all $j \in \mathbb{Z}$, $d(\phi^j y, \phi^j(\phi^p y)) \le d(\phi^j y, x_j) + d(x_j, \phi^j(\phi^p y)) \le \beta + \beta < \epsilon$. By expansiveness, $\phi^p y = y$. □

*Specification* We extend these pseudo-orbit acrobatics to specification — See DGS, p.193ff.

**Definition 19.1.** Let X be a compact metric space and $\phi : X \to X$ a homeomorphism.

(1) We say that the system $(X, \phi)$ has *weak specification* if given $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that given $x_1, x_2 \in X$, given intervals $I_1 = [a_1, b_1], I_2 = [a_2, b_2] \subset \mathbb{Z}$ with big enough gap (switching time) $a_2 - b_1 \ge K(\epsilon)$, and given integer $p > 0$ big enough that switching-back-time $p - (b_2 - a_1) \ge K(\epsilon)$, there is a point $x \in X$ with $\phi^p x = x$ such that $d(\phi^i x, \phi^i x_1) < \epsilon$ for $i \in I_1$ and $d(\phi^i x, \phi^i x_2) < \epsilon$ for $i \in I_2$.

(2) We say that $(X, \phi)$ has *specification* if we can do the same with any number $k \ge 1$ of orbit pieces : Given $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that for any $k \ge 1$, given $x_1, \dots, x_k \in X$, given intervals $I_j = [a_j, b_j] \subset \mathbb{Z}$, $j = 1, \dots, k$ with $a_{j+1} - b_j \ge K(\epsilon)$

for $j = 1, \dots, k-1$, and given an integer $p > 0$ such that $p - (b_k - a_1) \geq K(\epsilon)$, there is a point $y \in X$ with $\phi^p y = y$ such that $d(\phi^i y, \phi^i x_j) < \epsilon$ for all $i \in I_j$ and $j = 1, \dots, k$.

**Example 19.1** (Shift of finite type (SFT) and Higher Block Presentation)**.** We will show that every topologically mixing SFT has specification.

*Shift of Finite Type (SFT)*

Let $A = \{0, 1, \dots, r-1\}$ be a finite alphabet. Then $(A^{\mathbb{Z}}, \sigma) = \Sigma_r =$ the full $r$-shift, with the shift transformation $\sigma$ defined by $(\sigma\omega)_n = \omega_{n+1}$ for $n \in \mathbb{Z}$, $\omega = (\omega_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}$.

Define $d(\omega, \zeta) = 1/2^k$, where $k = \inf\{|m| : \omega_m \neq \zeta_m\}$ for any $\omega$, $\zeta \in A^{\mathbb{Z}}$.

A shift of finite type (SFT) determined by an $r \times r$ matrix $B$ of 0's and 1's (transition or incidence matrix) is $\Sigma_B = \{\omega \in A^{\mathbb{Z}} : B_{\omega_n \omega_{n+1}} = 1, \text{ for all } n \in \mathbb{Z}\}$. Then $\Sigma_B$ is a closed shift-invariant subset of $A^{\mathbb{Z}}$, and $(\Sigma_B, \sigma|_{\Sigma_B})$ is a *1-step SFT*.

The $r \times r$ matrix $B$ corresponds to a finite directed graph on $r$ vertices labeled $0, 1, \dots, r-1$ with directed edge from $i$ to $j$ if and only if $B_{ij} = 1$. Thus $\Sigma_B =$ the set of all doubly infinite paths on the associated graph, and $B_{ij} = 0$ means there is no edge from the symbol $i$ to the symbol $j$, and in any $\cdots\omega_{-1}\omega_0\omega_1\cdots \in \Sigma_B$, you never see the 2-block $ij$. More generally, we can give a finite list $S$ of blocks in

$$\{0, 1, \dots, r-1\}^* = \cup_{k\geq 0}\{0, 1, \dots, r-1\}^k$$

and define

$$\Sigma(S) = \{\omega \in A^{\mathbb{Z}} : \text{ no block in } S \text{ appears (as a consecutive string) in } \omega\}.$$

If $n =$ maximal length of blocks in $S$, then $\Sigma(S)$ is an *(n-1)-step SFT.*

Let $\Sigma_B$ be a 1-step SFT and fix $m > 0$. Define a new alphabet

$$A^m = \overbrace{A \times \cdots \times A}^{m} = \{m\text{-blocks of symbols on original alphabet}\}$$

and define the $m$-th higher block code $\zeta^{(m)} : \Sigma_B \to (A^m)^{\mathbb{Z}}$ by

$$\cdots\omega_0\omega_1\cdots\omega_{m-1}\omega_m\cdots \overset{\zeta^{(m)}}{\longmapsto} \cdots A_0A_1\cdots$$

where $A_j = \omega_j\omega_{j+1}\dots\omega_{j+m-1} \in A^m$.

Then, $\zeta^{(m)} : \Sigma_B \to \zeta^{(m)}(\Sigma_B)$ is continuous, 1-1, onto and commutes with $\sigma$. So, $\zeta^{(m)} : (\Sigma_B, \sigma) \to (\zeta^{(m)}(\Sigma_B), \sigma)$ is a topological conjugacy. Generally, for any $n, m \in \mathbb{N}$, if $\Sigma(S)$ is an $n$-step SFT, then $\zeta^{(m)}(\Sigma(S))$ is a higher-block presentation of $\Sigma(S)$ and a 1-step SFT on the large alphabet $A^m$.

For example, if $A = \{0, 1\}$ and $S = \{101\}$, then $\Sigma(S) \cong \zeta^{(2)}(\Sigma(S))$, which is a 1-step SFT on $A^2 = \{00, 01, 10, 11\}$. See Figure 40.

## 20. Specification in shifts of finite type. February 21 (*Notes by SS*)

**Proposition 20.1.** (1) *A SFT $\Sigma_B$ is topologically ergodic if and only if $B$ is irreducible, i.e., for all $i, j \in \{0, 1, \dots, r-1\}$, there exists $n \geq 1$ such that $(B^n)_{ij} > 0$, or equivalently, the graph is strongly connected, i.e., for any two vertices $i$, $j$, there exists a path from $i$ to $j$.*

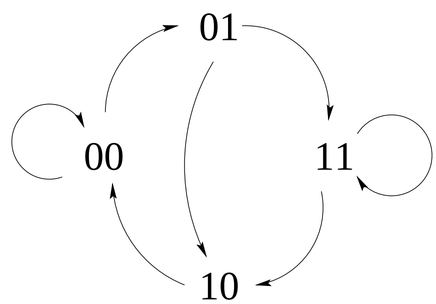



FIGURE 40. Higher block representation of $\Sigma(\{101\})$

(2) *A SFT $\Sigma_B$ is topologically mixing if and only if $B$ is primitive, i.e., there exists $n \geq 1$ such that $B^n > 0$, i.e., $(B^n)_{ij} > 0$ for all $i, j$, or equivalently, the graph is strongly connected and aperiodic.*

*SFT and specification*

**Proposition 20.2.** *If $(\Sigma_B, \sigma)$ is a topologically mixing SFT, then it has specification.*

*Proof.* Let $m$ be such that $(B^m)_{ij} > 0$ for all $i, j$ (from topological mixing for cylinder sets of length 1) i.e., we can get from any symbol $i$ to any symbol $j$ provided only that the path has length at least $m$. In order to see the idea, consider first a simple case, when $\epsilon = 1$. Then, choose the switching time $K(\epsilon) = m$.

Now suppose we are given $k \geq 1$, $x^1, \dots, x^k \in \Sigma_B$, intervals $I_j = [a_j, b_j] \subset \mathbb{Z}$, $j = 1, \dots, k$, with $a_{j+1} - b_j \geq m$, $j = 1, \dots, k-1$, and an integer $p \geq m + (b_k - a_1)$.

Since $j_1 = a_2 - b_1 + 1 > m$, there exists a path $u^1_1 \dots u^1_{j_1}$ of length $j_1$ from the symbol $u^1_1 = x^1_{b_1}$ to the symbol $u^1_{j_1} = x^2_{a_2}$. Similarly, for $l = 1, \dots, k$, since $j_l = a_{l+1} - b_l + 1 > m$ with $a_{k+1} = p + a_1$, there exists a path $u^l_1 \dots u^l_{j_l}$ of length $j_l$ from the symbol $u^l_1 = x^l_{b_l}$ to the symbol $u^l_{j_l} = x^{l+1}_{a_{l+1}}$, where $x^{k+1}_{a_{k+1}} = x^1_{a_1}$.

Then define $x^*$ as follows:

$$x^*_i = \begin{cases} x^j_i & \text{for } i \in I_j = [a_j, b_j] \text{ and } j = 1, \dots, k, \\ u^l_{i-b_l+1} & \text{for } b_l \leq i < a_{l+1} \text{ and } l = 1, \dots, k \text{ with } a_{k+1} = p + a_1. \end{cases}$$

Now $x^*_i$ is defined for $i = a_1, a_1 + 1, \dots, p + a_1 - 1$.

Let $x^* = \cdots (x^*_{a_1} \cdots x^*_{p+a_1-1})(x^*_{a_1} \cdots x^*_{p+a_1-1})(x^*_{a_1} \cdots x^*_{p+a_1-1}) \cdots$. This construction is given schematically in Figure 41.

Then $x^* \in \Sigma_B$, $\sigma^p x^* = x^*$, and for all $i \in I_j$, $j = 1, \dots, k$, $d(\sigma^i x^*, \sigma^i x^j) < 1/2^0 = 1$, since $(\sigma^i x^*)_0 = x^*_i = x^j_i = (\sigma^i x^j)_0$. Note $d(y, z) < 1 \Leftrightarrow y_0 = z_0$ (otherwise $d = 1$).

For arbitrary $\epsilon > 0$, choose $r$ such that $1/2^r < \epsilon$ and let $K(\epsilon) = m + 2r$.

Now suppose we are given $k \geq 1$, $x^1, \dots, x^k \in \Sigma_B$, intervals $I_j = [a_j, b_j] \subset \mathbb{Z}$, $j = 1, \dots, k$, with $a_{j+1} - b_j \geq m + 2r$, $j = 1, \dots, k-1$ and an integer $p \geq m + 2r + (b_k - a_1)$.

Since $a_{l+1} - b_l \geq m + 2r$ for $l = 1, \dots, k$, with $a_{k+1} = p + a_1$, $j_l = (a_{l+1} - r) - (b_l + r) + 1 > m$ for $l = 1, \dots, k$. So there exists a path $u^l_1 \dots u^l_{j_l}$ of length $j_l$ from the symbol $u^l_1 = x^l_{b_l + r}$ to the symbol $u^l_{j_l} = x^{l+1}_{a_{l+1} - r}$, where $x^{k+1}_{a_{k+1} - r} = x^1_{a_1 - r}$.

Then define $x^*$ as follows:

$$x_i^* = \begin{cases} x_i^j & \text{for } a_j - r \le i \le b_j + r \text{ and } j = 1, \dots, k, \\ u^l_{i-b_l-r+1} & \text{for } b_l + r \le i < a_{l+1} - r, l = 1, \dots, k \text{ with } a_{k+1} = p + a_1. \end{cases}$$

Now $x_i^*$ is defined for $i = a_1 - r, a_1 - r + 1, \dots, p + a_1 - r - 1$.

Let $x^* = \cdots (x^*_{a_1-r} \cdots x^*_{p+a_1-r-1})(x^*_{a_1-r} \cdots x^*_{p+a_1-r-1}) \cdots$.

Then $x^* \in \Sigma_B$, $\sigma^p x^* = x^*$, and for all $i \in I_j$, $j = 1, \dots, k$, $d(\sigma^i x^*, \sigma^i x^j) < 1/2^r < \epsilon$, since for $l \in [-r, r]$ and $i \in [a_j, b_j]$, $a_j - r \le i + l \le b_j + r$ and so $(\sigma^i x^*)_l = x^*_{i+l} = x^j_{i+l} = (\sigma^i x^j)_l$, i.e., $\sigma^i x^*$ and $\sigma^i x^j$ agree on the central $(2r+1)$-block (Note that $d(y, z) < 1/2^r \Leftrightarrow y$ and $z$ agree on the central $(2r+1)$-block). Hence it has specification. □

## 21. Specification in Axiom A systems. February 24 (*Notes by KJ*)

Last time we proved that a topologically mixing shift of finite type has specification. Today we show that *Specification cont.*

**Theorem 21.1.** *If $(X, \phi)$ is a topological dynamical system which is topologically mixing, expansive and has the pseudo-orbit shadowing property, then it has specification. Therefore, in an Axiom A system every elementary part of $(\Omega, \phi)$ has specification.*

*Proof.* (See DGS, pp 232 ff) The proof is similar to the proof of the corollary of the pseudo-orbit shadowing property: a tight enough pseudo-orbit which comes back to its beginning, has a periodic shadowing point. We do the same thing but for several orbit pieces at once.

Let $\beta \ge 0$ and $\beta <$ (expansive constant)/2. Let $\alpha$ be the pseudo-orbit tightness parameter that produces $\beta$-shadowing. Now we need to find $K(\epsilon)$.

Let $\mathcal{U} = \{U_1, U_2, \dots, U_r\}$ be a cover of $X$ by $\alpha$-balls. Using topological mixing, choose $K$ large enough so that if $n \ge K$ then $\phi^{-n} U_i \cap U_j \ne 0$ for all $i, j = 1, \dots, r$. This allows us to get from one part of the space to another, providing we allow a time at least K for the transition.

Turning to specification, suppose we are given the

$$I_i = [a_i, b_i], x_i \in X, i = 1 \dots k,$$

with gaps $a_i - b_i \ge K, p - (b_k - a_i) \ge K$ $(\epsilon = \beta)$.

Define $x_{k+1} = x_1$, $a_{k+1} = a_1 + p$.

Now we are going to make our pseudo-orbit. Define an $\alpha$-pseudo orbit as follows:

$$z_i = \phi^j x_j \text{ if } a_j \le i < b_j \text{ for all } j = 1, \dots, k.$$

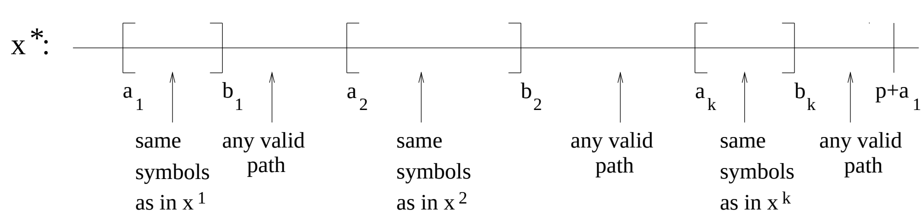



Figure 41. Construction of $x_i^*$ for $a_1 \le i \le p + a_1 - 1$

For $j = 1, \dots, k$ find $y_j \in \mathcal{U}(\phi^{b_j} x_j) \cap \phi^{-a_{j+1}+b_j}(\mathcal{U}(\phi a_{j+1} x_{j+1}))$, where $\mathcal{U}(x)$ is a member of $\mathcal{U}$ to which $x$ belongs. Since $a_{j+1} - b_j \geq K$, the intersection is nonempty because of the above argument based on topological mixing.

Further define

$$z_i = \phi^{i-b_j} y_j \text{ if } b_j \leq i < a_{j+1}.$$

Thus the distance between $\phi^{b_1} x_1$ and $y_1$ is less than $\alpha$, and so on:

$$d(\phi^{b_j - 1} x_j, y_j) < \alpha$$

for $j = 1, 2, \dots k$. We used strong mixing to get us from an $\alpha$-neighborhood of one point to an $\alpha$-neighborhood of another point with the orbit of one point.

At the end, we land within $\alpha$ of $\phi^{a_1} x_1$. This gives an $\alpha$-pseudo-orbit of length $p$. Repeat, to get an $\alpha$-pseudo-orbit $(z_i)_{i \in \mathbb{Z}}$ by defining $z_{i+p} = z_i$ for other $i$'s.

Now, by the pseudo-orbit shadowing property, there is an $x \in X$ such that $d(\phi^i x, z_i) < \beta$ for all $i \in \mathbb{Z}$. But we also have

$$\begin{aligned} d(\phi^i x, \phi^{i+p} x) &\leq d(\phi^i x, z_i) + d(z_i, \phi^{i+p} x) \\ &\leq d(\phi^i x, z_i) + d(z_{i+p}, \phi^{i+p} x) \\ &\leq 2\beta \\ &\leq \text{expansive constant (for all i)} \end{aligned}$$

and so $\phi^p x = x$. Of course the orbit pieces of x on the $I_i$ $\beta$-shadow those of the $x_i$.

□

**Remark 21.1.** Note that $(\Omega(\phi), \phi)$ may not be topologically mixing. The elementary parts are invariant sets, and the $x_i$'s could have come from any of them.

**Theorem 21.2** (DGS Chapter 21)**.** *Let $(X, \phi)$ be a topological dynamical system with weak specification (for example, an elementary part of the nonwandering set of an Axiom A system). Then the following are true:*

1. *The periodic points are dense (Proof: have a periodic point shadow one orbit piece of one point).*
2. *$(X, \phi)$ is topologically mixing (Proof: choose two orbit pieces, one in $U$ and one in $V$, and have a point orbit shadow the two pieces).*
3. *$(X, \phi^k), (X \times Y, \phi \times \psi)$ (where $(Y, \psi)$ has weak specification) and any factor of $(X, \phi)$ all have weak specification.*
4. *Exercise: $h_{top}(X, \phi) > 0$.*

(continued on the next day)

## 22. Consequences of specification. February 26 (*notes by KJ*)

*Results on specification*

**Theorem 22.1** (cont.)**.** *Let $(X, \phi)$ be a topological dynamical system with weak specification (for example, an elementary part of the nonwandering set of an Axiom A system). Then the following are true:*

(1) *The periodic points are dense (Proof: have a periodic point shadow one orbit piece of one point.)*
(2) $(X, \phi)$ *is topologically mixing (Proof: choose two orbit pieces, one in* $U$ *and one in* $V$*, and have a point orbit shadow the two pieces).*
(3) $(X, \phi^k), (X \times Y, \phi \times \psi)$ *(where* $(Y, \psi)$ *has weak specification) and any factor of* $(X, \phi)$ *all have weak specification.*
(4) *(Bowen)* $h_{top}(X, \phi) > 0$.
(5) *(Sigmund) In the compact metric space* $\mathcal{I}$ *(with respect to the weak* topology) of* $\phi$*-invariant Borel probability measures on* $X$*, the set* $\mathcal{E}$ *of ergodic invariant measures (which is equal to the set of extreme points of the compact convex set* $\mathcal{I}$*) is residual (a dense* $G_\delta$*).*

**Remark 22.1.** This result is extremely different from uniquely ergodic systems — in systems with weak specification, there are *many* ergodic measures.

(6) *(Sigmund – proof by Parthasarathy) The set of strongly mixing measures in* $\mathcal{I}$ *is first category in* $\mathcal{I}$*. [Remember that strongly mixing means that for all* $A, B \in \mathcal{B}, \mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$ *as* $n \to \infty$*. Page 199 of DGS has a reference to Parthasaranthy (1961). He must have proved this using a more general situation than specification.]*
(7) Every $\mu \in \mathcal{I}$ *has a generic point: There is* $x \in X$ *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\phi^k x) \to \int_X f d\mu \text{ for all } f \in C(X).$$

*Equivalently, there are* $x$*'s for which the averages of the point masses* $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{\phi^k x}$ *converge to* $\mu$ *in the weak* topology.*

**Remark 22.2.** Note that if $\mu$ is ergodic, it is not hard to find generic points. Given $f$, the Ergodic Theorem gives a set of full $\mu$-measure such that $\frac{1}{n}\sum f(\phi^k x) \to \int_X f d\mu$. Then we could get a set of full $\mu$ measure that would be good for all $f$ in a countable dense set in $C(X)$. To get these points to work for an arbitrary $g$, given $\epsilon > 0$, choose one of our good $f$ from the countable dense set with $|g - f| < \frac{\epsilon}{3}$. Then

$$\begin{aligned}\left|\frac{1}{n}\sum g(\phi^k x) - \int g d\mu\right| &\le \left|\frac{1}{n}\sum (g-f)(\phi^k x)\right| \\ &\quad + \left|\frac{1}{n}\sum f(\phi^k x) - \int f d\mu\right| + \left|\int (f-g) d\mu\right| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\end{aligned}$$

if $n$ is chosen so that $\left|\frac{1}{n}\sum f(\phi^k x) - \int f d\mu\right| < \frac{\epsilon}{3}$. This is a corollary of the theorem on p. 202 ff. of DGS, and tells us that we can find generic points for any measure, even the non-ergodic ones.

(8) *(Sigmund) The set of points* $x \in X$ *with* maximum oscillation*:*

$$V_\phi(x) = \text{the set of limit points of } \left\{\frac{1}{n}\sum_{k=0}^{n-1}\delta_{\phi^k x} : n = 1, 2, \ldots\right\} = \mathcal{I},$$

*is residual.*

**Remark 22.3.** For points of maximum oscillation, not even statistical description of the behavior of $\mathcal{O}(x)$ is possible. Thus according to statement (8) there is a residual, that is to say, a topologically typical, set of points whose orbits are beyond description, in that we can make no statement (even statistically) of what is going on. This is really bad chaos. With good chaos, you get an attractive measure that statistically describes long-term average behavior.

In systems with specification averages of point masses along a typical orbit shadow *every* measure.

This property, in combination with the Kolmogorov (algorithmic) complexity of the orbit, is discussed in Homer White's thesis.

For more on this theorem, see Chapter 22 in DGS.

**Theorem 22.2** (Bowen)**.** *Let $(X, \phi)$ be expansive and have specification. For each $n = 1, 2, \ldots$ let $P_n = \text{card}\{x \in X : \phi^n x = x\}$. Then*

(1) $h_{top}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log P_n$

(2) *$(X, \phi)$ has a unique measure of maximal entropy ($h_\mu(\phi) = h_{top}(\phi)$), ($(X, \phi)$ is called* intrinsically ergodic*) which is given by (among other ways) $\lim_{n\to\infty} \frac{1}{P_n} \sum_{\phi^n x = x} \delta_x$ in the weak* topology.*

In a toral automorphism, we have one basic set, which is topologically mixing, and Lebesgue measure is the unique measure of maximal entropy. As discussed earlier, the periodic points are the points with rational coordinates. The convergence to Lebesgue measure in statement 2 of Theorem 22.2 should remind us of Riemann integration of continuous functions. The same argument works for Haar measure on compact abelian groups.

Note that $P_n$ is finite for every $n$: if we had infinitely many fixed points, for example, we would defeat expansiveness. DGS gives a bound for the number of periodic points.

Our interest in this theorem is mainly for the elementary parts of $(\Omega, \phi)$.

