Hidden Markov Chains Found Again

(Continuous Images of Measures on Shifts of Finite Type).

Karl Petersen

University of North Carolina at Chapel Hill

Setting

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X, Y topologically mixing 1-step shifts of finite type

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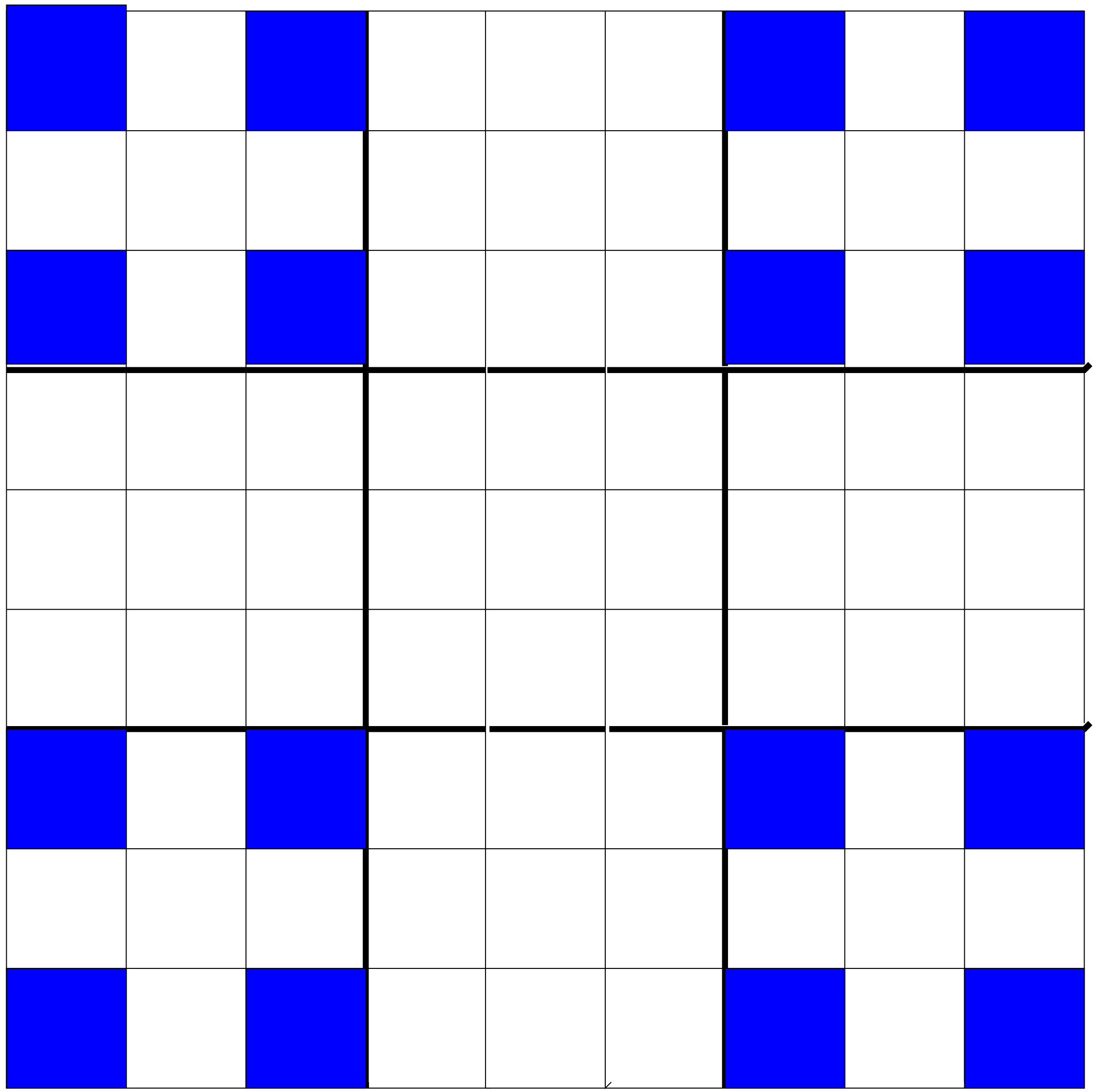
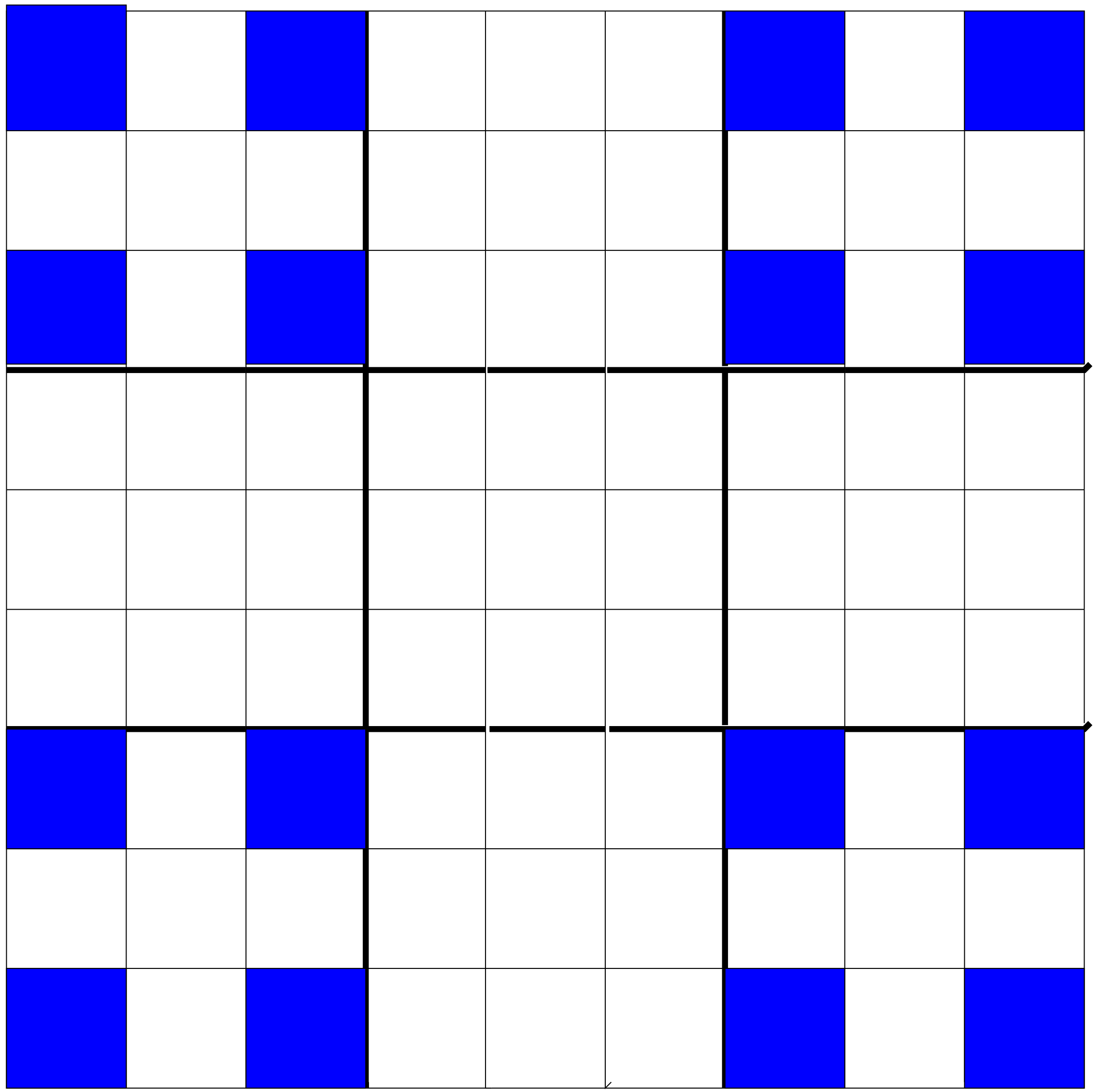
$\nu = \pi\mu$ on Y: $\nu(B) = \mu(\pi^{-1}B)$