DGS calls the unique measure in Theorem 22.2 the 'Bowen measure' — but it is *not* the same as the SRB measure.

### 22.1. **More consequences of the pseudo-orbit shadowing property.**

*Anosov Closing Lemma*

**Theorem 22.3** (Anosov Closing Lemma)**.** *If $(M, \phi)$ is an Anosov system (the entire manifold $M$ is a hyperbolic set) then the periodic points are dense in $\Omega(\phi)$. Consequently, $(M, \phi)$ is Axiom A.*

This is a loose end left over from the beginning weeks of the class.

*Proof.* We review previous arguments in the pseudo-orbit shadowing property and others to see where we used the density of periodic points in $\Omega(\phi)$.

- Stable Manifold Theorem: This holds true on hyperbolic sets, hence on all of $M$.
- Canonical coordinates $[x, y] \in W^s_\epsilon(x) \cap W^u_\epsilon(y) \cap \Omega$. We used expansiveness. It took effort to prove that $[x, y] \in \Omega$, where we used "homoclinic pinball" and density of periodic points. But to prove this theorem we don't require that $[x, y]$ be in $\Omega$, just in $M$.

- The proof of the pseudo-orbit shadowing property did not use that periodic points are dense in $\Omega$, just hyperbolicity and canonical coordinates.
- The corollary on closing up periodic orbits is also fine, since it just uses expansiveness and hyperbolicity.

Now, let $x \in \Omega(\phi)$ and let $\beta > 0$. . We are trying to show that there is $z \in M$ which is periodic and within $\beta$ of $x$. Choose $\alpha$ as in Corollary 19.1 and let $U = B_{\alpha/2}(x)$. We know that there is $y \in U$ such that $\phi^n y \in U$, since $x$ is in the nonwandering set. Then there is a $z$ which $\beta$ shadows $y, \phi y, \dots \phi^n y$ and is periodic, hence is within $\beta$ of $x$. □

## 23. Anosov Closing Lemma. February 28 (*Notes by KJ*)

### 23.1. More consequences of the pseudo-orbit shadowing property.

**Proposition 23.1.** *If $(M, \phi)$ is an Axiom A dynamical system, then there is a neighborhood $U$ of $\Omega(\phi)$ such that*

*Fundamental neighborhood*

$$\bigcap_{n\in\mathbb{Z}} \phi^n U = \Omega(\phi).$$

*Proof.* Take $\beta <$ (expansive constant)/2, $\alpha$ as in the pseudo-orbit shadowing theorem, and $\gamma < (\alpha + \text{(expansive constant)})/2$, and such that if $d(x, y) < \gamma$, then $d(\phi x, \phi y) < \alpha/2$. Let $U = \{y \in M : d(y, \Omega) < \gamma\}$.

(Showing that $\cap_{n\in\mathbb{Z}}\phi^n U \subset \Omega(\phi)$) Suppose that $y \in \cap_{n\in\mathbb{Z}}\phi^n U$. This means that for each $n \in \mathbb{Z}$ we can find $x_n \in \Omega$ with $d(\phi^n y, x_n) < \gamma$. Then the $x_n$'s are an $\alpha$-pseudo-orbit:

$$\begin{aligned} d(\phi x_n, x_{n+1}) &\le d(\phi x_n, \phi^{n+1} y) + d(\phi^{n+1} y, x_{n+1}) \\ &< \alpha/2 + \gamma < \alpha \end{aligned}$$

Now find $x \in \Omega$ whose orbit $\beta$-shadows the $x_n$'s. Then for every $n \in \mathbb{Z}$,

$$\begin{aligned} d(\phi^n y, \phi^n x) &\le d(\phi^n y, x_n) + d(x_n, \phi^n x) \\ &\le \gamma + \beta \\ &\le \text{(expansive constant)}. \end{aligned}$$

Therefore $y = x \in \Omega$.

The reverse inclusion is clear since $\Omega$ is $\phi$-invariant. □

**Remark 23.1.** Similarly, for each $j$ there is a neighborhood $E_j$ of $\Omega_j$ such that $\cap_{n\in\mathbb{Z}}\phi^n E_j = \Omega_j$. (In the proof above, make sure that $\alpha, \beta, \gamma$ are smaller than the fixed distance of the $\Omega_j$'s from each other.)

The proposition implies that if something is not in $\Omega$, it can't stay in $U$ all the time — at some time it gets outside of $U$.

**Proposition 23.2.** *In a general (compact) topological dynamical system $(X, \phi)$ with non-wandering set $\Omega$, for each $x \in X$,*

$$d(\phi^n x, \Omega) \to 0 \text{ as } n \to \infty$$
$$d(\phi^{-n} x, \Omega) \to 0 \text{ as } n \to \infty.$$

*If $(\Omega, \phi)$ is, as in the Axiom A case, the disjoint union of finitely many closed sets $\Omega_j$, then for each $x \in X$ there are $j(x), k(x)$ such that*

$$d(\phi^n x, \Omega_{j(x)}) \to 0 \quad \textit{and} \quad d(\phi^{-n} x, \Omega_{k(x)}) \to 0$$

*as $n \to \infty$.*

*Thus if*

$$W^s(\Omega_j) = \{x \in X : d(\phi^n x, \Omega_j) \to 0\}$$
$$W^u(\Omega_j) = \{x \in X : d(\phi^{-n} x, \Omega_j) \to 0\}$$

*as $n \to \infty$, then*

$$X = \sqcup_j W^s(\Omega_j) = \sqcup_j W^u(\Omega_j)$$

*where $\sqcup$ means disjoint union.*

*Proof.* Let $x \in X$ and let $\omega(x) = \{\phi^n x : n \geq 0\}'$ where $U'$ denotes the set of limit points of $U$. We will show that $\omega(x) \subset \Omega(\phi)$. (This will prove the first part of the Proposition: If the distance didn't go to 0, then there would be a neighborhood $U$ of $\Omega$ and infinitely many $n_i$'s with $\phi^{n_i} x \notin U$. Then there would be a limit point of $\omega(x)$ outside of $U$, hence not in $\Omega$.)

Let $z \in \omega(x)$, say $\phi^{n_i} x \to z, n_i \geq 0$. We will show that $z \in \Omega$, *i.e.* that every neighborhood of $z$ returns to itself in forward time. Let $V$ be a neighborhood of $z$. Then find $\phi^{n_i} x, \phi^{n_j} x \in V$ with $n_i < n_j$. Then $\phi^{n_i - n_j} V \cap V$ is not empty, since $\phi^{n_j} x \in V$ and $\phi^{n_j} x = \phi^{n_j - n_i} \phi^{n_i} x \in \phi^{n_j - n_i} V$. Therefore, $z \in \Omega$ and the first part is proved.

For the second statement, use the limit set $\alpha(x) = \{\phi^{-n} x, n \geq 0\}'$.

Now we want to show that each $\omega(x)$ is in one $\Omega_j$. For each $j$, define $U_j, V_j$ to be disjoint open neighborhoods of $\Omega_j, \cup_{k \neq j} \Omega_k$ respectively, so that $\overline{U_j} \cap \overline{V_j} = \emptyset$. Suppose we have infinitely many $\phi^n x \in U_j$ and $V_j$. Then we can find an increasing sequence $n_i$ with $\phi^{n_i} x \in U_j$ and $\phi^{n_i + 1} x \in V_j$ with $\phi^{n_i} x \to z \in \Omega_j$ and $\phi^{n_i+1} x \to w \in \overline{V_j} \subset U_j^c$ But then also $w = \phi(z)$, which is impossible by the invariance of $\Omega_j$.

□

## 24. More consequences of pseudo-orbit shadowing. Markov partitions. March 3 (*Notes by LK*)

We begin with one more consequence of pseudo-orbit shadowing.

**Proposition 24.1.** *Let $\Omega_j$ be a basic set of an Axiom A system $(M, \phi)$. Then*

1. $W^s(\Omega_j) = \bigcup_{x \in \Omega_j} W^s(x)$ *and* $W^u(\Omega_j) = \bigcup_{x \in \Omega_j} W^u(x)$.
2. *Given $\epsilon > 0$, for each $j$ there exists a neighborhood $U_j$ of $\Omega_j$ such that* $\bigcap_{k \geq 0} \phi^{-k} U_j \subset \bigcup_{x \in \Omega_j} W^s_\epsilon(x)$ *and* $\bigcap_{k \geq 0} \phi^k U_j \subset \bigcup_{x \in \Omega_j} W^u_\epsilon(x)$.

*Proof.*

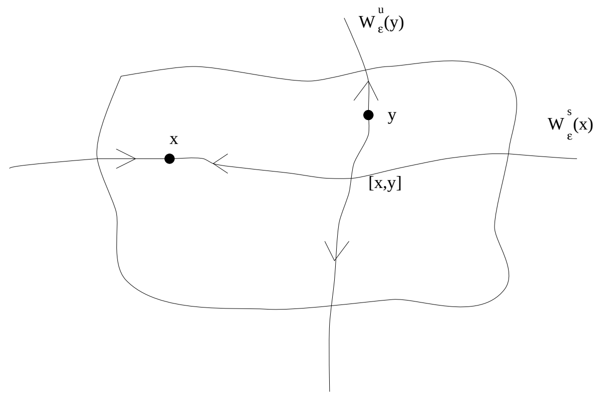



FIGURE 42. A rectangle $R$

(1) That $W^s(\Omega_j) \supset \bigcup_{x\in\Omega_j} W^s(x)$ is trivial. To show $W^s(\Omega_j) \subset \bigcup_{x\in\Omega_j} W^s(x)$, we pick a $y \in W^s(\Omega_j)$. Let $\epsilon > 0$, let $\beta = \epsilon/2$, let $\alpha$ be determined by the Pseudo-Orbit Shadowing Theorem for $\beta$, and let $\gamma < \alpha/2$. Pick $N$ large enough that $d(\phi^n y, \Omega_j) < \gamma$ for all $n > N$. This means we can find points $x_n \in \Omega_j$ with $d(\phi^n y, x_n) < \gamma$ for $n > N$. Since $\gamma < \alpha/2$, $(x_n)_{n\geq N}$ is an $\alpha$-pseudo-orbit in $\Omega_j$. Take $x \in \Omega$ that $\beta$-shadows $(x_n)_{n\geq N}$. If $\beta$ is small enough, we must have $x \in \Omega_j$, since its orbit $\beta$-shadows points of $\Omega_j$ and the $\Omega_j$ are closed and pairwise disjoint. Hence
$$\phi^N y \in W^s_\epsilon(\phi^N x) \subset W^s(\phi^N x)$$
and thus
$$y \in \phi^{-N} W^s(\phi^N x) = W^s(x).$$
The other direction follows similarly.

(2) Using $N = 0$ and $U_j = \{y \in M : d(y, \Omega_j) < \gamma\}$ in part (1), we showed that $\bigcap_{k\geq 0} \phi^{-k} U_j \subset \bigcup_{x\in\Omega_j} W^s_\epsilon(x)$.

□

24.1. **Markov partitions.** Fix a basic set $\Omega_j$ of an Axiom A system $(M, \phi)$ and an $\epsilon > 0$ small enough for the Stable Manifold Theorem to hold. We define a *rectangle* as a set $R \subset \Omega_j$ which has small enough diameter that if $x, y \in R$ then $[x, y]$ is defined and $[x, y] \in R$. See Figure 42. *Markov partitions*

We say that $R$ is *proper* if $R$ is closed and $R = \overline{\operatorname{int} R}$, where the interior of $R$ is relative to $\Omega_j$. A *Markov partition* of $\Omega_j$ is a finite family $R_1, \ldots, R_m$ of proper rectangles such that:

(1) all $R_i$ have small diameters (even compared to $\epsilon$),
(2) $\Omega_j = R_1 \cup \cdots \cup R_m$,
(3) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ if $i \neq j$,

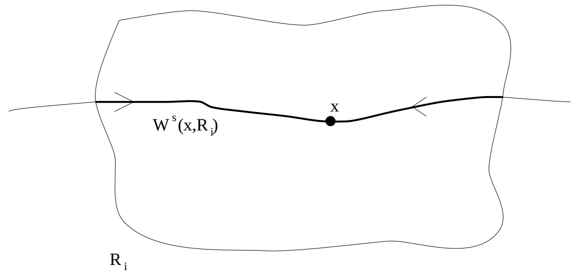



Figure 43. $W^s(x, R_i)$

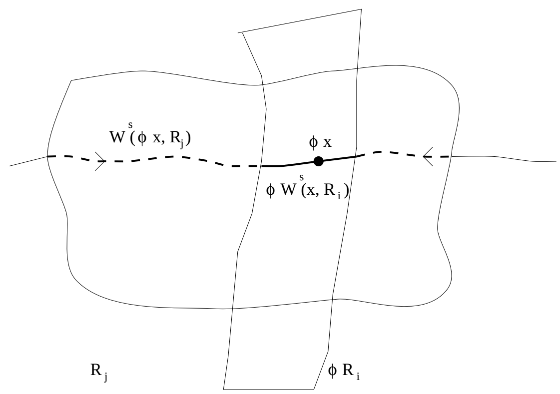



Figure 44. $\phi W^s(x, R_i)$

(4) Let $W^s(x,R) = W^s_\epsilon(x) \cap R$ and $W^u(x,R) = W^u_\epsilon(x) \cap R$. If $x \in \operatorname{int} R_i$ and $\phi x \in \operatorname{int} R_j$, then $\phi W^u(x,R_i) \supset W^u(\phi x, R_j)$ and $\phi W^s(x,R_i) \subset W^s(\phi x, R_j)$. See Figures 43 and 44.

Thus the image of each unstable section goes all the way across, in the unstable direction, any rectangle that it gets mapped to. We include an illustration in Figure 45 of rectangles that map badly and do not satisfy the last condition in the definition of a Markov partition.

## 25. Exercises on solenoid and Markov partitions. March 5 (*Notes by LK*)

**Exercise 4.**

(1) The solenoid. Let $S_1 = \mathbb{R}/\mathbb{Z} = [0,1]$ with $0 \sim 1$. Let $X = (S_1)^{\mathbb{Z}}$ be the space of all doubly-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ where $0 \le x_i < 1$ for all $i$. We endow $X$ with the product topology and let $\sigma\colon X \to X$ be the shift map. $X$ also has the product group structure and $\sigma$ is a continuous group automorphism. Let $G = \{x \in X \colon x_{n+1} = 2x_n \text{ for all } n\}$.

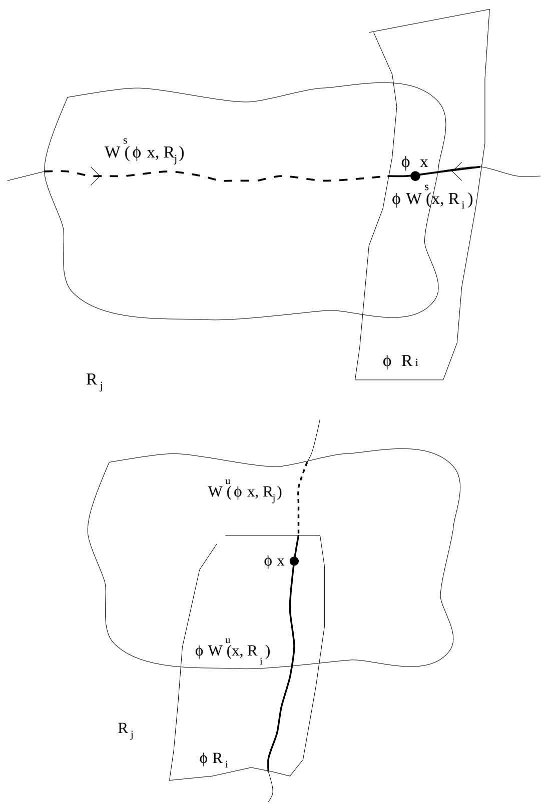



FIGURE 45. Rectangles that map badly

Show that $G$ is closed and $\sigma$-invariant. Also, show that $(G, \sigma)$ is an automorphism of a compact abelian group and is topologically conjugate to the solenoid system $(\Lambda, \phi)$. Can you use this conjugacy to find the stable and unstable sets of points?

(2) Determine whether the partitions that we used to code the closed invariant sets in the horseshoe and solenoid constructions are Markov partitions. Recall that we found the invariant sets as factors of symbolic systems starting with some partition of the original space.

*Notetaker's remark: We also completed the definition of a Markov partition in today's lecture. For clarity, I have moved this discussion into the March 3 lecture notes.*

## 26. EXISTENCE OF MARKOV PARTITIONS. MARCH 7 (*Notes by LK*)

We begin with a discussion of rectangles in shifts of finite type. Let $\Sigma_B \subset A^{\mathbb{Z}}$, where $B$ is an $r$-by-$r$ matrix with entries 0 or 1 and $A = \{0, 1, \dots, r-1\}$. We define canonical coordinates in the following manner: if $\omega, \xi \in \Sigma_B$ are close enough that $\omega_0 = \xi_0$ (i.e. $d(\omega, \xi) < 1$) then we let $[\omega, \xi] = \dots, \xi_{-2}, \xi_{-1}, \xi_0, \omega_1, \omega_2 \dots$. We note that $[\omega, \xi]$ is in $\Sigma_B$ because the transitions are allowed since $\omega_0 = \xi_0$. Further, this point is unique in $\Sigma_B$. If

$\rho = [\omega, \xi]$ then $\rho \in W_1^s(\omega) \cap W_1^u(\xi) = \{\rho\}$. That is,

$$d(\phi^k \rho, \phi^k \omega) \leq 1 \text{ for all } k \geq 0 \text{ and } d(\phi^k \rho, \phi^k \xi) \leq 1 \text{ for all } k \leq 0.$$

So a cylinder set $[a]$ is a rectangle in this setting. That is, if $\omega, \xi, \in [a]$, then $[\omega, \xi] \in [a]$. We next prove the following warm-up observation about rectangles in $\Omega_j$.

*Warm-up Lemma*

**Lemma 26.1.** *If, in a basic set $\Omega_j$ of an Axiom A system $(M, \phi)$, $R$ is a proper rectangle with small enough diameter compared to $\epsilon$, then*

$$\operatorname{int}_{\Omega_j} R = \bigcup_{x \in R} (\operatorname{int}_{W^u_\epsilon(x) \cap \Omega} W^u(x, R) \cap \operatorname{int}_{W^s_\epsilon(x) \cap \Omega} W^s(x, R)) \tag{3}$$

Figure 46 illustrates Lemma 26.1. This lemma can also be stated in terms of boundaries, as in Bowen:

$$\partial_{\Omega_j} R = \partial^s R \cup \partial^u R$$

where

$$\partial^s R = \{x \in R \colon x \notin \operatorname{int}_{W^u_\epsilon(x) \cap \Omega} W^u(x, R)\},$$
$$\partial^u R = \{x \in R \colon x \notin \operatorname{int}_{W^s_\epsilon(x) \cap \Omega} W^s(x, R)\}.$$

*Proof.* If $x \in \operatorname{int}_{\Omega_j} R$, then $W^u(x, R) = R \cap (W^u_\epsilon(x) \cap \Omega)$ is a neighborhood of $x$ in $W^u_\epsilon \cap \Omega$ and thus $x \in \operatorname{int}_{W^u_\epsilon(x) \cap \Omega} W^u(x, R)$. Similarly, $x \in \operatorname{int}_{W^s_\epsilon(x) \cap \Omega} W^s(x, R)$.

Conversely, suppose $x$ is in the right-hand side of Equation 3. We want to show if $y \in \Omega_j$ is close enough to $x$, then $y \in R$. For $y$ near $x$, we know $[x, y] \in W^s_\epsilon(x) \cap \Omega$ and $[y, x] \in W^u_\epsilon(x) \cap \Omega$. Further, $[x, y] \in W^s(x, R)$ and $[y, x] \in W^u(x, R)$ since canonical coordinates are continuous functions of $y$, and $x$ is in the (relative) interiors of these sets. Thus $[x, y]$ and $[y, x]$ are in $R$. Then $[[y, x], [x, y]] = y$ is also in $R$. □

*Markov partitions of toral automorphisms*

**Example 26.1** (Toral automorphisms have Markov partitions)**.** For a construction in the 2-dimensional case, see page 250 of Petersen's book. We will outline the procedure here. Let $A$ be a 2-by-2 integer matrix with determinant $\pm 1$. Then $A$ has one real eigenvalue greater than 1, and one real eigenvalue less than one. The resulting eigenspaces have irrational

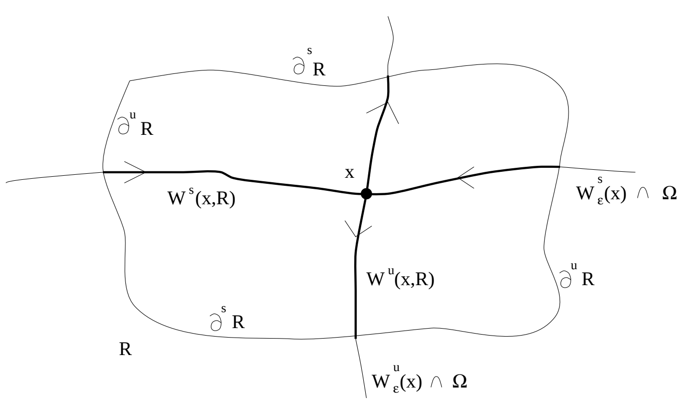



FIGURE 46. An illustration of Lemma 26.1

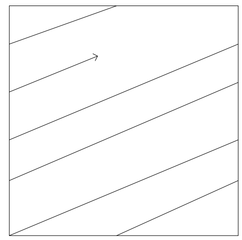


FIGURE 47. One eigenspace

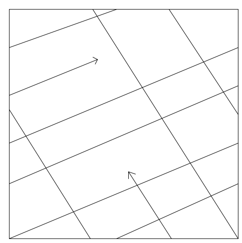


FIGURE 48. Both eigenspaces

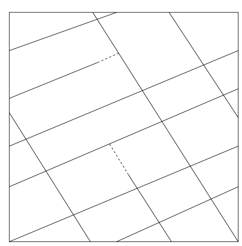


FIGURE 49. Extending the eigenspaces to obtain a Markov partition

slope. We use these eigenspaces to make the partitions. To get rectangles, we make sure that we stop extending the eigenspace when we intersect the other eigenspace. We give an illustration of this procedure in Figures 47, 48, and 49 and encourage the reader to examine the details in Petersen's book.

Lind and Marcus explicitly find a Markov partition for the toral automorphism

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We note that once you have a Markov system, you have a shift of finite type; sometimes it is easier to just work on this SFT. In fact, Lind and Marcus give an elementary definition of a Markov partition in terms of a coding to a SFT. Denker, Grillenberger, and Sigmund generalize these concepts to topological dynamical systems on page 241 of their book.

Markov partitions of $\Omega_j$

**Theorem 26.2** (Bowen)**.** *Let $(M, \phi)$ be an Axiom A diffeomorphism and $\Omega_j$ a basic set. Then there are Markov partitions of $\Omega_j$ into rectangles of arbitrarily small diameter.*

*Proof.* Let $\epsilon$ be as in the Stable Manifold Theorem (Theorem 14.1). Take a positive $\beta$ much less than $\epsilon$ and less than $\frac{1}{2}$ the expansive constant. Let $\alpha > 0$ be as in the Pseudo-Orbit Shadowing Theorem (Theorem 18.1) for $\beta$. Choose $\gamma < \alpha/2$ such that $d(x, y) < \gamma$ implies that $d(\phi x, \phi y) < \alpha/2$.

Choose a finite $\gamma$-dense set $P = \{p_1, p_2, \dots, p_s\}$ in $\Omega_j$. Look at the symbolic space of *$\alpha$-pseudo-orbits* using $P$.

$$\Sigma(P) = \{\omega \in P^{\mathbb{Z}} \colon d(\phi\omega_i, \omega_{i+1}) < \alpha \text{ for all } i \in \mathbb{Z}\}.$$

This is actually a SFT. Give $\Sigma(P)$ the product topology. There is a map

$$\theta \colon \Sigma(P) \to \Omega_j$$

defined by

$$\theta(\omega) = \text{ the unique } x \text{ in } \Omega_j \text{ whose orbit } \beta\text{-shadows } (\omega_i)_{i\in\mathbb{Z}}.$$

We need to check the existence and uniqueness of such an $x$. If $\omega \in \Sigma(P)$, then the orbit of $\omega$ defines an $\alpha$-pseudo-orbit and there exists a point $x$ $\beta$-shadowing it by the Pseudo-Orbit Shadowing Theorem. Uniqueness follows because if there is another such $y$ which $\beta$-shadows $(\omega_i)$, then $d(\phi^i x, \phi^i y) < 2\beta$ for all $i$, but $\beta$ is less then half the expansive constant. Hence $x = y$.

Clearly, $\theta\sigma = \sigma\phi$, so that the following diagram commutes:

$$\begin{array}{ccc} \Sigma(P) & \xrightarrow{\sigma} & \Sigma(P) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ \Omega_j & \xrightarrow{\phi} & \Omega_j \end{array}$$

We next show that $\theta$ is onto. Given $x \in \Omega_j$, since $P$ is $\gamma$-dense we can choose $\omega_i \in P$ within $\gamma$ of $\phi^i x$ for all $i$. Then $\theta(\omega) = x$ because the orbit of $x$ and the orbit of $\theta(\omega)$ both $\beta$-shadow the $(\omega_i)_{i\in\mathbb{Z}}$.

## 27. Start the proof of existence of Markov partitions. March 17 *(Notes by NP)*

*Editor's Note:* There are some overlaps in the notes concerning the next several weeks. (Several times the lecturer retraced his steps, and several notetakers expanded their write-ups for coherence.)

This week we will be continuing the proof of Bowen's theorem proving the existence of Markov partitions for the nonwandering set of an Axiom A diffeomorphism (Theorem 26.2).

We prove that $\theta$ is continuous by contradiction—suppose it is not. Then there exists an $\eta > 0$ such that for all $n = 1, 2, \dots$ we can find points $\omega^{(n)}, \zeta^{(n)} \in \Sigma(P)$ such that

$$\omega^{(n)}(i) = \zeta^{(n)}(i) \text{ for } |i| \le n,$$

but

$$d(\theta\omega^{(n)}, \theta\zeta^{(n)}) \ge \eta.$$

Figure 50 illustrates the central blocks of $\omega^{(n)}$ and $\zeta^{(n)}$.

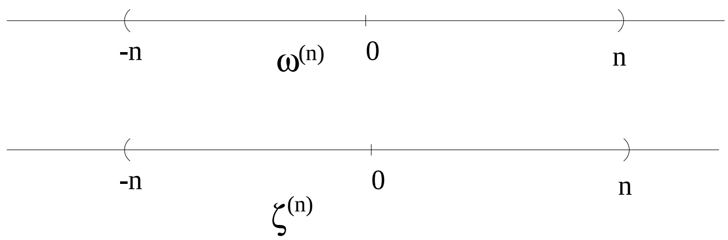



FIGURE 50. $\omega^{(n)}$ and $\zeta^{(n)}$ are the same on a central block.

If $x^{(n)} = \theta(\omega^{(n)})$ and $y^{(n)} = \theta(\zeta^{(n)})$, we can pass to a subsequence and assume $x^{(n)} \to x$ and $y^{(n)} \to y$, so that $d(x, y) \geq \eta$.

Now $x^{(n)}$ $\beta$-shadows $\omega^{(n)}$ and $y^{(n)}$ $\beta$-shadows $\zeta^{(n)}$, and $\omega^{(n)}(i) = \zeta^{(n)}(i)$ for $|i| \leq n$, so

$$d(\phi^i x^{(n)}, \phi^i y^{(n)}) \leq 2\beta \qquad \text{for } |i| \leq n.$$

This implies that

$$d(\phi^i x, \phi^i y) \leq 2\beta < \text{ expansive constant}$$

for all $i$, and this contradicts the expansiveness of $\phi$.

We have shown that $\theta$ is a continuous factor map from a shift of finite type onto a basic set. Next we shall show that this mapping also maps certain rectangles in $\Sigma(P)$ onto rectangles in $\Omega_j$.

Consider the cylinder set $[p_i] = \{\omega : \omega_0 = p_i\}$ in $\Sigma(P)$. It is a rectangle since for any

$$\omega, \zeta \in [p_i], \quad [\omega, \zeta] = ..., \zeta_{-2}, \zeta_{-1}, \zeta_0, \omega_1, \omega_2, ...,$$

which is clearly in $W_1^s(\omega) \cap W_1^u(\zeta) \cap [p_i]$. (Cylinder sets corresponding to longer blocks are also rectangles in $\Sigma(P)$).

Define $T_i$ to be the image under $\theta$ of the rectangle $[p_i]$; we will show it is also a rectangle for all $i = 1, 2, ..., r$. These rectangles in $\Omega_j$ are crucial to the construction of the Markov partition of $\Omega_j$—refinements of them form the said partition.

Note that $T_i$ is closed and the diameter of $T_i$ is less than $2\beta$, because any point $y \in \Omega_j$ which beta-shadows a point in $[p_i]$ must be within $\beta$ of $p_i$, since this is the time-zero entry in the pseudo-orbit.

Let $x, y \in T_i$, and suppose that

$$\omega = ..., \omega_{-1}, p_i, p_i', ... \quad \stackrel{\theta}{\to} \quad x \text{ , and}$$
$$\zeta = ..., \zeta_{-1}, p_i, p_i'', ... \quad \stackrel{\theta}{\to} \quad y.$$

We claim that

$$\theta[\omega, \zeta] = [\theta\omega, \theta\zeta] = [x, y],$$

implying that $[x, y] \in \theta[p_i] = T_i$. We know that

$$[\omega, \zeta] = ..., \zeta_{-1}, \zeta_0, \omega_1, ...$$
$$= ..., \zeta_{-1}, p_i, p_i', ...,$$

so

$$d(\phi^m\theta[\omega,\zeta],\phi^m\theta\omega)\leq 2\beta \quad \text{for } m\geq 0 \text{ , and}$$
$$d(\phi^m\theta[\omega,\zeta],\phi^m\theta\zeta)\leq 2\beta \quad \text{for } m\leq 0.$$

Thus $\theta[\omega,\zeta]\in W^s_{2\beta}(x)\cap W^u_{2\beta}(y)=\{[x,y]\}$. We have ascertained that $T_i$ is a rectangle in $\Omega_j$.

Now we will check the Markov mapping property for the $T_i$'s. The following is a proof that if $x\in T_i$ and $\phi x\in T_k$, then

$$\phi W^s(x,T_i)\subset W^s(\phi x,T_k).$$

The proof for the unstable manifolds is analogous.

Let $y\in W^s(x,T_i)=W^s_\epsilon(x)\cap T_i$. Look upstairs in the shift space; does it follow that the preimages of $x$ and $y$ under $\theta$ are identical to the right of zero? That is, can we show that

$$\omega=...\omega_{-1}\dot{p}_ip_k... \quad \overset{\theta}{\to} \quad x \text{ , and}$$
$$\zeta=...\zeta_{-1}\dot{p}_ip_k,... \quad \overset{\theta}{\to} \quad y?$$

Since $y\in W^s_\epsilon(x), y=[x,y]=\theta[\omega,\zeta]$. So we find that

$$\begin{aligned}\phi y=\phi[x,y]&=\phi[\theta\omega,\theta\zeta]=\theta\sigma[\omega,\zeta]\\&=\theta(...\zeta_{-1}\zeta_0\dot{\omega}_1\omega_2...)\\&=\theta(...\zeta_{-1}p_i\dot{p}_k\omega_2...)\in T_k.\end{aligned}$$

Noting that $\phi y\in W^s_\epsilon(\phi x)$ because $\phi W^s_\epsilon(x)\subset W^s_\epsilon(\phi x)$, we see that $y\in W^s(x,T_i)$ implies that $\phi y\in W^s(\phi x,T_k)$, finishing the proof of the Markov mapping property for the stable manifolds. Using an analogous argument for the unstable manifolds completes the proof that the $T_i$'s have the Markov mapping property.

We have obtained a family of rectangles of small diameter which satisfy the Markov mapping property. However, this family may not be proper or essentially disjoint. We must refine the rectangles to obtain these properties so that a Markov partition is formed.

We will begin the next part of the proof in the next class period, by creating an essentially disjoint family of subrectangles of the $T_i$'s.

## 28. More of the proof. March 19 *(Notes by NP)*

Recall that $\{T_k=\theta[p_k],k=1,2,...r\}$ is a family of rectangles which satisfy the Markov mapping property. We will extract refinements of these rectangles which retain these properties but which also are proper and essentially disjoint. For any $m,k$ with $T_m\cap T_k\neq\emptyset$, define four subsets of $T_m$ as follows:

$$\begin{aligned}T^1_{m,k}&=\{x\in T_m:W^u(x,T_m)\cap T_k\neq\emptyset,W^s(x,T_m)\cap T_k\neq\emptyset\}\\T^2_{m,k}&=\{x\in T_m:W^u(x,T_m)\cap T_k=\emptyset,W^s(x,T_m)\cap T_k\neq\emptyset\}\\T^3_{m,k}&=\{x\in T_m:W^u(x,T_m)\cap T_k\neq\emptyset,W^s(x,T_m)\cap T_k=\emptyset\}\\T^4_{m,k}&=\{x\in T_m:W^u(x,T_m)\cap T_k=\emptyset,W^s(x,T_m)\cap T_k=\emptyset\}.\end{aligned}$$

Figure 51 illustrates these refinements.

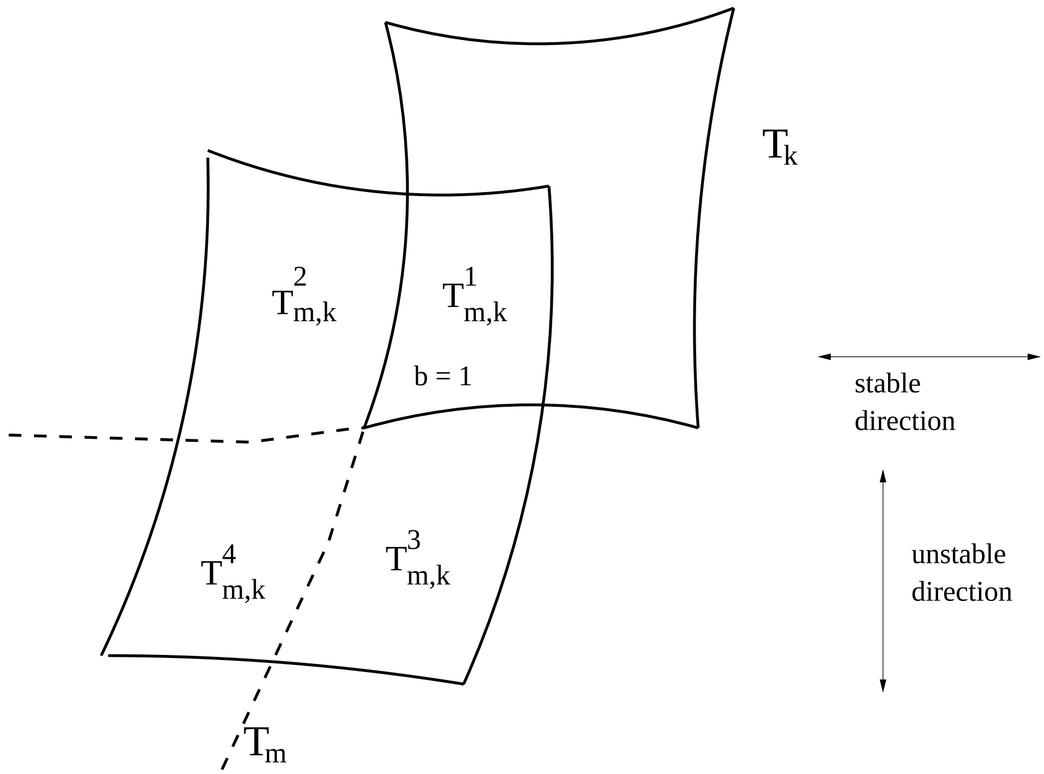



FIGURE 51. The refinements of $T_m$ in terms of $T_k$.

We show that these $T^b_{m,k}$'s are rectangles; take $x, y \in T^b_{m,k}$, then $z = [x, y] \in T^b_{m,k}$ since

$$W^s(z, T_m) \text{ intersects } T_k \text{ if and only if } W^s(x, T_m) \text{ does, and}$$
$$W^u(z, T_m) \text{ intersects } T_k \text{ if and only if } W^u(y, T_m) \text{ does.}$$

Then each $\overline{T^b_{m,k}}$ is also a rectangle, and, by the warm-up remark, so is each $\operatorname{int}(\overline{T^b_{m,k}})$ (if it is not empty).

Now we must avoid boundaries (which are probably topologically horrible); we will show that the set of all points whose stable and unstable sets do not hit the corresponding boundaries of any neighboring rectangles is open and dense in $\Omega_j$. Since it will be shown that the interiors of the $T^b_{m,k}$'s cover this open dense set, we can make a partition of the open dense set. The closures of the rectangles forming this partition will be proper and essentially disjoint and will ultimately be our Markov partition.

Each $T_k$ is closed, so $\partial T_k$ is nowhere dense, hence $\bigcup_k \operatorname{int} T_k$ is open and dense in $\Omega_j$. Discard any $T_k$ for which $\operatorname{int} T_k = \emptyset$.

Define the 'good set' of points to be

$$G = \{x \in \bigcup_k \operatorname{int} T_k : x \in \operatorname{int} T_k, d(T_k, T_m) < 4\beta \Rightarrow W^s_\epsilon(x) \cap \partial^s T_m = \emptyset,$$
$$W^u_\epsilon(x) \cap \partial^u T_m = \emptyset\}.$$

$G$ is *open*, because $\partial^s T_k$ and $\partial^u T_k$ are closed, and $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ vary continuously with $x$. We show that $G$ is also *dense* in $\Omega_j$.

Let $k$ and $m$ be such that $d(T_m, T_k) < 4\beta$, and define

$$G^s_{m,k} = \{x \in \operatorname{int} T_k \ : \ W^s_\epsilon(x) \cap \partial^s T_m = \emptyset\}.$$

We will show that $G^s_{m,k}$ is dense in $T_k$. A similar proof holds for the sets $G^u_{m,k}$. Then, since $G = \bigcup_k \bigcap_m (G^s_{m,k} \cap G^u_{m,k})$, a finite union of finite intersections, we find that $G$ is open and dense.

To prove that $G^s_{m,k}$ is dense in $T_k$, note that if $y \in \operatorname{int} T_m$, then $W^s_\epsilon(y) \cap \partial^s T_m = \emptyset$. We prove this by contradiction, assuming that there is a $z \in W^s_\epsilon(y) \cap \partial^s T_m$. Since $\partial^s T_m = \{z \in T_m \ : \ z \notin \operatorname{int}_{W^u_\epsilon(z)\cap\Omega}(W^u_\epsilon(z) \cap T_m)\}$, we can find $z' \notin T_m$ with $z' \in W^u_\epsilon(z)$ and $z'$ arbitrarily close to $z$. We can use canonical coordinates in $G^s_{m,k}$ since $\beta$ is much smaller than $\epsilon$, and we see that the point $y' = [z', y]$ is close to $y$. If it is close enough to $y$, which can be achieved by letting $z'$ be appropriately close to $z$, then it must be in $\operatorname{int} T_m$. But then, as can be seen in Figure 52, $z' = [y', z]$ must be in $T_m$, which is impossible. So we see that any point

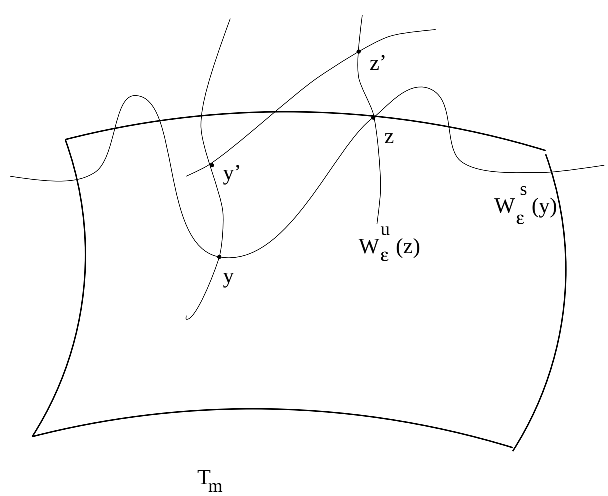



FIGURE 52. If $y \in \operatorname{int} T_m$, then $W^s_\epsilon(y) \cap T_m = \emptyset$.

in the interior of $T_m$ cannot have stable or unstable sets intersecting the stable or unstable boundaries of $T_m$, respectively.

Given $x \in T_k$, find $x_1 \in \operatorname{int} T_k$ with $x_1 \approx x$. (This approximation can be chosen to be arbitrarily accurate). Fixing $T_m$ within a distance $4\beta$ of $T_k$, we will move $x_1$ slightly if necessary to find $x_2 \in G^s_{m,k}$ with $x_2 \approx x$ (again with arbitrary accuracy). If $W^s_\epsilon(x_1) \cap T_m = \emptyset$, then $x_1 \in G^s_{m,k}$ and we do not need to adjust $x_1$ at all. Otherwise, we will find an $x_2 \approx x_1$

which does have this property, as follows. Suppose $z \in W^s_\epsilon(x_1) \cap \partial^s T_m$. Find $y \approx z$ with $y \in \mathrm{int} T_m$, as seen in Figure 53. By the preceding paragraph, $W^s_\epsilon(y) \cap \partial^s T_m = \emptyset$. Let $x_2 = [y, x_1]$, so $x_2 \approx x_1$ and $x_2 \in G^s_{m,k}$ (if $y$ is close enough to $z$).

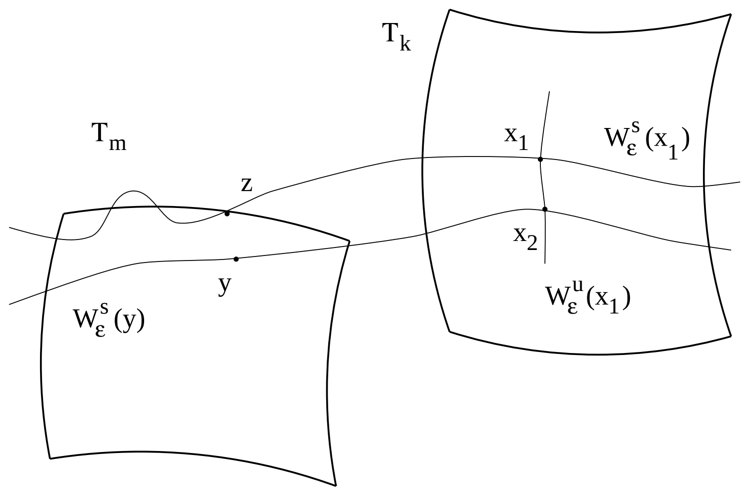



FIGURE 53. If $W^s_\epsilon(x_1) \cap T_m \neq \emptyset$, there is a point $x_2$ arbitrarily close to $x_1$ with $W^s_\epsilon(x_1) \cap T_m = \emptyset$.

This shows that any point $x \in T_k$ can be approximated by elements of $G^s_{m,k}$, making this set dense in $T_k$. Using an analogous argument to show that $G^u_{m,k}$ is dense in $T_k$, it is clear that the set $G^s_{m,k} \cap G^u_{m,k}$ is dense in $T_k$. In fact, $\bigcap_m G^s_{m,k} \cap G^u_{m,k}$ is a dense set. Since $\Omega_j = \bigcup_k T_k$, we see that the set

$$G = \bigcup_k \bigcap_m G^s_{m,k} \cap G^u_{m,k}$$

is dense in $\Omega_j$.

## 29. Proof continued. March 24 (*Notes by MA*)

This week we continue the proof of the existence of Markov partitions for the nonwandering set $\Lambda$ of an Axiom A diffeomorphism $(M, \phi)$. Let us first summarize what has been done so far.

Using the Stable Manifold Theorem (Theorem 7.2) followed by the Pseudo-Orbit Shadowing Theorem (Theorem 18.1) we obtained an SFT $\Sigma(P)$ which maps onto $\Omega_c$, a basic set, through a continuous factor map $\theta$. In $\Sigma(P)$ we have cylinder sets $[p_k]$, which are rectangles, and map to rectangles $T_k$ in $\Omega_c$. These cover $\Omega_c$ and satisfy the Markov Mapping Property. However this family of rectangles need not be proper nor need they be essentially disjoint. The purpose of this week's notes is to construct a Markov partition by properly cutting up the rectangles $T_k$ into subrectangles.

29.1. **A candidate for a Markov partition.** Suppose two rectangles $T_m$ and $T_k$ intersect. We cut $T_m$ into smaller rectangles $T^b_{m,k}$ as follows:

$$\begin{aligned} T^1_{m,k} &= \{x \in T_m : W^u(x, T_m) \cap T_k = \emptyset, W^s(x, T_m) \cap T_k = \emptyset\}, \\ T^2_{m,k} &= \{x \in T_m : W^u(x, T_m) \cap T_k = \emptyset, W^s(x, T_m) \cap T_k \neq \emptyset\}, \\ T^3_{m,k} &= \{x \in T_m : W^u(x, T_m) \cap T_k \neq \emptyset, W^s(x, T_m) \cap T_k = \emptyset\}, \\ T^4_{m,k} &= \{x \in T_m : W^u(x, T_m) \cap T_k \neq \emptyset, W^s(x, T_m) \cap T_k \neq \emptyset\}. \end{aligned}$$

In this manner we obtain at most 4 subrectangles, for some may be empty. Successively cutting in this manner we obtain a new set of rectangles whose only intersections are along boundaries. We can ignore these boundaries by restricting ourselves to a 'good set' $G$ defined as follows:

$$G = \{x \in \Omega_c : W^s_\epsilon(x) \cap \partial^s T_k = \emptyset, W^u_\epsilon(x) \cap \partial^u T_k = \emptyset \\ \text{for all } k \text{ such that } T_k \text{ hits some } T_n \text{ to which } x \text{ belongs}\}.$$

Last week we showed that $G$ was open and dense in $\Omega_c$. We now establish some other properties of $G$.

First we show that $G \subset \cup \left(T^b_{m,k}\right)^\circ$. If $x \in G$, then $x \in T_m$ for some $m$ but $x \notin \partial T_m$ so $x \in T^\circ_m$. By the definition of $G$, $W^s_\epsilon(x) \cap \partial^s T_k = \emptyset$ and $W^u_\epsilon(x) \cap \partial^u T_k = \emptyset$ for all $k$ such that $T_k$ hits $T_m$. Now if $W^u_\epsilon(x, T_m)$ hits $T_k$ then it must hit $T^\circ_k$. Similarly if it misses then it must hit $(T^c_k)^\circ$. This puts $x$ in the interior of some $T^b_{m,k}$.

Next we show that each $G \cap \left(T^b_{m,k}\right)^\circ$ is an open rectangle. Clearly each is open. Suppose $x, y \in G \cap \left(T^b_{m,k}\right)^\circ$. Then $[x, y] \in T^b_{m,k}$ since $T^b_{m,k}$ is a rectangle. So $[x, y] \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$ which misses boundaries, so $[x, y] \in \left(T^b_{m,k}\right)^\circ$. We must still show that $[x, y] \in G$. Suppose otherwise. Then we can assume, without loss of generality, that $W^s_\epsilon([x, y]) \cap \partial^s T_n \neq \emptyset$ for some $T_n$ which hits $T_m$. But $W^s_\epsilon([x, y]) = W^s_\epsilon(x)$ so $W^s_\epsilon(x) \cap \partial^s T_n \neq \emptyset$ contradicting at $x \in G$.

Finally, since the sets $\left\{G \cap \left(T^b_{m,k}\right)^\circ\right\}$ partition $G$, we will use these sets as our Markov partition. Relabel these open rectangles as $R_1, ..., R_m$. They are pairwise disjoint by definition, and from the beginning of the construction (using cylinder sets) we see that these rectangles can be constructed with arbitrarily small diameter. Since $G$ is open and dense in $\Omega_c$, we see that the closures $\overline{R_i}$ will cover $\Omega_c$. We claim that these closures form a Markov partition for $\Omega_c$.

To show that these closures form a Markov partition for $\Omega_b$, we need only show that the rectangles are proper, essentially disjoint, and have the Markov Mapping Property (MMP).

Since $R_i$ is open and $\overline{R_i}^\circ$ is the largest open set lying in $\overline{R_i}$, $R_i \subset \overline{R_i}^\circ$. From this is follows that $\overline{R_i} \subset \overline{\overline{R_i}^\circ}$ so

$$\overline{R_i} \subset \overline{\overline{R_i}^\circ} \subset \overline{\overline{R_i}} = \overline{R_i}$$

and hence $\overline{\overline{R_i}^\circ} = \overline{R_i}$. Therefore the rectangles are proper.

To see that the rectangles $\overline{R_i}$ are essentially disjoint, note that for $i \neq j$, $\overline{R_i}^\circ \subset \overline{R_i} \subset R_j^c$ since $R_i \subset R_j^c$ and $R_j^c$ is closed. Similarly $\overline{R_j}^\circ \subset \overline{R_j} \subset \left(\overline{R_i}^\circ\right)^c$ since by the first step, $R_j \subset \left(\overline{R_i}^\circ\right)^c$ and $\left(\overline{R_i}^\circ\right)^c$ is closed . So $\overline{R_i}^\circ \cap \overline{R_j}^\circ = \emptyset$.

It remains to be shown that the rectangles $\overline{R_i}$ satisfy the Markov Mapping property. Since the boundaries may cause problems in the proof, we show that if MMP holds for $x \in G$ then MMP holds for all $x \in \Omega_c$.

Suppose MMP holds on $G$. Let $x \in \overline{R_i}^\circ$ and $\phi x \in \overline{R_j}^\circ$. Take $x' \in \overline{R_i}^\circ \cap \phi^{-1}\left(\overline{R_j}^\circ\right) \cap G$ very close enough to $x$. We must show that $\phi W^u\left(x, \overline{R_i}\right) \supset W^u\left(\phi x, \overline{R_j}\right)$ and $\phi W^s\left(x, \overline{R_i}\right) \subset W^s\left(\phi x, \overline{R_j}\right)$. We will prove the inclusion for the stable part since the unstable part will follow similarly.

Note that

$$W^s\left(x, \overline{R_i}\right) = \left\{[x, y] : y \in W^s\left(x', \overline{R_i}\right)\right\}$$

(this does not depend on $x'$ being in the good set $G$) since if $z \in W^s\left(x, \overline{R_i}\right)$ then $z = [x, [x', z]]$. The reverse inclusion is clear.

Since MMP holds on $G$ (i. e. $\phi W^s\left(x', \overline{R_i}\right) \subset W^s\left(\phi x', \overline{R_j}\right)$) we have

$$\begin{aligned} \phi\left\{[x, y] : y \in W^s\left(x', \overline{R_i}\right)\right\} &= \left\{[\phi x, \phi y] : y \in W^s\left(x', \overline{R_i}\right)\right\} \\ &\subset \left\{[\phi x, z] : z \in W^s\left(\phi x', \overline{R_j}\right)\right\}. \end{aligned}$$

Thus

$$\phi W^s\left(x, \overline{R_i}\right) \subset \left\{[\phi x, z] : z \in W^s\left(\phi x', \overline{R_j}\right)\right\}.$$

Finally, from an earlier argument, we have that

$$\left\{[\phi x, z] : z \in W^s\left(\phi x', \overline{R_j}\right)\right\} = W^s\left(\phi x, \overline{R_j}\right)$$

and so we have the inclusion (for the stable part).

To complete the proof of existence we now need only show that MMP holds on the good set $G$.

Suppose $x \in \overline{R_i}^\circ \cap \phi^{-1}\left(\overline{R_j}^\circ\right) \subset G$. We need to show that $\phi W^s\left(x, \overline{R_i}\right) \subset W^s\left(\phi x, \overline{R_j}\right)$. In order to do this, we will show that $G \cap \phi^{-1}G$ is dense in $W^s\left(x, \overline{R_i}\right)$, and so we will then only need to show that $\phi\left(W^s\left(x, \overline{R_i}\right) \cap G \cap \phi^{-1}G\right) \subset W^s\left(\phi x, \overline{R_j}\right)$.

Note that $G \cap \phi^{-1}G$ is dense in $\Omega_c$ (both $G$ and $\phi^{-1}G$ are residual). Suppose $t \in W^s\left(x, \overline{R_i}\right)$. Take $z \in G \cap \phi^{-1}G \cap \overline{R_i}$ very close to $t$. Now $z$ may not be in $W^s\left(x, \overline{R_i}\right)$, but $w = [x, z]$ is. By continuity $w$ is close to $t$. Here we need to remember that the initial rectangles $R_i$ (without closure) were rectangles in $G$ so that $w = [x, z] \in R_i \subset G$. Further, if $\phi x, \phi z \in G$, then $\phi w = [\phi x, \phi z] \in G$ by the definition of $G$. (Note: $x \in R_i \cap \phi^{-1}R_j$, so $\phi x \in R_j \subset G$.) Therefore, $G \cap \phi^{-1}G$ is dense in $W^s\left(x, \overline{R_i}\right)$.

To show that $\phi\left(W^s\left(x, \overline{R_i}\right) \cap G \cap \phi^{-1}G\right) \subset W^s\left(\phi x, \overline{R_j}\right)$, note that $\phi W^s_\epsilon(x) \subset W^s_\epsilon(\phi x)$, so we only need to show $\phi\left(W^s\left(x, \overline{R_i}\right) \cap G \cap \phi^{-1}G\right) \subset \overline{R_j}$. Take $y \in W^s\left(x, \overline{R_i}\right) \cap G \cap \phi^{-1}G$. We must show that $\phi x$ and $\phi y$ lie in the same rectangles $T^b_{m,k}$, for then $\phi y \in \overline{R_j}$.

If $x \in T_k \cap \phi^{-1}T_m$, then $y \in W^s_\epsilon(x) \cap \overline{R_i} \subset W^s_\epsilon(x) \cap T_k = W^s(x, T_k)$. But since the rectangles $T_k$ have MMP we have $\phi W^s(x, T_k) \subset W^s(\phi x, T_m)$ and so $\phi y \in W^s(\phi x, T_m) \subset$

$T_m$. Similarly, if $y \in T_k \cap \phi^{-1} T_m$ then $\phi x \in T_m$. Thus $\phi x$ and $\phi y$ lie in the same rectangles $T_r$.

We now need to show that $\phi x$ and $\phi y$ lie in the same rectangles $T^b_{m,k}$.

Suppose $\phi x, \phi y \in T_m$ and $T_m \cap T_k \neq \emptyset$. We have already seen that $W^s(\phi x, T_m) = W^s(\phi y, T_m)$ so we only need check

$$W^u(\phi x, T_m) \cap T_k \neq \emptyset \text{ if and only if } W^u(\phi y, T_m) \cap T_k \neq \emptyset.$$

Assume $\phi z \in W^u(\phi x, T_m) \cap T_k$ for some $z$. Then by MMP, whenever $x \in T_i$ (and hence also $y \in T_i$), $\phi z \in \phi W^u(x, T_i)$ and so $z \in W^u(x, T_i) \subset T_i$. Let $z'' = [z, y]$. Then $\phi z'' = [\phi z, \phi y] \in W^s_\epsilon(\phi z) \cap W^u_\epsilon(\phi y)$ and $\phi z'' \in T_m$ since $T_m$ is a rectangle. Once we get $\phi z'' \in T_k$ we will have the desired result $W^u(\phi y, T_m) \cap T_k = W^u_\epsilon(\phi y) \cap T_m \cap T_k \neq \emptyset$.

Since $x$ and $y$ lie in the same $\overline{R_i}^{\circ}$ we have that $x$ and $y$ lie in the same rectangle $T_i$ as above (different index $i$) so $z'' = [z, y] \in W^s_\epsilon(z) \cap T_i = W^s(z, T_i)$. This gives $\phi z'' \in \phi W^s(z, T_i) \subset W^s(\phi z, T_k)$ by the MMP for the $T_i$, since $\phi z \in T_k$; therefore $\phi z'' \in T_k$.

## 30. Near the end of the proof. March 31 (*Notes by PS*)

Final Details in Markov Proof

To finish the proof on the existence of Markov partitions, we need to show that $\phi z'' \in T_m$. What we are in the process of showing is:

$$x \in \text{int}\overline{R_i} \cap \phi^{-1}(\text{int}\,\overline{R_i}) \cap G \Rightarrow \phi(W^s(x, \overline{R_i})) \subset W^s(\phi(x), \overline{R_j})$$

We took

$$y \in W^s(x, \overline{R_i}) \cap G \cap \phi^{-1}(G)$$

and we are showing that

$$\phi(y) \in W^s(\phi(x), \overline{R_j}).$$

In fact, such $y$ are dense in $W^s(x, \overline{R_i})$. We showed that $\phi(x)$ and $\phi(y)$ are always in the same $T_m$. We are trying to show that $\phi(x)$ and $\phi(y)$ are always in the same $T^b_{m,k}$. Recall that the interiors of $T^b_{m,k}$ partition $G$ into $\text{int}(\overline{R_\nu})$.

We assume that

$$W^u(\phi(x), T_k) \cap T_m \neq \emptyset$$

and try to show that

$$W^u(\phi(y), T_k) \cap T_m \neq \emptyset,$$

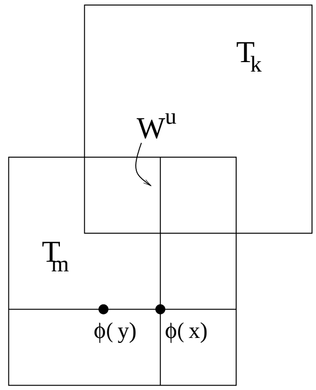
because $T^b_{m,k}$ is defined by these intersection.

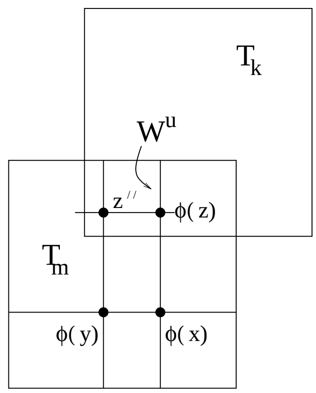
We took $z$ such that

$$\phi(z) \in W^u(\phi(x), T_k) \cap T_m \subset T_k \cap T_m$$

for $k \neq m$. Put $z'' = [z, y]$. Then

$$\phi z'' = [\phi(z), \phi(y)] \in W^s_\epsilon(\phi(z)) \cap W^s_\epsilon(\phi(y))$$




Thus, $\phi(z'') \in T_k$ because $\phi(z), \phi(y) \in T_k$.

Now, $\phi(z'') \in W^u(\phi(z), T_k)$. We need only find some $\phi(z'') \in T_m$. Note that

$$\phi(z) \in W^u(\phi(x), T_k) \subset \phi(W^u(x, T_i))$$

for some $i$ by the Markov Mapping property. Pulling back through $\phi^{-1}$, $z \in T_i$ and $z \in T_t$ for some $t \neq i$, where the inverse images of $T_i$ and $T_t$ meet $T_k$ and $T_m$ respectively.

The pseudo-orbits would appear as follows:

$$\cdots p_i p_k \cdots$$

and

$$\cdots p_t p_m \cdots$$

Now,

$$z \in W^u(x, T_i) \cap T_t.$$

Therefore

$$W^u(y, T_i) \neq \emptyset$$

because $x, y \in T^b_{i,t}$. Take

$$z' \in W^u(y, T_i) \cap T_t.$$

Then

$$z'' = [z, y] = [z, z']$$

since $W^u(y) = W^u(z')$. But $[z, z'] \in W^s(z, T_t)$, since both $z, z' \in T_t$, which is a rectangle.

Therefore:

$$\phi(z'') \in \phi(W^s(z, T_t)) \subset W^s(\phi(z), T_m) \subset T_m$$

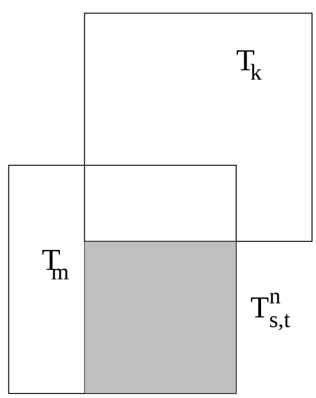
where $\phi(W^s(z, T_t)) \subset W^s(\phi(z), T_m)$ follows from the Markov mapping property.

It is conceivable that we might have a problem with our subshift. In particular, can we ensure $t \neq i$ ? If not, we simply extend our original alphabet of $p_i$ so that there are enough points making this possible.

We may thus assume that $P$ contains enough points so that whenever $\omega \in \Sigma(p)$, then there is a $\tilde{\omega} \in \Sigma(p)$, with $\tilde{\omega}(0) \neq \omega(0)$ but $\theta(\omega) = \theta(\tilde{\omega})$. We can ensure this by "splitting" each point of $P$ into two very nearby points.

## 31. End of the proof. April 2 (*Notes by PS*)

We have another lingering question in the Markov Proof: How can we know $\phi(z'') \in T_m$? According to Bowen (p. 82), you might have $q_0{}' = p_t = p_s$, which causes problems with the statement "Now $z \in W^u(x, T_s) \cap T_t$ so that $x, y$ are in the same $T^n_{s,t}$", since $T^n_{s,t}$ is only defined for $s \neq t$. If $s = t$, $T^n_{s,t} = T_s = T_t$.




We know that $\phi(z) \in W^u(\phi(x), T_k) \cap T_m$ and $\phi(x), \phi(y) \in$ the same $T_k$. We showed that $z \in W^u(x, T_i) \subset T_i$. Setting $z'' = [z, y]$ gives

$$\phi(z'') = [\phi(x), \phi(y)] \in T_k \cap W^s_\epsilon(\phi(z)) \cap W^u_\epsilon(\phi(y)) = W^s(\phi(z), T_k) \cap W^u_\epsilon(\phi(y)) \subset T_k$$

Now, $x \in T_i \Rightarrow y \in T_i$. Therefore,

$$z'' = [z, y] \in W^s_\epsilon(z) \cap T_i = W^s(z, T_i).$$

So,

$$\phi(z'') \in \phi(W^s(z,T_i)) \subset W^s(\phi(z), \text{any } T \text{ for which } \phi(z) \in T)$$

(the containment holds by the Markov property—for example, $\phi(W^s(z,T_i) \subset W^s(\phi(z),T_m))$.

**Theorem 31.1.** *Given $R = \{R_1,\ldots,R_r\}$ is a Markov partition for $\Omega_b$ into proper closed essentially disjoint rectangles and*

$$A_{i,j} = \begin{cases} 1 & \text{if } \operatorname{int} R_i \cap \phi^{-1}(\operatorname{int} R_j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

*If $\omega \in \Sigma_A \subset \{1,\ldots,r\}^Z$, we can define a factor map $\pi : (\Sigma_A,\sigma) \to (\Omega_b,\phi)$ by*

$$\pi(\omega) = \bigcap_{j\in\mathbb{Z}} \phi^{-j}\big(R_{\omega_j}\big)$$

*which is one-to-one over the residual set:*

$$Y = \Omega \setminus \bigcap\nolimits_{j\in\mathbb{Z}} \phi^{-j}\big(\cup_{i=1}^{r}(\partial R_i)\big)$$

Thus, the dynamical systems are "almost" the same topologically. There are only countable many such systems, in a sense: finitely many $R_i$ and finitely many choices for the matrix $A$, so there are countably many over all.

**Lemma 31.2.** *Let $\epsilon$ be small enough so that canonical coordinates are well-defined. Let*

$$C \subset W^u_\epsilon(x) \cap \Omega_b$$

*and*

$$D \subset W^s_\epsilon(x) \cap \Omega_b.$$

*Then $[C,D]$ is a proper rectangle if and only if $\overline{\operatorname{int} C} = C$ and $\overline{\operatorname{int} D} = D$ (with respect to the relative topologies of $W^u_\epsilon(x) \cap \Omega_b$ and $W^s_\epsilon(x) \cap \Omega_b$ )*

*Proof.* First note that $[C,D]$ is closed and a rectangle. To see that it is closed, if $[c_n,d_n] \in [C,D]$, then by taking subsequences, assume $c_n \to c \in C$ and $d_n \to d \in D$. By the continuity of $[,]$ we have $[c_n,d_n] \to [c,d] \in [C,D]$.

To see that it is a rectangle, if $[c_1,d_1]$ and $[c_2,d_2]$ are in $[C,D]$, then $[c_1,d_1] \in W^s(c_1)$ and $[c_2,d_2] \in W^u(d_2)$ so that $[[c_1,d_1],[c_2,d_2]] = [c_1,d_2] \in [C,D]$.

**Step 1**: Assume $\overline{\operatorname{int} C} = C$ and $\overline{\operatorname{int} D} = D$. Let $[u,v] \in [C,D]$ and try to find $[c,d] \in \operatorname{int}[C,D]$ with $[c,d]$ near $[u,v]$. If we can do this, then $[C,D] \subset \overline{\operatorname{int}[C,D]}$ and is thus a proper rectangle.

Now, pick $c$ near $u$ and $d$ near $v$ with $c \in \operatorname{int} C$ and $d \in \operatorname{int} D$.

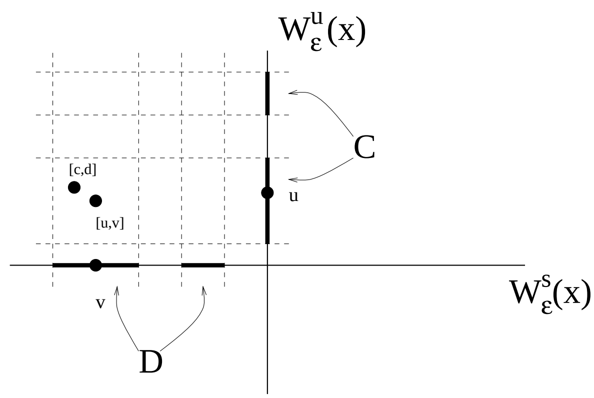
**Claim**: $[u,v] \approx [c,d]$ because of the continuity of $[,]$.

**Claim**: $[c,d] \in \operatorname{int}[C,D]$.

Recall the earlier remark that had as a consequence that if $R$ is a rectangle, then $\operatorname{int} R$ is a rectangle as well. Thus

$$\operatorname{int}_{\Omega_b} R = \bigcup\nolimits_{y\in R} (\operatorname{int}_{(W^s_\epsilon(y)\cap R)} W^s(y,\Omega) \cap \operatorname{int}_{(W^u_\epsilon(y)\cap R)} W^u(y,\Omega)$$

But $y = [c,d]$ is in the right-hand side of the above equation. For example, $[c,d] \in \operatorname{int}_{(W^s_\epsilon(y)\cap\Omega)} W^s(y,R)$ because $d \in \operatorname{int} D$.




If $y' \in W^s(y,R)$ is sufficiently near $y$, then $d' = [d,y'] \approx d$ will still be in $D$, since $d \in \text{int}D$. And then $y' = [c,d'] \in [C,D]$.

**Step 2**: Conversely, assume $[C,D]$ is proper. Given $c \in C, d \in D$, we want to find $u \in \text{int}C, v \in \text{int}D$ with $u \approx c, v \approx d$.

Find $y \in \text{int}[C,D]$ so that $y \in \text{int}W^s(y,[C,D]) \cap \text{int}W^u(y,[C,D])$, with $y$ near to $[c,d]$. Let $x = [c,d]$. Then

$$u = [y,x] \approx [[c,d],x] = c$$

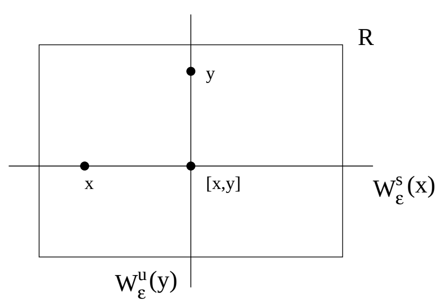
and

$$[x,y] \approx [x,[c,d]] = d.$$

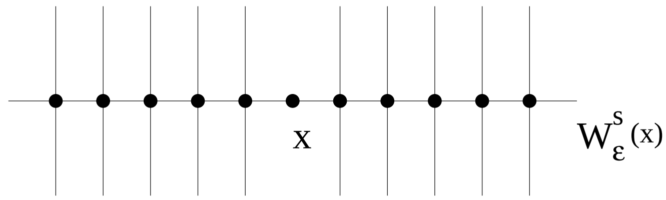
**Claim**: $[y,x] \in \text{int}C$. Let $y = [p,q]$ with $p \in C$ and $q \in D$, so that $[y,x] = [[p,q],x] = p \in C$. Moving $[p,q]$ slightly still leaves us in $[C,D]$, so moving $p$ slightly and still leaves us in $C$. □

## 32. More end of the proof. April 4 (*Notes by PS*)

**Question**: Are rectangles connected?



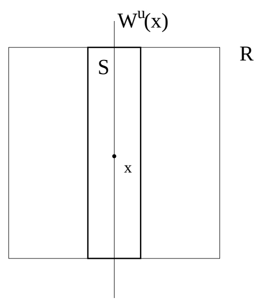

Take a rectangle $R \subset \Omega_b(x) \subset \Omega$. Note that $\Omega$ is often very ugly, and its relative topology can be very bad. Intersection with $\Omega$ can ruin connectivity. Also, rectangles typically have terrible boundaries. $W^s_\epsilon(x)$ can hit $R$ repeatedly. For example, for hyperbolic automorphisms of the 3-torus, boundaries of rectangles in the Markov partition contain no rectifiable arcs.




## 33. Back to coding

**Definition 33.1.** Let $S$ and $R$ be closed rectangles with $S \subset R$. Then $S$ is called a *u-subrectangle* of $R$ if

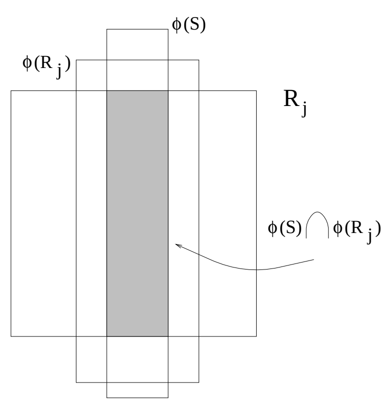
(1) $S \neq \emptyset$.
(2) $S$ is proper (i.e. $S = \overline{\mathrm{int}_\Omega S}$ )
(3) If $s \in S$, then $W^u(s, S) = W^u(s, R)$. (That is, $S$ extends all the way across $R$ in the unstable direction.)




The key to getting nonempty intersections is to map u-subrectangles into u-subrectangles using the Markov property.

Let $R = \{R_1, \ldots, R_n\}$ be a Markov partition of $\Omega_b$ of proper closed rectangles with $\mathrm{int}R_i \cap \mathrm{int}R_j \neq \emptyset$ for $i \neq j$.

**Lemma 33.1.** *If $S$ is a u-subrectangle of $R_i$ and $\mathrm{int}R_i \cap \phi^{-1}(\mathrm{int}R_j) \neq \emptyset$ (i.e. $A_{i,j} = 1$ ), then $\phi(S) \cap R_j$ is a u- subrectangle of $R_j$.*



This is the key to showing that for $\omega \in \Sigma_A$, then $\pi(\omega) = \cap_{j\in Z}\phi^{-j}\big(R_{\omega_j}\big) \neq \emptyset$. Note we have nested sets $\cap_{-n}^{n}\phi^{-j}\big(R_{\omega_j}\big) \supset \cap_{-n-1}^{n+1}\phi^{-j}\big(R_{\omega_j}\big) \supset \dots$

Bowen uses this to induct on $\cap_{j=1}^{n}\phi^{n-j}\big(R_{\omega_j}\big) = R_{\omega_n} \cap \phi\left(\prod_{j=1}^{n-1}\phi^{n-1-j}\left(R_{\omega_j}\right)\right)$ where $\prod_{j=1}^{n-1}\phi^{n-1-j}\big(R_{\omega_j}\big)$ plays the role of the u-subrectangle $S$. The above Lemma makes this induction work properly.

Note: so far we haven't used the fact that the rectangle is proper. The rectangles must be proper because the Markov mapping property is only known to hold for $x \in \mathrm{int}R_i \cap \phi^{-1}(\mathrm{int}R_j)$, so that we know that

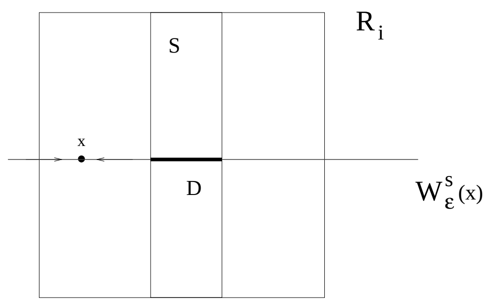
$\phi(W^s(x, R_i)) \subset W^s(\phi(x), R_j)$ and

$\phi(W^u(x, R_i)) \supset W^u(\phi(x), R_j)$

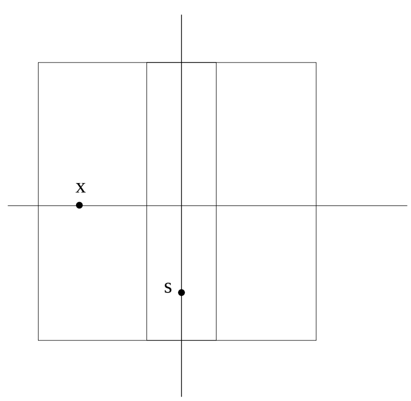
Caution: $\partial R_i$ can (a) be horrible, (b) can intersect $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ in an ugly manner and (c) map under $\phi$ in an awful way.

*Proof of Lemma 33.1.* Let
$$x \in \mathrm{int}R_i \cap \phi^{-1}(\mathrm{int}R_j).$$
Let $D = W^s(x, R_i) \cap S$.

a) First we show that $D$ is relatively proper (i.e. $D = \overline{\mathrm{int}_{W^s_\epsilon(x)\cap\Omega} D}$ and $D \neq \emptyset$.




Note $S = \bigcup_{y\in D} W^u(y, R_i)$ implies $D \neq \emptyset$.




To see $D \supset \overline{\mathrm{int}_{W^s_\epsilon(x)\cap\Omega} D}$:
$$y \in D \subset S$$

implies

$$W^u(y,R_i) = W^u(y,R_j) \subset S.$$

To see $D \subset \overline{\mathrm{int}_{W^s_\epsilon(x)\cap\Omega} D}$:

If $s \in S$, then $[x,s] \in D$ because it is in each of $W^s_\epsilon(x), R_i$ and $W^u_\epsilon(s)$ and thus in $W^u(s,R_i) = W^u(s,S) \subset S$.

Furthermore, $S = [W^u(x,R_i), D]$:

(1) $S \supset [W^u(x,R_i), D]$ is clear.

(2) If $s \in S$, write $s = [[s,x],[x,s]]$, where $[s,x] \in W^u(x,R_i)$ and $[x,s] \in D$.

Since $S$ is proper, $D$ is relatively proper via the preceding lemma ($[C,D]$ is proper if and only if $C$ and $D$ are relatively proper.

b) Next show that $\phi S \cap R_j \neq \emptyset$:

We claim that

$$\phi S \cap R_j = \bigcup_{y\in D} \phi(W^u(y,R_i)) \cap R_j = \bigcup_{y\in D} W^u(\phi(y),R_j).$$

To check this: if $y \in D \subset W^s(x,R_i)$ then (since a$x \in \mathrm{int}R_i$)

$$\phi(y) \in \phi(W^s(x,R_i)) \subset W^s(\phi(x),R_i) \subset R_j$$

by the Markov Mapping property.

If

$$y \in \mathrm{int}_{W^s_\epsilon(x)\cap\Omega} D$$

then

$$\phi(W^u(y,R_i)) \supset W^u(\phi(y),R_j)$$

Thus

$$\phi(W^u(y,R_i)) \cap R_j \supset W^u(\phi(y),R_j)$$

Take $y \in \mathrm{int}R_i$ and $\phi(y) \in \mathrm{int}R_j$. Since such $y$'s are dense in $D$, we get

$$\phi(S) \cap R_j = \bigcup_{y\in D} \phi(W^u(y,R_i)) \cap R_j = \bigcup_{y\in D} W^u(\phi y,R_j)$$

which is dense because $D$ is proper. Now $y \in \mathrm{int}D$ implies $y \in \mathrm{int}R_i$, so that

$$\phi(S) \cap R_j = \bigcup_{y'\in\phi(D)} W^u(y',R_j) \neq \emptyset$$

since $\phi(D) \neq \emptyset$.

This is proper because

$$\bigcup_{y'\in\phi(D)} W^u(y',R_j) = [W^u(\phi(x),R_j),\phi(D)]$$

as in part (a). Thus $\phi(S)\cap R_j$ is proper. We need only check a few more things to complete the Lemma. □

## 34. Obtaining symbolic dynamics. April 7 (*Notes by RJ*)

The lemma stated last time comes from Bowen's (3.14).

After the end of the proof of the Theorem on Existence of Markov partitions, add:

**Remark 34.1.** The proof shows that if $\mathrm{int}R_i \cap \phi^{-1}\mathrm{int}R_j \neq \emptyset$, then for all $x' \in R_i \cap \phi^{-1}R_j$,

$$\phi W^s(x', R_i) \subset W^s(\phi x', R_j)$$

and

$$\phi W^u(x', R_i) \supset W^u(\phi x', R_j).$$

(We used a point $x \in \mathrm{int}\, R_i \cap \phi^{-1}\,\mathrm{int}\, R_j \cap G$ to prove the tough claim. This did not need the $R_i$ to be proper.) As long as rectangle interiors overlap, our statements are okay; this justifies the stronger statements of the mapping property made by Mañé and Bowen. For example, if $x' \in R_i \cap \phi^{-1}R_j$, then

$$\begin{aligned} \phi W^s(x', \overline{R_i}) &= \{[\phi x', \phi y] : y \in W^s(x, \overline{R_i})\} \\ &\subset \{[\phi x', z] : z \in W^s(\phi x, \overline{R_j})\} \\ &= W^s(\phi x', \overline{R_j}). \end{aligned}$$

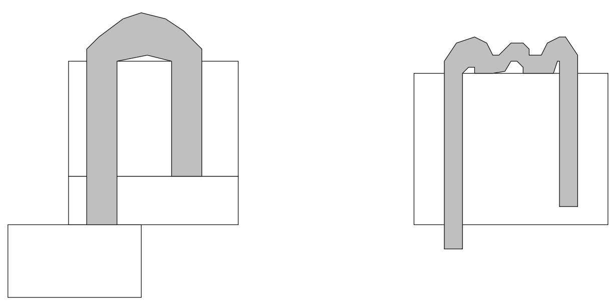


Figure 54. The argument shows that as long as we're in the interior, the whole thing maps correctly but boundary points might have problems.

**Lemma 34.1.** *If $S$ is a $u$-subrectangle of $R_i$ and $\mathrm{int}\, R_i \cap \phi^{-1}\,\mathrm{int}\, R_j \neq \emptyset$, then $\phi S \cap R_j$ is an $u$-subrectangle of $R_j$.*

*Proof.* Let $x \in R_i \cap \phi^{-1}R_j$.

(1) $S = \bigcup_{y \in D} W^u(y, R_i) = [W^u(x, R_i), D]$, where $D = W^s(x, R_i) \cap S$.

This is so because if $s \in S$, then $[x, s] \in D$, and $s = [[s, x], [x, s]]$. The reverse inclusion is easy. Thus $D$ is nonempty and relatively (in $W^s_\epsilon(x) \cap \Omega$) proper, since $S$ is.

(2) $\phi S \cap R_j = \bigcup_{y \in D} \phi W^u(y, R_i) \cap R_j = \bigcup_{y \in D} W^u(\phi y, R_j)$.

The reason for this is that if $y \in D \subset W^s(x, R_i)$, then

$$\phi y \in \phi W^s(x, R_i) \subset W^s(\phi x, R_j) \subset R_j,$$

by the preceding remark. Also, by applying the remark to $y$,

$$\phi W^u(y, R_i) \supset W^u(\phi y, R_j),$$

hence

$$\phi W^u(y, R_i) \cap R_j = W^u(\phi y, R_j).$$

So, as in (1), $\phi S \cap R_j = [W^u(\phi x, R_j), \phi D]$.

$W^u(\phi x, R_j)$ is relatively proper, by the preceding lemma, because $R_j$ is proper and

$$R_j = [W^u(\phi x, R_j), W^s(\phi x, R_j)].$$

(As in the previous calculation, $s = [[s, \phi x], [\phi x, s]]$.)

(3) $\phi : W^s_\epsilon(x) \cap \Omega \hookrightarrow W^s_\epsilon(\phi x, \Omega)$ with open image.

Let $y \in W^s_\epsilon(x) \cap \Omega$. If $\phi z$ is close enough to $\phi y$, then $\phi^k z$ is $\epsilon$-close to $\phi^k x$ for all $k \geq 0$. For $z$ is within $\epsilon$ of $x$ if the neighborhood of $\phi y$ in which $\phi z$ lies is small enough. And $\phi^k z$ is within $\epsilon$ of $\phi^k x$ for $k \geq 1$ because $z \in W^s_\epsilon(x)$ and so $\phi z \in W^s_\epsilon(\phi x)$. Therefore, $\phi D$ is relatively proper (in $W^s_\epsilon \cap \Omega$). And hence, $\phi S \cap R_j$ is proper.

We note further that $\phi S \cap R_j$ is nonempty because the representation in (2) shows that $\phi S \cap R_j \supset 0 \neq \emptyset$.

(4) Now we check the $u$-subrectangle condition: if $y \in \phi S \cap R_j$, then $W^u(y, R_j) = W^u(y, \phi S \cap R_j)$. It is clear that $W^u(y, R_j) \supset W^u(y, \phi S \cap R_j)$. It remains then to show $W^u(y, R_j) \subset \phi S \cap R_j$. If $y \in \phi S \cap R_j$, then (by (2)) $y \in W^u(y', R_j)$ for some $y' \in \phi D_j$, so

$$W^u(y, R_j) = W^u(y', R_j) \subset \phi S \cap R_j.$$

(This is okay since the diameter of $R_i$ is very small compared to $\epsilon$.)

□

The point of this machinery is to be able to prove the theorem on Symbolic Dynamics.

*Proof.* (For $\omega \in \Sigma_A$, $\pi\omega = \bigcap_{j \in \mathbb{Z}} \phi^{-j} R_{\omega_j}$ is a factor map $\Sigma_A \to \Omega_b$ which is one-to-one over the residual set $Y = \Omega_b \setminus \bigcup_{j \in \mathbb{Z}} \phi^{-j}(\bigcup_{i=1}^r \partial R_i)$.)

We show first that for each $n = 1, 2, \ldots$, $\bigcap_{j=-n}^n \phi^{-j} R_{\omega_j} \neq \emptyset$.

Let $a_1 \ldots a_n$ be an allowed word in $\Sigma_A$ (i.e., $A_{a_i a_{i+1}} = 1$ for all $i = 1, 2, \ldots n-1$). Note that for $n > 1$

$$\bigcap_{j=1}^{n} \phi^{n-j} R_{a_j} = \phi(\bigcap_{j=1}^{n-1} \phi^{n-1-j} R_{a_j}) \cap R_{a_n},$$

and $R_{a_1}$ is proper and nonempty. Using induction, assume that $S = \bigcap_{j=1}^{n-1} \phi^{n-1-j} R_{a_j}$ is a (proper and nonempty) $u$-subrectangle of $R_{a_{n-1}}$. What about $\phi S \cap R_{a_n}$? Since $\operatorname{int} R_{a_i} \cap \phi^{-1} \operatorname{int} R_{a_{i+1}} \neq \emptyset$ for all $i$ and $A_{a_{n-1} a_n} = 1$, then $\phi s \cap R_{a_n}$ is a (nonempty and proper) $u$-subrectangle of $R_{a_n}$. Thus $\bigcap_{j=1}^n \phi^{-j} R_{a_j} \neq \emptyset$. □

Note: Just to get this intersection nonempty, the "proper stuff" was not used.

## 35. Symbolic dynamics. Entropy. April 9 (*Notes by RJ*)

Let $R = \{R_1, R_2, \dots, R_r\}$ be a Markov partition of $\Omega_b$.

(1) We showed that if $a_1 a_2 \dots a_n$ is an allowed word in $\Sigma_A$ (i.e. $\operatorname{int} R_{a_j} \cap \phi^{-1} R_{a_{i+1}} \neq \emptyset$ for all $i$), then $\bigcap_{j=1}^n \phi^{-j} R_{a_j} \neq \emptyset$, by induction using the first lemma.

Consequently, if $\omega \in \Sigma_A$, then for all $n \geq 0$,

$$\bigcap_{j=-n}^{n} \phi^{n-j} R_{\omega_j} \neq \emptyset,$$

and hence

$$\bigcap_{j=-n}^{n} \phi^{-j} R_{\omega_j} = \phi^{-n} \bigcap_{j=1}^{2n+1} \phi^{(2n+1)-j} R_{\omega_{j-n-1}} \neq \emptyset.$$

**Remark 35.1.** This did not use the stuff in the lemma about "proper." But, we did get that not only is $\bigcap_{j=-n}^{n} \phi^{-j} R_{\omega_j} \neq \emptyset$, but also it's proper (by induction, using the lemma), hence its interior is nonempty, so for all $n$,

$$\bigcap_{j=-n}^{n} \operatorname{int} \phi^{-j} R_{\omega_j} \neq \emptyset.$$

Using compactness,

$$\pi\omega = \bigcap_{j=-\infty}^{\infty} \phi^{-j} R_{\omega_j} \neq \emptyset \text{ for all } \omega \in \Sigma_A.$$

(2) $\pi\omega$ is just one point. For if $x,\ y \in \pi\omega$, then $\phi^j x$, $\phi^j y$ are in the same $R_{\omega_j}$ for all $j \in \mathbb{Z}$. Hence $d(\phi^j x, \phi^j y) \leq$ expansive constant for all $j$, hence $x = y$. Thus $\pi : \Sigma_A \to \Omega_b$ is defined as a map.

(3) $\pi\sigma = \phi\pi$. This is obvious.

(4) $\pi$ is continuous, by the same proof that showed $\theta : \Sigma(p) \to \Omega_b$ is continuous. That proof only used the fact that if $\xi_j = \omega_j$ for $|j| \leq n$, then $x = \Pi\xi$ and $y = \Pi\omega$ have $d(\phi^j x, \phi^j y)$ small ($< 2\beta$) for $|j| \leq n$. (If $\pi$ weren't continuous, we could find sequences agreeing on arbitrarily long blocks whose images remained at least some distance apart. Take a limit point of this sequence, etc.)

(5) The set $Y = \Omega_b \setminus \bigcup_{-\infty}^{\infty} \phi^{-j}(\bigcup_{k=1}^{r} \partial R_k)$ is residual because the $R_k$'s are proper, hence have nowhere dense boundaries. We show that $\pi$ maps $\Sigma_A$ one-to-one onto $Y$. Each point $y \in Y$ has a uniquely determined itinerary $\omega \in \Sigma_A$: $\omega_j$ is the unique $k$ such that $\phi^j y \in \operatorname{int} R_k$. (The orbit $\mathcal{O}(y)$ never hits any $\partial R_k$, and $\operatorname{int} R_k \cap \operatorname{int} R_{k'} = \emptyset$ for $k \neq k'$.) So for any $y$, there is one and only one $\omega \in \Sigma_A$ such that $\pi\omega = y$.

(6) $\pi(\Sigma_A) = \Omega_b$, since $Y$ is dense in $\Omega_b$ and $\pi(\Sigma_A)$ is closed.

**Corollary 35.1.** *$\sigma : \Sigma_A \to \Sigma_A$ is topologically transitive. If $(\Omega_b, \phi)$ is topologically mixing, then so is $(\Sigma_A, \sigma)$.*

*Proof.* $(\Omega_b, \phi)$ is known to be topologically transitive and is topologically mixing if it has only one elementary part. This is the proof that requires the "proper" machinery. Take cylinder sets $[a_1, \dots, a_r] = C$; $[b_1, \dots, b_r] = D \subset \Sigma_A$. (Recall that a typical basic open set is $\{\omega \in \Sigma_A : \omega_i = a_i \text{ for } |i| \le r\}$.) Then $\pi C \supset \bigcap_{i=-r}^{r} \phi^{-r} \operatorname{int} R_{a_i}$, a nonempty open set $C_0$ in $\Omega_b$. (Similarly, for $\pi D \supset D_0$.) Say $\phi^n C_0 \cap D_0 \neq \emptyset$. (There exist such $n \neq 0$. If $(\Omega_b, \phi)$ is topologically mixing, then this happens for $|n| \ge N(C_0, D_0)$.) Then, since $\pi$ is onto,

$$\begin{aligned} \emptyset \neq \pi^{-1}(\phi^n C_0 \cap D_0) &= \sigma^n \pi^{-1} C_0 \cap \pi^{-1} D_0 \\ &\subset \sigma^n C \cap D. \end{aligned}$$

That proves it. □

**Remark 35.2.** We can also use this "proper machinery" to show that $\pi : \Sigma_A \to \Omega_b$ is at most $r^2$ to 1 (where $r$ is the size of the alphabet, the number of rectangles in the Markov partition).

35.1. **Entropy, pressure, equilibrium states, Gibbs states.** The first half of Bowen is a good reference for the upcoming material. One may also wish to see Petersen's book as a reference on entropy; and Walters is a good resource for pressure and stuff.

Let $X$ be a compact metric space, and let $\phi$ be a homeomorphism on $X$. Then $(X, \phi)$ is a topological dynamical system.

We consider these definitions of the topological entropy, $h_{\text{top}}(X, \phi) = h_{\text{top}}(\phi) = h(\phi) = h$.

(1) *Open-cover definition (Adler-Konheim-McAndrew)*

Let $\mathcal{U}$ be an open cover of $X$ and denote by $N(\mathcal{U})$ the minimum possible number of elements in any subcover of $\mathcal{U}$. Let $H(\mathcal{U}) = \log N(\mathcal{U})$.

Now,

$$h_{\text{top}}(\mathcal{U}, \phi) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U} \vee \phi^{-1}\mathcal{U} \vee \cdots \vee \phi^{-n+1}\mathcal{U})$$

(this exists by sub-additivity). The topological entropy is then

$$h_{\text{top}}(X, \phi) = \sup_{\mathcal{U}} h_{top}(\mathcal{U}, \phi).$$

This definition was motivated by the definition of entropy for measures (in turn drawing from Shannon's definitions of entropy and capacity in information theory). If $\mu$ is a $\phi$-invariant measure on $X$, then for a finite, measurable partition $\alpha = \{A_1, \dots, A_r\}$ of $X$, define:

$$H(\alpha) = -\sum_{i=1}^{r} \mu(A_i) \log \mu(A_i),$$

$$h_\mu(\alpha, \phi) = \lim_{n\to\infty} \frac{1}{n} H(\alpha \vee \phi^{-1}\alpha \vee \cdots \vee \phi^{-n+1}\alpha),$$

and

$$h_\mu(\phi) = \sup_{\alpha} h_\mu(\alpha, \phi).$$

Here $\alpha \vee \phi^{-1}\alpha \vee \cdots \vee \phi^{-n+1}\alpha$ represents the common refinement of the partitions involved.

## 36. Entropy, pressure, Gibbs measures. April 11 (*Notes by RJ*)

Again, we discuss a topological dynamical system $(X, \phi)$. We can consider a family of metrics on $(X, \phi)$ given by

$$D_n(x, y) = \sup_{0 \le k \le n-1} d(\phi^k x, \phi^k y).$$

36.1. **Bowen's definitions of topological entropy.** We introduce two other definitions for $h_{\text{top}}(X, \phi)$, involving the notions of $n, \epsilon$-separated and $n, \epsilon$-spanning sets.

**Definition 36.1.** For $\epsilon > 0$, $n = 1, 2, 3, \dots$, a set $A \subset X$ is called *$n, \epsilon$-separated* if for all $x$, $y \in A$, $x \neq y$, there exists an integer $k$ with $0 \le k \le n-1$ such that $d(\phi^k x, \phi^k y) \ge \epsilon$.

**Definition 36.2.** For $\epsilon > 0$, $n = 1, 2, 3, \dots$, a set $B \subset X$ is called *$n, \epsilon$-spanning* if every point of $X$ is within $\epsilon$ of $\phi^k x$, for some $x \in B$, $0 \le k \le n-1$.

Then we have

$$h_{\text{top}}(X, \phi) = \lim_{\epsilon \searrow 0} \uparrow \limsup_{n \to \infty} \frac{1}{n} (\log \sup\{\operatorname{card} A : \ A \text{ is } n, \epsilon\text{-separated}\})$$

and

$$h_{\text{top}}(X, \phi) = \lim_{\epsilon \searrow 0} \uparrow \limsup_{n \to \infty} \frac{1}{n} (\log \inf\{\operatorname{card} B : \ B \text{ is } n, \epsilon\text{-spanning}\}).$$

Note that $\sup\{\operatorname{card} A : \ A \text{ is } n, \epsilon\text{-separated}\}$ is equal to the maximum possible number of disjoint open $\epsilon/2$-balls with respect to $D_n$ (i.e., how many such balls can be *packed* into $X$) and that $\inf\{\operatorname{card} B : \ B \text{ is } n, \epsilon\text{-spanning}\}$ is the minimum possible number of $\epsilon$-balls with respect to $D_n$ that *cover* $X$.

*Variational Principle for entropy*

**Theorem 36.1.** (Dinaburg, Goodman, and Goodwyn):

$$h_{top}(X, \phi) = \sup_{\mu \in \mathcal{M}(X,\phi)} h_\mu(X, \phi).$$

Recall that $\mathcal{M}(X, \phi)$ is the set of $\phi$-invariant Borel probability measures on $X$. (See DGS, p.138; KEP, p.239.) $\mathcal{M}(X, \phi)$ is compact and metrizable in the weak$^\star$- topology.

Any measure that achieves this supremum is called *maximal*. In general, there may not be any such maximal measures, but in an *expansive* system, they always exist. (See DGS, p.139, Walters, p.224.) Also, more generally, maximal measures exist in any system for which $\mu \to h_\mu(\phi)$ is upper semi-continuous (*USC*). If the supremum is attained, it might not be unique; then we say we have a *phase transition*.

If there is only one maximal measure, then the system is called *intrinsically ergodic*.

A continuous function $V : X \to \mathbb{R}$ is called a *potential function*. Let $S_n V(x) = \sum_{k=0}^{n-1} V(\phi^k x)$ for $n \ge 1$. The *pressure* of $V$ and $\phi$ is defined to be

$$P(V, \phi) = \lim_{\epsilon \searrow 0} \uparrow \limsup_{n \to \infty} \frac{1}{n} (\log \sup\{\sum_{x \in A} e^{S_n V(x)} : A \text{ is } n, \epsilon\text{-separated}\}),$$

or equivalently,

$$P(V, \phi) = \lim_{\epsilon \searrow 0} \uparrow \limsup_{n \to \infty} \frac{1}{n} (\log \inf\{\sum_{x \in B} e^{S_n V(x)} : B \text{ is } n, \epsilon\text{-spanning}\}).$$

The relationship between these definitions and those for topological entropy can be seen by noting that card $A = \sum_{x \in A} 1 = \sum_{x \in A} e^0$. Thus $h_{\text{top}}(\phi) = P(0, \phi)$.

**Theorem 36.2.** (Ruelle, Walters)

*Variational Principle for pressure*

$$P(V, \phi) = \sup_{\mu \in \mathcal{M}(X,\phi)} \{h_\mu(\phi) + \int_X V d\mu\}.$$

(See Bowen, Walters.)

$\langle V \rangle$ represents the expected, or average, value of the energy. (There may be a negative thrown in from the physics.) In this way, $P(V, \phi) = \sup(-\text{“free energy”})$. So the pressure is related to minimizing the free energy.

The difference between entropy and average energy tells us how much “free energy” there is in a system. And, somehow, natural systems want to minimize this quantity.

Any measure that achieves this supremum is called an *equilibrium state*. Again, if $(X, \phi)$ is expansive (or $\mu \to h_\mu(\phi)$ is USC), then equilibrium states exist.

If there is more than one equilibrium state, then we say we have a “phase transition.” Nature gives us many good illustrations of phase transitions. For example, consider springtime in the Arctic. As the earth gets a little warmer, the solid ice begins melting into the water. Yet, simultaneously, it is still cold enough that some water continues to freeze. Similarly, consider a large chunk of magnetic material. At room temperature, the molecules in this material all point the same way and are magnetized. However, if you heat it up, the material will completely demagnetize. The atoms will act in a Bernoulli, completely chaotic state—pointing up, down, or in any random direction. Somewhere between solid ice and flowing water, between a powerful magnet and a chunk of dead rock—when a system just doesn't seem to quite know what to do—we have a phase transition.

36.2. **Gibbs Measures.** Gibbs measures were brought into ergodic theory by Sinai around 1967. Previous significant work had been done by Dobrushin and several others going back to Gibbs, finding physically significant measures as descriptions of equilibrium states of systems with interacting components.

Let $\Sigma_A$ be a (two-sided) shift of finite type (SFT) on $\{1, \ldots, r\}^{\mathbb{Z}}$. For a continuous function $V : \Sigma_A \to \mathcal{R}$, define the *n-range variation* of $V$ to be

$$\text{var}_n V = \sup\{|V(x) - V(y)| : x, y \in \Sigma_A, x_k = y_k \text{ for } |k| \leq n\}.$$

How fast does the dependence of the function on coordinates near the center decay? One condition for fairly rapid decay is the following. We say that $V$ has *summable variation* if $\sum_{n=0}^{\infty} \text{var}_n V < \infty$.

**Definition 36.3.** A measure $\mu$ on $\Sigma_A$ is called a *Gibbs measure* with potential function $V$ if there exist constants $c > 0$ and $P$ such that

$$c \leq \frac{\mu[x_0, \ldots, x_n]}{\exp\{-nP + S_n V(x)\}} \leq \frac{1}{c}$$

for all $x \in \Sigma_A$ and all $n \geq 0$.

Thus the idea of Gibbs measure is a generalization of that of Markov measure. A Markov measure $\mu$ with transition matrix $(M_{ij})$ is a Gibbs measure with potential function $V(x) = \log M_{x_0 x_1}$ and pressure $P = 0$. This is because

$$e^{S_n V(x)} \asymp M_{x_0 x_1} M_{x_1 x_2} \dots M_{x_{n-2} x_{n-1}} \asymp \mu[x_0 x_1 \dots x_{n-1}].$$

## 37. Motivations from physics. April 14 (*Notes by SS*)

**Definition 37.1.** Let $\Sigma_A$ = topologically mixing SFT $\subset \{1, \dots, r\}^{\mathbb{Z}}$. Then a (shift-invariant) probability measure $\mu$ on $\Sigma_A$ is called a *Gibbs measure* with potential function $V \in C(\Sigma_A)$ and pressure $P \in \mathbb{R}$ if there is a constant $c > 0$ such that

$$\frac{1}{c} \leq \frac{\mu[x_0 x_1 \cdots x_{n-1}]}{\exp\{-nP + S_n V(x)\}} \leq c$$

for all $x \in \Sigma_A$ and all $n \geq 1$, where $S_n V(x) = \sum_{k=0}^{n-1} V(\sigma^k x)$.

**Remarks 37.1.**

(1) A Markov measure on $\Sigma_A$, say 1-step with stochastic transition matrix $(M_{ij})$, is a Gibbs measure with potential function $V(x) = \log(M_{x_0 x_1})$ and pressure $P = 0$ :

$$\mu[x_0 \cdots x_{n-1}] = p_{x_0} M_{x_0 x_1} M_{x_1 x_2} \cdots M_{x_{n-2} x_{n-1}}$$

and

$$S_n V(x) = \textstyle\sum_{k=0}^{n-1} \log(M_{x_k x_{k+1}}) = \log(M_{x_0 x_1} \cdots M_{x_{n-1} x_n})$$

so that

$$\exp\{-nP + S_n V(x)\} = M_{x_0 x_1} \cdots M_{x_{n-1} x_n} \asymp \mu[x_0 \cdots x_{n-1}].$$

So, a Gibbs measure is a nice generalization of a Markov measure.

(2) For a Gibbs measure $\mu$,

$$\frac{\mu[x_0 x_1 \cdots x_{n-1}]}{\mu[y_0 y_1 \cdots y_{n-1}]} \asymp \exp[S_n V(x) - S_n V(y)].$$

If $(x, y) \in R_A$ = Gibbs equivalence relation (which means, there exists $K$ such that $x_k = y_k$ for all $k$ with $|k| \geq K$), then (if $V$ depends on just a few coordinates, or has its dependence on far-out coordinates fall off quickly, say $\sum_{n=0}^{\infty} \mathrm{var}_n V < \infty$), there exists the limit $\rho_\mu(x, y)$, called the Radon-Nikodym derivative of the equivalence relation $R_A$:

$$\rho_\mu(x, y) = \lim_{n \to \infty} \frac{\mu[x_{-n} \cdots x_n]}{\mu[y_{-n} \cdots y_n]} = \exp \underbrace{\sum_{k=-\infty}^{\infty} [V(\sigma^k x) - V(\sigma^k y)]}_{\text{a finite sum}}$$

where $[x_{-n} \cdots x_n]$ is a configuration = block or word.

*Idea from statistical mechanics* : Fix a finite region $F$ and suppose the coordinates of $x$ are fixed outside of $F$. Then, the relative probabilities of the different possible configurations within $F$ are supposed to be given in a Gibbsian way, determined by the energies of the configurations :

$$\frac{P(C_1)}{P(C_2)} \sim \frac{e^{E(C_1)}}{e^{E(C_2)}}.$$

See J. Gibbs, Y. Sinai, D. Ruelle, R. Dobrushin, or C. Preston's "Gibbs states on countable sets".

(3) Suppose that we have a physical system that can be in situations $1, \cdots, Q$.

For example, the phase space $X$ (e.g., $X = \mathbb{R}^{6N}$ = (position, momentum)-space for an ideal gas of $N$ particles, or constant-energy surface in $\mathbb{R}^{6N}$) is decomposed into many cells with associated (system) energies $E_1, E_2, \ldots, E_Q$ and probabilities $p_i = \mu(E_i), i = 1, \ldots, Q$. By the *state* of the system, we mean the probability measure $\mu$, which describes the probabilities of the observable (macro)configurations of the system.

How do we choose the $p_i$ so as to minimize the Helmholtz free energy :

$$\frac{1}{kT} \underbrace{\sum_{i=1}^{Q} p_i E_i}_{\langle E \rangle} - h_\mu$$

where $k$ = Boltzmann's constant, $T$ = temperature and $h_\mu = \sum_{i=1}^{Q} p_i \log p_i$ ?

There is reason to believe that nature prefers such states, i.e., choices of the $p_i$. We might call this the Physical Variational Principle.

If the total energy $\langle E \rangle$ is fixed, then this is the same as maximizing entropy $h_\mu$.

In general, our system might be in contact with a much larger one (say a heat source or sink), and so its expected energy might be allowed to change.

From a small calculation, the best $\mu$ has $p_i \sim e^{-\beta E_i}$ (the same $\beta = 1/kT$ for all $i$).

For fixed energy, maximizing entropy yields the Maxwell-Boltzmann velocity distribution, in which

$$p_i = \frac{e^{-E_i/kT}}{\sum_j e^{-E_j/kT}}.$$

The denominator $Z = \sum_j e^{-E_j/kT}$, a normalizing constant, is the partition function (or Zustandssumme). ( See Bowen, p.4, p.34 and Walters, p.217, p.227.)

## 38. Maximizing entropy or free energy. April 16 (*Notes by SS*)

**Example 38.1.** Let $X = \mathbb{R}^{6N}$ (3 coordinates of position, momentum for each particle) in an ideal gas of $N$ particles. Then cut (macro)phase (position-momentum) space $\mathbb{R}^6$ into cells with associated energies $E_1, E_2, \ldots, E_r$. Then a probability measure $\mu$ on $X$ determines $p_i = \mu(E_i)$ and a macro-configuration $\vec{n} = (n_1, n_2, \ldots, n_r)$ gives the numbers $n_i$ of particles in the various cells.

Maybe $E_i = \frac{1}{2}mV_i^2$, if there is only kinetic energy; or there may be potential energy (due to position), or interaction energies among the particles. If $E_i = \frac{1}{2}mV_i^2$, then $p_i \sim e^{-V_i^2}$, and the Physical Variational Principle yields the Maxwell-Boltzmann velocity distribution for a hard-sphere gas:

The best $p_i$'s are

$$\frac{e^{-E_i/kT}}{\sum_j e^{-E_j/kT}}.$$

Another approach seeks to maximize the relative probability $W$ of a state $\mu$. Note that here we are talking about the probability of occurrence of some probability measure that describes probabilities of observable configurations of the system. Associated with $\mu$ are the expected values of the occupation number $n_i$ of the different macro-configurations, with associated energies $E_i$. So we ask how to choose the $n_i$ so as to maximize $W$, or, equivalently, $\log W$. If $W$ is the relative probability of a state $\vec{n}$, then $W \sim$ the number of micro-configurations that produce the macro-configuration $\vec{n}$, so that, considering all micro-configurations to be equiprobable,

$$W = \frac{N_1!N_2!\cdots N_r!}{N!}.$$

In the fixed energy case, if $p_i = N_i/N$, then by Stirling's Formula maximizing $W$ is the same as maximizing the thermodynamic entropy $S = k \log W$, and this is the same as maximizing $h_\mu = -\sum p_i \log p_i$.

By the way, what is temperature? It's a subtle quantity: if $U = \sum n_i E_i$ is the internal energy of the system, then the temperature $T = \partial U/\partial S$.

## 39. Existence and uniqueness of equilibrium states. April 18 (*Notes by SS*)

**Exercise 5.** Recall that an attractor (in a topological dynamical system $(X, \phi)$) is a closed $\phi$-invariant set $\Lambda$ for which there is a neighborhood $U$ of $\Lambda$ such that $\phi U \subset U$ and $\cap_{n\geq 0}\phi^n U = \Lambda$.

(1) Does it change the definition to require the existence of such an open $U$?
(2) Does it change the definition to require that there exists such an open $U$ with $\phi\bar{U} \subset U$?

We call an attractor *tight* if for any $\eta > 0$, we can find such a $U$ ($\Lambda \subset$ open $V \subset U, \phi U \subset U, \cap_{n\geq 0}\phi^n U = \Lambda$) with $U \subset \{y \in X : d(y, \Lambda) < \eta\}$.

(3) Is every attractor tight?
(4) What if $\Lambda = \Omega_b =$ a basic set in an Axiom A system?

**Example 39.1.** Suppose that our system consists of particles of types $\{1, \cdots, r\}$ located at the points of the integer (1-dimensional) lattice in $\mathbb{R}$. A configuration of the system is a point $x \in \{1, \cdots, r\}^{\mathbb{Z}}$, i.e., at each point of $\mathbb{Z}$, we have a symbol from $\{1, \cdots, r\}$ :

$$\text{micro-configuration } x = \cdots x_{-2}x_{-1}x_0x_1x_2\cdots$$

Suppose that we have a function $U$ of one coordinate ($U(x) = U(x_0)$) which gives the (potential) energy due to particle $x_0$ being at 0, i.e.,

presence of symbol $x_0$ at 0 contributes energy $U(x) = U(x_0)$;

presence of symbol $x_k$ at $k$ contributes energy $U(\sigma^k x) = U(x_k)$.

Also suppose that there is an interaction energy between each pair of particles due to the presence of $x_k$ at $k$ and $x_n$ at $n$,

$$I(n-k; x_k, x_n).$$

Then cut $X = \Sigma_A$ into cells $[x_{-m} \cdots x_m]$ and estimate the energy involved with a cell :

$$E_m[x_{-m} \cdots x_m] = \sum_{k=-m}^{m} [U(\sigma^k x) + \sum_{-m \le k < n \le m} I(n-k; x_k, x_n)],$$

or taking account also the interactions of the $x_k, -m \le k \le m$ with symbols outside the range $[-m, m]$,

$$\tilde{E}_m[x_{-m} \cdots x_m] = \sum_{k=-m}^{m} \underbrace{[U(\sigma^k x) + \frac{1}{2} \sum_{j \ne k} I(k-j; x_j, x_k)]}_{-V(\sigma^k x)}.$$

According to the Physical Variational Principle, good probabilities to try to assign to the cylinder sets $[x_{-m} \cdots x_m] \subset \Sigma_A$ are

$$\mu_m[x_{-m} \cdots x_m] \sim \frac{e^{-\tilde{E}_m[x_{-m} \cdots x_m]/kT}}{\sum_{B=[y_{-m} \cdots y_m]} e^{-\tilde{E}_m(B)/kT}}.$$

Thinking about the definition of pressure shows that

$$Z = \sum_{B=[y_{-m} \cdots y_m]} e^{S_{2m+1} V(\sigma^{-m} y)} \sim e^{(2m+1)P},$$

therefore

$$\mu_m[x_{-m} \cdots x_m] \sim \frac{e^{S_{2m+1} V(\sigma^{-m} x)}}{e^{(2m+1)P(V,\phi)}},$$

and hence

$$\frac{\mu_m[x_{-m} \cdots x_m]}{\exp\{-(2m+1)P + S_{2m+1} V(\sigma^{-m} x)\}} \asymp 1.$$

We hope that these measures $\mu_m$ converge as $m \to \infty$ to a Gibbs measure on $\Sigma_A$ which is an equilibrium state for $V$ (sometimes they don't exist or are not unique).

**Theorem 39.1.** *(Sinai, Ruelle, Bowen) If $V : \Sigma_A \to \mathbb{R}$ has summable variation (e.g., is Hölder continuous), then there is a unique equilibrium state $\mu_V$ on $\Sigma_A$ with potential function $V$, and $\mu_V$ is also the unique Gibbs measure with potential function $V$.*

To prove this, we follow the idea of Walters, Ruelle's operator theorem and $g$-measures, Trans. A.M.S. 214 (1975), 375-387.

First, we can reduce to $\Sigma_A^+ =$ 1-sided SFT $\subset \{1, \cdots, r\}^{\mathbb{N}}$ with the same allowed blocks, because we have the following :

**Proposition 39.2.** *If $V : \Sigma_A \to \mathbb{R}$ has summable variation, then there are $V_1$ with summable variation and continuous $u : \Sigma_A \to \mathbb{R}$ such that $V = V_1 + u - u\sigma$ (i.e., $V$ and $V_1$ are cohomologous) and $V_1(x)$ only depends on $x_0 x_1 \cdots$, i.e.,*

$$x_k = y_k \text{ for all } k \ge 0 \text{ implies that } V_1(x) = V_1(y).$$

*(And then $V$ and $V_1$ have the same Gibbs states.)*

Let $g \in C(\Sigma_A^+)$ and $g > 0$ with $\sum_{y\in\sigma^{-1}x} g(y) = 1$ for all $x \in \Sigma_A^+$.

Then define $L = L_g : C(\Sigma_A^+) \to C(\Sigma_A^+)$ by

$$Lf(x) = \sum_{y\in\sigma^{-1}x} g(y)f(y) \text{ for} f \in C(\Sigma_A^+),$$

and let $L^* : C(\Sigma_A^+)^* \to C(\Sigma_A^+)^*$ be the dual operator on measures i.e.,

$$L^*\mu(f) = \int f d(L^*\mu) = \int Lf d\mu = \mu(Lf).$$

Then, a measure $\mu$ on $\Sigma_A^+$ is called a *g-measure* if $L^*\mu = \mu$.

**Theorem 39.3.** *(Keane, Inv. Math. 16 (1972), 309-324) Let $\Sigma_A^+$ be a 1-sided topologically mixing SFT and $g \in C(\Sigma_A^+)$ with $\sum_{n=1}^\infty \text{var}_n \log g < \infty$. Then for all $f \in C(\Sigma_A^+), L^n f$ converges uniformly to a constant $\mu(f)$. This defines a measure $\mu$ on $\Sigma_A^+$, which is the unique g-measure.*

**Theorem 39.4.** *(Ledrappier, ZW 30 (1974), 185-202) Let $g \in C(\Sigma_A^+)$ be such that $g > 0$ and $\sum_{y\in\sigma^{-1}x} g(y) = 1$ for all $x \in \Sigma_A^+$, and let $\mu \in C(\Sigma_A^+)^*$. Denote by $\mathcal{B}$ the Borel $\sigma$-algebra of $\Sigma_A^+$. Then the following are equivalent:*

(1) *$\mu$ is a g-measure on $\Sigma_A^+$,*
(2) *$\mu$ is $\sigma$-invariant and for all $f \in L^1(\mu)$,*

$$E_\mu(f \mid \sigma^{-1}\mathcal{B})(x) = \sum_{z\in\sigma^{-1}\sigma x} g(z)f(z) \text{ for } \mu\text{-a.e. } x,$$

(3) *$\mu$ is $\sigma$-invariant and an equilibrium state for $V = \log g$.*

Note that using $f = \chi_{\{x_0=a\}}$ in (2) the right-hand side is just $g(a, x_1, x_2, \cdots)$ and so $g$ is a sort of Radon-Nikodym derivative of $\mu$:

$$g(x) = \lim_{A\searrow\{x\}} \frac{\mu(A)}{\mu(\sigma A)} = \lim_{n\to\infty} \frac{\mu[x_0x_1\cdots x_n]}{\mu[x_1\cdots x_n]} \text{ for } \mu\text{-a.e. } x.$$

## 40. Ruelle's Operator Perron-Frobenius Theorem, g-measures. April 21, 1997 (*Notes by KJ*)

40.1. **From last time. . .** We have $\Sigma_A^+$, a one-sided mixing shift of finite type with transition matrix $A$.

A *g-function* is defined to be a function $g \in \mathcal{C}(\Sigma_A^+)$ such that $g > 0$ and $\Sigma_{y\in\sigma^{-1}x} g(y) = 1$. Define the operator $L = L_g : \mathcal{C}(\Sigma_A^+) \to \mathcal{C}(\Sigma_A^+)$ by

$$Lf(x) = \Sigma_{y\in\sigma^{-1}x} g(y)f(y).$$

The operator $L$ has an adjoint $L^* : \mathcal{C}(\Sigma_A^+)^* \to \mathcal{C}(\Sigma_A^+)^*$. We say $\mu$ is a *g-measure* if $L^*\mu = \mu$.

**Theorem 40.1** (Keane. Inv Math, 1972)**.** *Let $\Sigma_A^+$ and $g$ be as above and assume that*

$$\Sigma_{n=0}^\infty \text{var}_n(\log g) < \infty.$$

*Then for every $f \in \mathcal{C}(\Sigma_A^+)$, $L^n f$ converges uniformly to a constant $\mu(f)$ and $\mu$ is the unique g-measure.*

**Theorem 40.2** (Ledrappier. ZW, 1974)**.** *Let $\Sigma_A^+$ and $g$ be as above. Then the following are equivalent:*

(1) *$\mu$ is a $g$-measure on $\Sigma_A^+$.*
(2) *$\mu$ is $\sigma$-invariant and for every $f \in L^1(\mu)$ $E_\mu(f|\sigma^{-1}\mathcal{B})(x) = \Sigma_{z\in\sigma^{-1}\sigma x} g(z)f(z)$ almost everywhere $d\mu$.*
(3) *$\mu$ is $\sigma$-invariant and an equilibrium state for $V = \log g$.*

**Remark 40.1.** For a $g$-measure $\mu$, $P(\log g, \sigma) = h_\mu(\sigma) + \int \log g \; d\mu = 0$.

**Remark 40.2.** $L(f \circ \sigma) = f$.

40.2. **Infinite-dimensional extension of the Perron-Frobenius Theorem for nonnegative matrices.** Recall the Perron-Frobenius Theorem for nonnegative matrices (see Parry and Tuncel or Lind and Marcus). In all that follows, we will assume that the shift of finite type is aperiodic. The theorem says that there exists a unique largest positive eigenvalue $\lambda$ with positive left and right eigenvectors $r$ and $l$. The entries in the $n$'th power of the matrix are asymptotic to $\lambda^n$, and in fact are between two constant multiples of $\lambda^n$.

We can normalize that matrix by replacing the nonzero entries with $r_i/\lambda r_j$ in the $i,j$ entry. From this we get a stochastic matrix.

If you start out with a $0,1$ matrix for a shift of finite type, then the stochastic matrix gives you the measure of maximal entropy. If your function is of two coordinates, say $e^{V(x_1,x_2)}$, then the stochastic matrix gives the equilibrium measure for that $V$.

**Theorem 40.3** (Ruelle's operator Perron-Frobenius Theorem)**.** *Let $\Sigma_A^+$ be a topologically mixing shift of finite type. Let $V \in \mathcal{C}(\Sigma_A^+)$ with $\Sigma_{n=0}^\infty \operatorname{var}_n V < \infty$. Define $\mathcal{L} = \mathcal{L}_V : \mathcal{C}(\Sigma_A^+) \to \mathcal{C}(\Sigma_A^+)$ by*

$$\mathcal{L}_V f(x) = \sum_{y\in\sigma^{-1}x} e^{V(y)} f(y).$$

*(Note that this $\mathcal{L} = \mathcal{L}_g$ in the previous notation, with $g = e^{V(x)}$, but this is probably not a $g$-function, since it probably does not sum to one.) Then there are $\lambda > 0$, the "positive eigenvalue," $\rho \in \mathcal{C}(\Sigma_A^+)$ with $\rho > 0$, the "left eigenvector," and $\nu \in \mathcal{C}(\Sigma_A^+)^*$ a positive measure on $\Sigma_A^+$, the "right eigenvector," such that*

$$\mathcal{L}\rho = \lambda\rho, \mathcal{L}^*\nu = \lambda\nu, \frac{\mathcal{L}^n f}{\lambda^n} \to \nu(f)\rho$$

*for all $f \in \mathcal{C}(\Sigma_A^+)$ uniformly on $\Sigma_A^+$ as $n \to \infty$.*

*Ingredients of proof (see Walters, Trans. AMS 214 (1985), 375-387, for details):*

(1) Use the Schauder-Tychonoff Theorem (continuous self-maps of compact convex sets in locally convex spaces have fixed points) to get $\nu$ as fixed point of the operator $\nu \to \mathcal{L}^*\nu/(\mathcal{L}^*\nu)\mathbb{I}$ where $\mathbb{I}$ represents the constant function. In order for the Schauder-Tychonoff Theorem to apply, use the Borel measures as the convex set.

Take $\lambda = \mathcal{L}^*\nu(\mathbb{I})$. This method gives both the measure and the eigenvalue.

(2) Use a similar method to get $\rho$.
(3) The hard part is to get asymptoticity. For this, use the previous theorem about $g$-measures, taking

$$g = \frac{e^V \rho}{\lambda\rho(\sigma)}$$

(in analogy to the finite-dimensional plan.) Then use the unique $g$-measure $\mu_g(f) = \nu(f\rho)$ and its convergence properties from Keane and Ledrappier.

□

As a direct application, we obtain most of the following theorem. For the rest, see Bowen, pp. 21-27, and Walters, reference above.

**Theorem 40.4** (Existence of Gibbs measures and equilibrium states)**.** *Let $\Sigma_A^+$ be a one-sided topologically mixing SFT, $V \in \mathcal{C}(\Sigma_A^+)$ with $\Sigma_{n=1}^{\infty} \operatorname{var}_n V < \infty$. Then there is a unique equilibrium state $\mu_V$ for $V$. It is the unique Gibbs state for $V$ and is given by $\mu_V(f) = \nu(f\rho)$ with $\nu, \rho$ as given in the Ruelle Theorem. $\mu_V$ is the unique $g$-measure for $g = e^V \rho/\lambda\rho(\sigma)$. Also $(\Sigma_A^+, \sigma, \mu_V)$ is exact, hence strongly mixing, and its natural extension is Bernoulli (since the time-zero partition is weakly Bernoulli).*

**Corollary 40.5.** *Similarly on $(\Sigma_A, \sigma)$.*

This follows from a trick developed by Sinai to replace an arbitrary $V$ by one that only depends on positive coordinates:

**Lemma 40.6.** *If $V : \Sigma_A \to \mathbb{R}$ has summable variation, then there is $u \in \mathcal{C}(\Sigma_A)$ such that $W = V + u - u \circ \sigma$ has the property that if $x_k = y_k$ for all $k \geq 0$ then $W(x) = W(y)$. (Then $V$ and $W$ have the same Gibbs states, justifying our working on $\Sigma_A^+$)*

There remain three major theorems to present:

- **Theorem 1:** Hölder-continuous potential functions on basic sets of Axiom $A$ systems have unique equilibrium states.
- **Theorem 2:** In a $\mathcal{C}^2$ Axiom $A$ system, almost every point of the manifold $M$ is attracted in forward time to some basic set (a "tight attractor").
- **Theorem 3:** The asymptotic statistics of the orbit of Lebesgue-a.e. point of $M$ are described by a certain equilibrium measure (the SRB measure) on the attractor which it approaches.

Due to lack of time, we will have to skip many details of their proofs.

Ruelle proved the following theorem for Anosov systems, and Bowen proved it for Axiom A systems.

**Theorem 40.7** (Theorem 1—Ruelle, Bowen)**.** *Let $\Omega_b$ be a basic set in an Axiom A system $(M, \varphi)$. Let $V : \Omega_b \to \mathbb{R}$ be Hölder continuous ($|V(x) - V(y)| \leq c\, d(x,y)^\eta$ for some $c, \eta \geq 0$ for all $x, y \in \Omega_b$). Then there is a unique equilibrium state $\mu_V$ for $V$. $\mu_V$ is ergodic. In addition, if $(\Omega_b, \varphi)$ is topologically mixing, then $\mu_V$ is Bernoulli ( i.e., $(\Omega_b, \varphi, \mu_V)$ is measure-theoretically isomorphic to a two-sided Bernoulli shift).*

*Proof.* We use the fact that we have great symbolic dynamics using the map $\pi$, which is one-to-one on a residual set.

We show next time that $V_1 = V \circ \pi$ has summable variation on $\Sigma_A$. We then need to get this function to depend on only half of the coordinates (or more precisely, replace it by a cohomologous one with this property).

This will give us $\mu_{V_1}$ on $\Sigma_A$, which is the unique equilibrium state for $V_1$. We then use $\pi$ to push $\mu_{V_1}$ down: let $\mu_V = \mu_{V_1}\pi^{-1}$. $\mu_{V_1}$ is fully supported: it gives positive measure to open sets.

Noting that the factor map $\pi$ is actually an isomorphism of the systems $(\Sigma_A.\sigma, \mu_{V_1})$ and $(\Omega_b, \varphi, \mu_V)$, the assertions will follow fairly easily.

□

(Continued on next day)

## 41. Symbolic dynamics yields existence of equilibrium states on basic sets. April 23, 1997 (*Notes by KJ*)

41.1. **Sketch of proof of the existence of equilibrium states on basic sets.** We treat the case when $(\Omega_b, \phi)$ is topologically mixing; for the non-mixing case consider $\Omega_b = X_1 \cup ... \cup X_m$ and replace $\phi$ by $\phi^m$.

Take a Markov partition $\mathcal{R} = \{R_1, \ldots, R_r\}$ into sets of small diameter, and use it to get the coding $\pi : \Sigma_A \to \Omega_b$. Let $V_1 = V \circ \pi : \Sigma_A \to \mathbb{R}$. Then $V$ Hölder implies that $V_1$ has summable variation, as follows: Suppose we have $\omega, \xi \in \Sigma_A$ with $\omega_k = \xi_k$ for $|k| \leq n$. If

$$\begin{aligned} x &= \pi\omega, \text{ and} \\ y &= \pi\xi, \end{aligned}$$

then $\phi^j x, \phi^j y$ are in the same $R_i$ for $|j| \leq n$. Therefore $d(x, y)$ has to be kind of small, much smaller than the maximum diameter. It is exponentially small in $n$ (by Hirsch-Pugh), i.e., there exists $a \in (0, 1)$ such that $d(x, y) \leq a^n$, for all $n$. Hence $|V_1(\omega) - V_1(\xi)| = |V(x) - V(y)| \leq c(a^n)^\rho$, and $\operatorname{var}_n V_1 \leq c(a^\rho)^n$ is summable in $n$.

By the previous theorem on the existence of equilibrium and Gibbs states on a shift of finite type (Theorem 40.4), there exists a unique equilibrium (Gibbs) state $\mu_{V_1}$ for $V_1$ on $\Sigma_A$. Define $\mu_V$ on $\Omega_b$ by $\mu_V(E) = \mu_{V_1}(\pi^{-1}E)$. Since $\pi$ is a factor mapping it commutes with the shift $\sigma$ and $\phi$. Since $\Omega_b$ is topologically mixing, $\mu_V$ is a $\phi$ invariant Borel probability measure on $\Omega_b$ and is measure-theoretically mixing.

We have a factor mapping

$$\pi : (\Sigma_A, \sigma, \mu_{V_1}) \to (\Omega_b, \phi, \mu_v)$$

between measure-preserving systems. We claim that $\pi$ is actually an isomorphism. To show this, we check that the set where $\pi$ is not 1-1 has measure zero.

For

$$x \notin \bigcup_{j \in \mathbb{Z}} \phi^j \bigcup_{i=1}^{r} (\partial^s R_i \cup \partial^u R_i),$$

$\pi^{-1}\{x\}$ is exactly one point. Define $\partial^s \mathcal{R} = \cup_{i=1}^r \partial^s R_i$, and define $\partial^u \mathcal{R}$ similarly. We claim that $\mu_{V_1}(\pi^{-1}\partial^s \mathcal{R}) = \mu_{V_1}(\pi^{-1}\partial^u \mathcal{R}) = 0$, hence $\pi$ is 1-1 on the set of full measure

$$\Sigma_A \backslash \cup_{j \in \mathbb{Z}} \phi^j \pi^{-1}(\partial^s \mathcal{R} \cup \partial^u \mathcal{R}).$$

We show here that $\mu_{V_1}(\pi^{-1}\partial^s \mathcal{R})$ has measure zero; the other case is similar.

Let $D_s = \pi^{-1}\partial^s \mathcal{R}$, which is a proper closed subset of $\Sigma_A$ because $\partial^s \mathcal{R} \subset \Omega_b$, the boundaries are closed, and $\pi$ is continuous. Note that $\sigma D_s \subset D_s$ since $\phi \partial^s \mathcal{R} \subset \partial^s \mathcal{R}$ by an earlier fact, section 3.15, p. 84 in Bowen. This forces it to have measure zero: let $D = \cap_{n \geq 0} \sigma^n D_s$. Then $D$ is a proper, closed, $\sigma$-invariant set. This forces $\mu_{V_1}(D) = 0$ or 1 since $\mu_{V_1}$ is ergodic. But $D^c$ is open and not empty since $\mu_{V_1}$ is a Gibbs measure and hence fully supported on $\Sigma_A$, so $\mu_{V_1}(D^c) > 0$. Therefore $\mu_{V_1}(D) = 0$ and $\mu_{V_1}(D_s) = 0$ (since $\mu_{V_1}(D) = \lim \mu_{V_1}\sigma^n(D_s)$). This gives us a measure-theoretical isomorphism, which tells us what we need.

Now we check that $\mu_V$ is an equilibrium state for $V$:

$$\begin{aligned} P(V,\phi) &\geq h_{\mu_V}(\phi) + \int_{\Omega_b} V d\mu_V \text{ (by Variational Principle)} \\ &= h_{\mu_{V_1}}(\sigma) + \int_{\Sigma_A} V_1 d\mu_{V_1} \text{ (by isomorphism)} \\ &= P(V_1,\sigma) \text{ (since } \mu_{V_1} \text{ is an equilibrium state for } V_1) \\ &\geq P(V,\phi). \end{aligned}$$

The last inequality follows from the fact that $(\Omega_b, \phi)$ is a factor of $(\Sigma_A, \sigma)$ and pressure, like entropy, cannot increase under factor mappings (see Bowen, Prop 2.13, p. 55). Therefore, $P(V,\phi) = h_{\mu_V}(\phi) + \int_{\Omega_b} V d\mu_V$, which means that $\mu_V$ is an equilibrium state for $V$.

Now we check for uniqueness. Suppose that $\mu$ on $\Omega_b$ is also an equilibrium state for $V$. Lift $\mu$ to $\nu$, a $\sigma$-invariant measure on $\Sigma_A$, using Lemma 4.1. Then

$$\begin{aligned} h_\nu\sigma + \int_{\Sigma_A} V_1 d\nu &\geq h_\mu\phi + \int_{\Omega_b} V d\mu \text{ (since the integrals are equal)} \\ &= P(V,\phi) \text{ (since } \mu \text{ is an equilibrium state)} \\ &= P(V_1,\sigma) \text{ (from above).} \end{aligned}$$

Therefore $\nu$ is an equilibrium state for $V_1$, hence $\nu = \mu_{V_1}$, so $\mu = \nu\pi^{-1} = \mu_V$ by our definition.

We summarize the idea of the argument: Good symbolic dynamics presents a residual invariant set as a tight topological factor of an SFT, so a Gibbs or equilibrium state on $\Sigma_A$ gives a Gibbs or equilibrium state on $\Omega_b$.

41.2. **Theorems 2 and 3.** We now move on to sketching the proofs of the remaining Theorems 2 and 3:

Lebesgue-almost every point of $M$ is attracted to one of the basic sets, and the statistics of Lebesgue-almost every orbit are given by SRB measure, which is an equilibrium state.

The measure we want comes from applying Theorem 1 to a good $V$.

Assume $(M,\phi)$ is Axiom A and $\phi$ **is a $\mathcal{C}^2$ Axiom A diffeomorphism**. It is known that certain parts of the following *do not hold* in the $\mathcal{C}^1$ case. We have that the basic set $\Omega_b$ is hyperbolic, so for $p \in \Omega_b$,

$$D_p\phi : E_p^u \to E_{\phi p}^u$$

where $E_p^u$ is the expanding subspace. Define $J_u(p)$ to be the Jacobian of this map, the determinant of $D_p\phi$ in local coordinates (recall that $M$ is an oriented Riemannian manifold with an adapted metric).

Recall that $\int_A J_\phi\omega = \int_{\phi A}\omega$, where $\omega$ is the volume form, and $J\phi$, the Jacobian, is the local volume distortion by $\phi$. We use the Jacobian above to define the special potential function $V_u(p) = -\log|J_u(p)|$. Note that $V_u \leq 0$, since $D_p\phi$ expands on $E^u_p$. Then $V - u$ is Hölder on $\Omega_b$ (by the Hirsch-Pugh theorem), and therefore by the preceding theorem there exists a unique equilibrium state for it:

$$\mu_{V_u} = \mu_{SRB},$$

the Sinai-Ruelle Bowen measure on $\Omega_b$. (This measure is actually independent of the choice of metric on $M$, since changing the metric will replace $V_u$ by a function cohomologous to $V_u$ + a constant.)

## 42. Finding attractors in Axiom A systems. April 24 (*Notes by LK*)

We begin with an Axiom A system $(M, \phi)$, a basic set $\Omega_b \subset \Omega(\phi) \subset M$, and we construct a special equilibrium state to go with this system. We let

$$D_p\phi \colon E^u_p \to E^u_{\phi p}$$

and define the Jacobian of this map $J_u(p) = g(p) =$ local volume distortion factor. We use the potential function

$$V_u(x) = -\log J_u(x)$$

for $x \in \Omega_b$. Since the map is expanding on this unstable space, we have that $J_u(x) \geq 1$ and hence $V_u(x) \leq 0$.

Recall that we defined the metric

$$D_n(x, y) = \sup_{0 \leq k \leq n-1} d(\phi^k x, \phi^k y).$$

We will be working with balls with respect to this metric and finding the measure of these balls with respect to $m$, the manifold measure given by the volume form, and $\mu_{SRB} = \mu_{V_u}$, the SRB-measure. The following lemma lets this process begin.

**Lemma 42.1.** *(Volume Lemma) For small $\epsilon > 0$, there exists $c_\epsilon > 1$ such that*

$$\frac{1}{c_\epsilon} \leq \frac{m(B_{D_n}(x, \epsilon))}{\exp[S_n V_u(x)]} \leq c_\epsilon,$$

*for all $x \in \Omega_b$ and all $n \geq 1$.*

The key ingredient to the proof of the Volume Lemma is the chain rule. We multiply the Jacobian along an orbit of a point, and then compare that to the Lebesgue measure of the set that is within $\epsilon$ of $x$ and stays within distance $\epsilon$ of the orbit of $x$.

We see hints of the connections between entropy (and pressure), volume growth, dimension, and Lyapunov exponents. There are many formulas relating these concepts, such as those of Young, Newhouse, and Yomdin.

We use the Volume Lemma to get the following proposition.

**Proposition 42.2.** (1) *On $(\Omega_b, \phi)$, we have*

$$P(V_u, \phi) = \lim_{n \to \infty} \frac{1}{n} \log m\Big(\bigcup_{x \in \Omega_b} B_{D_n}(x, \epsilon)\Big) \leq 0.$$

(2) *If $m(\bigcup_{x \in \Omega_b} W^s_\epsilon(x)) > 0$, then $P(V_u, \phi) = 0$ (and hence $h_{\mu_{SRB}}(\phi) = -\int_{\Omega_b} V_u d\mu_{SRB}$).*

*Proof.* (Proof of 1) If $A$ is a maximal $n,\delta$-separated set in $\Omega_b$, then each $B_{D_n}(x,\epsilon)$, for $x \in \Omega_b$, is contained in some $B_{D_n}(y,\epsilon+\delta)$ for some $y \in A$ (by the definition of $n,\delta$-separated and the definition of $D_n$). Thus, from the Volume Lemma,

$$m(\bigcup_{x\in\Omega_b} B_{D_n}(x,\epsilon)) \le c_{\epsilon+\delta} \sum_{y\in A} \exp(S_n V_u(y)).$$

There is also a reverse inequality that shows that these two things are comparable. We take the limit as $n \to \infty$ of $\frac{1}{n}$ times the log of both sides. The right-hand side gives the pressure, and the left-hand side gives the desired formula.

(Proof of 2) Let us use the abbreviated notation

$$\bigcup_{x\in\Omega_b} W^s_\epsilon(x) = W^s_\epsilon(\Omega_b);$$

then

$$W^s_\epsilon(\Omega_b) \subset \bigcup_{z\in\Omega_b} B_{D_n}(z,\epsilon)$$

(since $W^s_\epsilon(x) \subset B_{D_n}(x,\epsilon)$). If $m(W^s_\epsilon(\Omega)) = \eta > 0$, then for all $n \ge 0$,

$$m(\bigcup_{z\in\Omega_b} B_{D_n}(z,\epsilon)) \ge \eta > 0,$$

so (from (1)),

$$P(V_u,\phi) \ge \lim \frac{1}{n} \log \eta = 0.$$

□

**Remark 42.1.** Whenever $P(V,\phi) = 0$ and $\mu$ is the equilibrium state for $V$ ($\mu = \mu_V$), we have such a formula for $h_\mu(\phi) = -\int V d\mu_V$. In particular, we always have an integral formula for entropy for $g$-measures.

**Example 42.1.** A Markov measure is an equilibrium state with $P = 0$, hence

$$h_\mu(\sigma) = -\int \log M_{x_0 x_1} d\mu(x) = -\sum_{i,j} p_i M_{ij} \log M_{ij}.$$

Recall that $\Omega_b$ is an *attractor* if there exists a neighborhood $U$ of $\Omega_b$ such that $\phi U \subset U$ and $\bigcap_{n\ge 0} \phi^n U = \Omega_b$. We say that $\Omega_b$ is a *tight attractor* if for all $\eta > 0$, we can find such

$$U \subset B_\eta(\Omega_b) = \{y \in M : d(y,\Omega_b) < \eta\}.$$

We give an illustration of a possible non-tight attractor. Let $\Lambda$ be an attractor lying in the plane $L$ with some dense orbit. Suppose there are an attracting fixed point (AFP) and repelling fixed point (RFP) outside of $L$. Suppose there is a point $y$ which is close to RFP whose orbit gets close to $\Lambda$ but then heads towards AFP. Let $p = \phi^k(y)$ be the point close to $\Lambda$. We illustrate this in Figure 55.

Take a neighborhood $B$ of $\Lambda$ that also includes the one point $p$, but none of the rest of the orbit of $y$. Let $U = B \cup \phi B \cup \phi^2 B \ldots$. Then $\phi U \subset U$, and $U$ contains points at a significant distance from $\Lambda$. We illustrate $B$ and $\phi B$ in Figure 56.

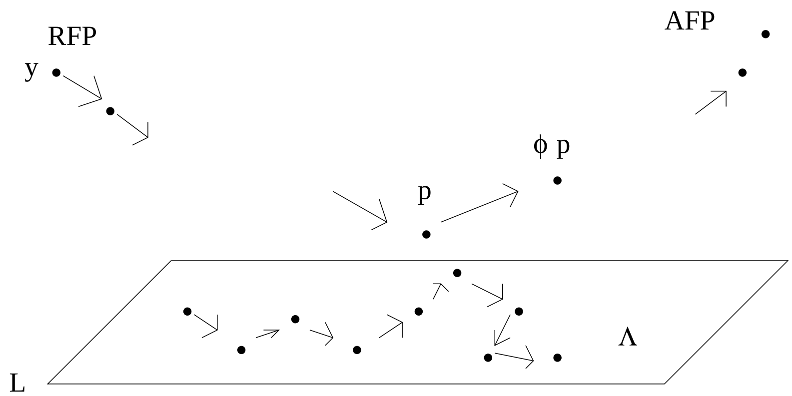



Figure 55. Attractor $\Lambda$

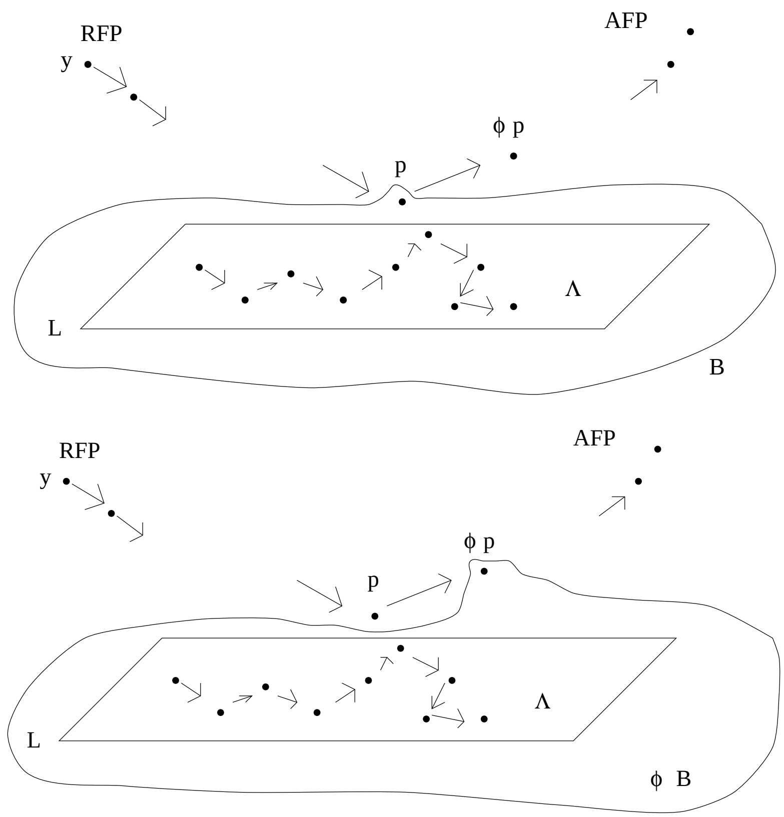



Figure 56. $B$ and $\phi B$

Then $\bigcap_{n\geq 0}\phi^n U = \Lambda$. If we have infinitely many such "outside orbits" swooping close to $\Lambda$, then we cannot have a tight attractor. We will see that the $\Omega_b$'s that are not tight attractors have Lebesgue measure 0.

**Proposition 42.3.** *$\Omega_b$ is a tight attractor if and only if $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$.*

*Proof.* Recall that a neighborhood of $\Omega_b$ is a set which contains an open set that contains $\Omega_b$. Suppose that $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$. Let $\eta > 0$, and choose $k$ so that $\lambda^k\epsilon < \eta$ (recall $\lambda \in (0,1)$ from the Stable Manifold Theorem), and let $U = \phi^k W^s_\epsilon(\Omega_b)$. Since $W^s_\epsilon(\Omega_b) \subset B_\epsilon(\Omega_b)$, and also

$$\phi W^s_\epsilon(x) \subset W^s_\epsilon(\phi x),$$

we have $\phi U \subset U$. Also, $\Omega_b = \bigcap_{n\geq 0} \phi^n U$, because points of $W^s_\epsilon(\Omega_b)$ have their forward iterates approach $\Omega_b$ exponentially fast.

Conversely, look at the proof of Proposition 24.1. Set $\eta$ to be the $\gamma$ found in this proof. Find

$$U \subset \{y \in M\colon d(y, \Omega_b) < \eta\} = B_\eta(\Omega_b)$$

with

$$\phi U \subset U \text{ and } \Omega_b = \bigcap_{n\geq 0} \phi^n U.$$

Then for all $n \geq 0$, $\phi^n U \subset U \subset B_\eta(\Omega_b)$, so $U \subset \phi^{-n} B_\eta(\Omega_b)$. Thus

$$\Omega_b \subset U \subset \bigcap_{n\geq 0} \phi^{-n} B_\eta(\Omega_b) \subset W^s_\epsilon(\Omega_b)$$

by Proposition 24.1. Therefore, $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$. □

We would like to determine which of the $\Omega_b$'s are tight attractors. To answer this, we will need two other technical lemmas.

**Lemma 42.4.** (1) *If there exists an $x \in \Omega_b$ such that $W^u_\epsilon(x) \subset \Omega_b$, then $\Omega_b$ is a tight attractor.*
(2) *If $\Omega_b$ is not a tight attractor, then there exists $\gamma > 0$ such that for all $x \in \Omega_b$, there exists $y \in W^u_\epsilon(x)$ with $d(y, \Omega_b) > \gamma$.*

*Proof.* $1 \Rightarrow 2$ is easy by compactness of $\Omega_b$. To prove 1, we use that

$$U_x = \bigcup_{y\in W^u_\epsilon(x)} W^s_\epsilon(y)$$

is a neighborhood of $x$ in $M$. We try to find a neighborhood $B_\delta(\Omega_b) \subset W^s_\epsilon(\Omega_b)$ and use the previous proposition. We use the fundamental neighborhood $U$ of the nonwandering set for which $\bigcap_{j\in\mathbb{Z}} \phi^j U = \Omega(\phi)$. Take a periodic point $p$ in $U_x$ and look at $X_p = \overline{W^u(p) \cap \Omega}$. Using this machinery, we are able to produce a neighborhood of $\Omega$ that is in $W^s_\epsilon(\Omega_b)$. □

**Lemma 42.5.** *For small $\epsilon, \delta > 0$, there exists a constant $c(\epsilon, \delta)$ such that if $x \in \Omega_b$ and $y \in B_{D_n}(x, \epsilon)$, then,*

$$m(B_{D_n}(y, \epsilon) \geq c(\epsilon, \delta) m(B_{D_n}(x, \epsilon)).$$

**Theorem 42.6.** *Let $\Omega_b$ be a basic set of a $C^2$ Axiom A system $(M, \phi)$. The following are equivalent:*

(1) *$\Omega_b$ is a tight attractor.*
(2) *$m(W^s(\Omega_b)) > 0$.*
(3) *$P(V_u, \phi|_{\Omega_b}) = 0$.*

*Proof.* 1 ⇒ 2: Recall

$$W^s(\Omega_b) = \bigcup_{x\in\Omega_b} W^s(x) = \bigcup_{x\in\Omega_b}\bigcup_{n\geq 0} \phi^{-n} W^s_\epsilon(\phi^n x).$$

If $\Omega_b$ is a tight attractor, then $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$ and hence has positive Lebesgue measure (since $m$ is positive on open sets). Hence $m(W^s(\Omega_b)) > 0$ because $W^s(\Omega_b) \supset W^s_\epsilon(\Omega_b)$.

2 ⇒ 3: Use statement 2 of Proposition 42.2, which said that

$$m(\bigcup_{x\in\Omega_b} W^s_\epsilon(x)) > 0 \Rightarrow P(V_u, \phi) = 0.$$

We have

$$W^s(\Omega_b) = \bigcup_{n\geq 0} \phi^{-n}(\bigcup_{x\in\Omega_b} W^s_\epsilon(\phi^n x)) = \bigcup_{n\geq 0} \phi^{-n}(\bigcup_{x\in\Omega_b} W^s_\epsilon(x)) = \bigcup_{n\geq 0} \phi^{-n} W^s_\epsilon(\Omega_b).$$

By nonsingularity, if $m(W^s_\epsilon(\Omega_b)) = 0$ then $m(W^s(\Omega_b)) = 0$; therefore $m(W^s_\epsilon(\Omega_b)) > 0$ and hence $P(V_u, \phi|_{\Omega_b}) = 0$.

3 ⇒ 1: If $\Omega_b$ is not a tight attractor, then find $\gamma$ as in Lemma 42.4 such that for all $x \in \Omega_b$, there exists $y \in W^u_\epsilon(x)$ such that $d(y, \Omega_b) > \gamma$. In fact, we will choose $x$'s from some $n, \epsilon$-separated set in $\Omega_b$ and find the corresponding $y(x, n)$'s with those $\gamma$'s.

From Lemma 42.5 and Proposition 42.2, we can show

$$P(V_u, \phi) = \lim_{n\to\infty} \frac{1}{n} \log m(\bigcup_{z\in\Omega_b} B_{D_n}(z, \epsilon)) < 0.$$

This contradicts our assumption that $P(V_u, \phi) = 0$, and hence $\Omega_b$ is a tight attractor. □

The following corollary summarizes the global dynamics of the system $(M, \phi)$.

**Corollary 42.7** (Theorem 2)**.** *For $m$-a.e. $p \in M$, there is $b(p)$ such that $d(\phi^n p, \Omega_{b(p)}) \to 0$ as $n \to \infty$; that is, the orbit of almost every point of $M$ approaches, in forward time, a basic set which is a tight attractor.*

*Proof.* Recall from Proposition 23.2

$$M = \bigcup_{b=1}^{r} W^s(\Omega_b),$$

and this equals, up to a set of $m$-measure 0,

$$\bigcup_{\Omega_b \text{ tight attr.}} W^s(\Omega_b).$$

□

## 43. Attracting measures. April 28 (*Notes by KS*)

We begin by saying a little more about Lemma 42.4 from last time:

**Lemma** (Lemma 42.4)**.** *If there exists an $x \in \Omega_b$ with $W^u_\epsilon(x) \subset \Omega_b$ then $\Omega_b$ is a* tight attractor*.*

*Sketch of proof:* Recall that to show $\Omega_b$ is a tight attractor, we need to show that $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$.

Take $x_0 \in \Omega_b$ such that $W^u_\epsilon(x_0) \subset \Omega_b$ and look at the set

$$U_{x_0} = \bigcup_{y \in W^u_\epsilon(x_0)} W^s_\epsilon(y).$$

This is a neighborhood of $x_0$ in $M$ (see Hirsch-Pugh).

Let $p \in U_{x_0}$ be a periodic point (recall that periodic points are dense) and let $m$ be such that $\phi^m(p) = p$. For small $\beta > 0$, we have $W^u_\beta(p) \subset U_{x_0}$ (and actually $W^u_\beta(p) \subset \Omega_b$—let $z \in W^u_\beta(p)$, then $z \in W^s_\epsilon(y)$ for some $y$, and $\phi^j(z)$ is close to $\Omega$ for all $j \in \mathbb{Z}$. Then if $\epsilon$ and $\beta$ are small enough, by our results on Fundamental Neighborhoods (Proposition 23.1), $z \in \Omega$ and hence $z \in \Omega_b$).

Now,

$$W^u(p) = \cup_{k \geq 0} \phi^{km} W^u_\beta(p) \subset \Omega_b,$$

since $\Omega_b$ is $\phi$-invariant, so $X_p = \overline{W^u(p) \cap \Omega} \subset \Omega_b$. Also, for some $N$,

$$\Omega_b = X_p \cup \phi X_p \cup \cdots \cup \phi^N X_p.$$

If $x \in Y = \cup_{k=0}^N \phi^k W^u(p)$, then $W^u_\epsilon(x) \subset \Omega_b$ (since $d(\phi^{-n} z, \phi^{-n} x) \leq \epsilon$ for all $n \geq 0$ and $d(\phi^{-n} x, \phi^{-n} \phi^k p) \to 0$ for some $k = 0, \ldots, N$ imply $z \in W^u(\phi^k p) = \phi^k W^u(p) \subset \Omega_b$).

As above, we deduce that $U_x \subset W^s_\epsilon(\Omega_b)$, and thus $\cup_{x \in Y} U_x \subset W^s_\epsilon(\Omega_b)$. Even though $Y$ is dense in $\Omega_b$, this is not yet quite enough to show that $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$. However, because of the Hirsch-Pugh result, according to which $W^u_\epsilon(x)$ and $W^s_\epsilon(y)$ vary continuously with $x$ and $y$, one can find a single $\delta > 0$ such that $B_\delta(x) \subset U_x$ for each $x$ in $Y$.

Then $\Omega_b \subset \cup_{x \in Y} B_\delta(x) \subset W^s_\epsilon(\Omega_b)$, so that $W^s_\epsilon(\Omega_b)$ is a neighborhood of $\Omega_b$, and hence $\Omega_b$ is a tight attractor. □

We now move to the *Sinai-Ruelle-Bowen Theorem* (Theorem 3 in our list). This theorem tells about the asymptotic statistics of the orbits. (Note that we already know something from the Corollary 42.7 (Theorem 2).)

**Theorem 43.1** (Theorem 3)**.** *Let $\Omega_b$ be a basic set in a $C^2$ Axiom A system $(M, \phi)$. Assume that $\Omega_b$ is a tight attractor. Then for $m$ (Lebesgue measure) a.e. $x \in W^s(\Omega_b)$ (and hence in the basin of $\Omega_b$), for each $f \in \mathcal{C}(\Omega_b)$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\phi^k x) \longrightarrow \int_{\Omega_b} f \, d\mu_{SRB}. \tag{4}$$

*Proof. Part 1.* Main idea: Show that $m$ and $\mu_{SRB}$ have the same asymptotics for the sets $B_{D_n}(p, \epsilon) = \{y \in M : d(\phi^k p, \phi^k y) < \epsilon \text{ for all } k = 0, 1, \ldots, n-1\}$.

We know that

$$m(B_{D_n}(p,\epsilon)) \asymp \exp(S_n V_u(p)),$$

by the Volume Lemma, and also,

$$\mu_{SRB}(B_{D_n}(p,\epsilon)) \asymp \exp(S_n V_u(p)),$$

since $\mu_{SRB}$ is a Gibbs measure (in the topologically mixing case—without topological mixing, it is still true. See Bowen, Section 4.4).

So for $x \in \Omega_b$ (or $W^s(\Omega_b)$), we have by the Ergodic Theorem (recall that $\mu_{SRB}$ is an ergodic measure):

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\phi^k x) \longrightarrow \int_{\Omega_b} f d\mu_{SRB}$$

for $\mu_{SRB}$-a.e. $x$.

We would like to show that this convergence holds for a.e. $x$ with respect to $m$. We can do this by using the measure asymptotics to estimate the $m$-measure of the "bad" set of $x \in W^s(\Omega_b)$ in terms of the $\mu_{SRB}$-measure.

*Part 2.* Show that the theorem holds for just *one* $f \in \mathcal{C}(\Omega_b)$—we do it for each $f$ in a countable dense set in $\mathcal{C}(\Omega_b)$, take the union of all the resulting "bad" sets of $m$-measure 0, then use standard approximation arguments to extend to all $f \in \mathcal{C}(\Omega_b)$.

*Part 3.* Let $f \in \mathcal{C}(\Omega_b)$ and let $\delta > 0$. Set

$$C_n(\delta) = \{p \in M : \left|A_n f(p) - \bar{f}\right| > \delta\},$$

where

$$A_n f(p) = \frac{1}{n}\sum_{k=0}^{n-1} f(\phi^k p) \text{ and } \bar{f} = \int_{\Omega_b} f d\mu_{SRB}.$$

Let $E_\delta = \cap_{m\geq 1} \cup_{n\geq m} C_n(\delta)$ (note that the intersection is decreasing). The set $E_\delta$ is the set of $p$ which are $\delta$-bad for infinitely many $n$. We know that $\mu_{SRB}(E_\delta) = 0$, and we want to show that $m(E_\delta \cap W^s(\Omega_b)) = 0$ as well.

Choose $\epsilon > 0$ small enough so that $|f(x) - f(y)| < \delta$ when $d(x,y) < \epsilon$ (using uniform continuity), and small enough that the requirements for canonical coordinates, etc., are met.

*Part 4.* Fix $m > 0$ and define $R_m, R_{m+1}, \ldots$ inductively by letting $R_m$ be a maximal family of disjoint $D_m$-balls of radius $\epsilon$ in $\Omega_b \cap C_m(2\delta)$ and $R_{m+1}$ be a maximal family of disjoint $D_{m+1}$-balls of radius $\epsilon$ in $\Omega_b \cap C_{m+1}(2\delta)$ which are also disjoint from the balls in $R_m$. Let $R_n' \subset \Omega_b \cap C_n(2\delta)$ denote the set of centers of the balls in $R_n$.

Then

$$W^s_\epsilon(\Omega_b) \cap \cup_{n=m}^{\infty} C_n(3\delta) \subset \cup_{n=m}^{\infty} \cup_{x\in R_n'} B_{D_n}(x, 2\epsilon). \tag{5}$$

(If $y$ is an element of the left-hand side of (5), then $y \in W^s_\epsilon(z)$ for some $z \in \Omega_b$. But since $y \in C_n(3\delta)$ for some $n \geq m$, and $|A_n f(z) - A_n f(y)| < \delta$, we have $z \in C_n(2\delta)$. By maximality of $\{R_n\}$, for some $k$ between $m$ and $n$, and some $x \in R_k'$, $B_{D_n}(z,\epsilon) \cap B_{D_k}(x,\epsilon) \neq \emptyset$. Thus

$$y \in B_{D_n}(z,\epsilon) \subset B_{D_k}(z,\epsilon) \subset B_{D_k}(x, 2\epsilon),$$

which is what we needed to show.)

Therefore, by the Volume Lemma,

$$m(W^s_\epsilon(\Omega_b) \cap \cup_{n=m}^{\infty} C_n(3\delta)) \leq \sum_{n=m}^{\infty} \sum_{x \in R'_n} C_{2\epsilon} \exp(S_n V_u(x)). \tag{6}$$

We now use the $\mu_{SRB}$-measure to estimate the right-hand side of (6) from above.

*Part 5.* Let $U_m = \cup_{n=m}^{\infty} \cup_{x \in R'_n} B_{D_n}(x, \epsilon)$. Since $x \in R'_n \subset C_n(2\delta)$, each $B_{D_n}(x,\epsilon) \subset C_n(\delta)$, so $U_m \subset \cup_{n=m}^{\infty} C_n(\delta)$. Therefore, by the Ergodic Theorem,

$$0 = \mu_{SRB}(E_\delta) = \lim_{m\to\infty} \mu_{SRB}\left(\cup_{n=m}^{\infty} C_n(\delta)\right),$$

so $\mu_{SRB}(U_m) \to 0$.

*Part 6.* There exist positive constants $b_\epsilon$ and $C$ such that

$$\begin{aligned}\mu_{SRB}(U_m) &\geq b_\epsilon \sum_{n=m}^{\infty} \sum_{x \in R'_n} \exp(S_n V_u(x)) \\ &\geq C m(W^s_\epsilon(\Omega_b) \cap \cup_{n=m}^{\infty} C_n(3\delta)).\end{aligned}$$

In the topologically mixing case, the first inequality holds using the Gibbs condition on $\mu_{SRB}$. If $(M, \phi)$ is not topologically mixing, then the inequality still holds (see Bowen, 4.4). The second inequality holds by Part 4.

Therefore, as $m \to \infty$, we have $m(W^s_\epsilon(\Omega_b) \cap \cup_{n=m}^{\infty} C_n(3\delta)) \to 0$ and so $m(W^s_\epsilon(\Omega_b) \cap E_{3\delta}) = 0$.

*Part 7.* Let $\delta' > 3\delta$ and $n \geq 0$. Then the set $E_{\delta'} = \{\overline{\lim}\, |A_n f - \bar{f}| \geq \delta'\}$ is $\phi$-invariant, so $E_{\delta'} \cap W^s(\Omega_b) = E_{\delta'} \cap \cup_{n\geq 0} \phi^{-n} W^s_\epsilon(\Omega_b)$. Now, for each $n$, since $\phi$ preserves sets of $m$-measure 0, and

$$E_{\delta'} \cap \phi^{-n} W^s_\epsilon(\Omega_b) \subset \phi^{-n}(E_{3\delta} \cap W^s_\epsilon(\Omega_b)),$$

we have $m(E_{\delta'} \cap \phi^{-n} W^s_\epsilon(\Omega_b)) = 0$. Thus, $m(E_{\delta'} \cap W^s(\Omega_b)) = 0$.

*Part 8.* Finally, the set of $x$ for which (4) does *not* hold is equal to $\cup_{n=1}^{\infty} E_{1/n}$. Since $m(E_{1/n} \cap W^s(\Omega_b)) = 0$, we conclude that the convergence holds for $m$-a.e. $x$ in $W^s(\Omega_b)$.

□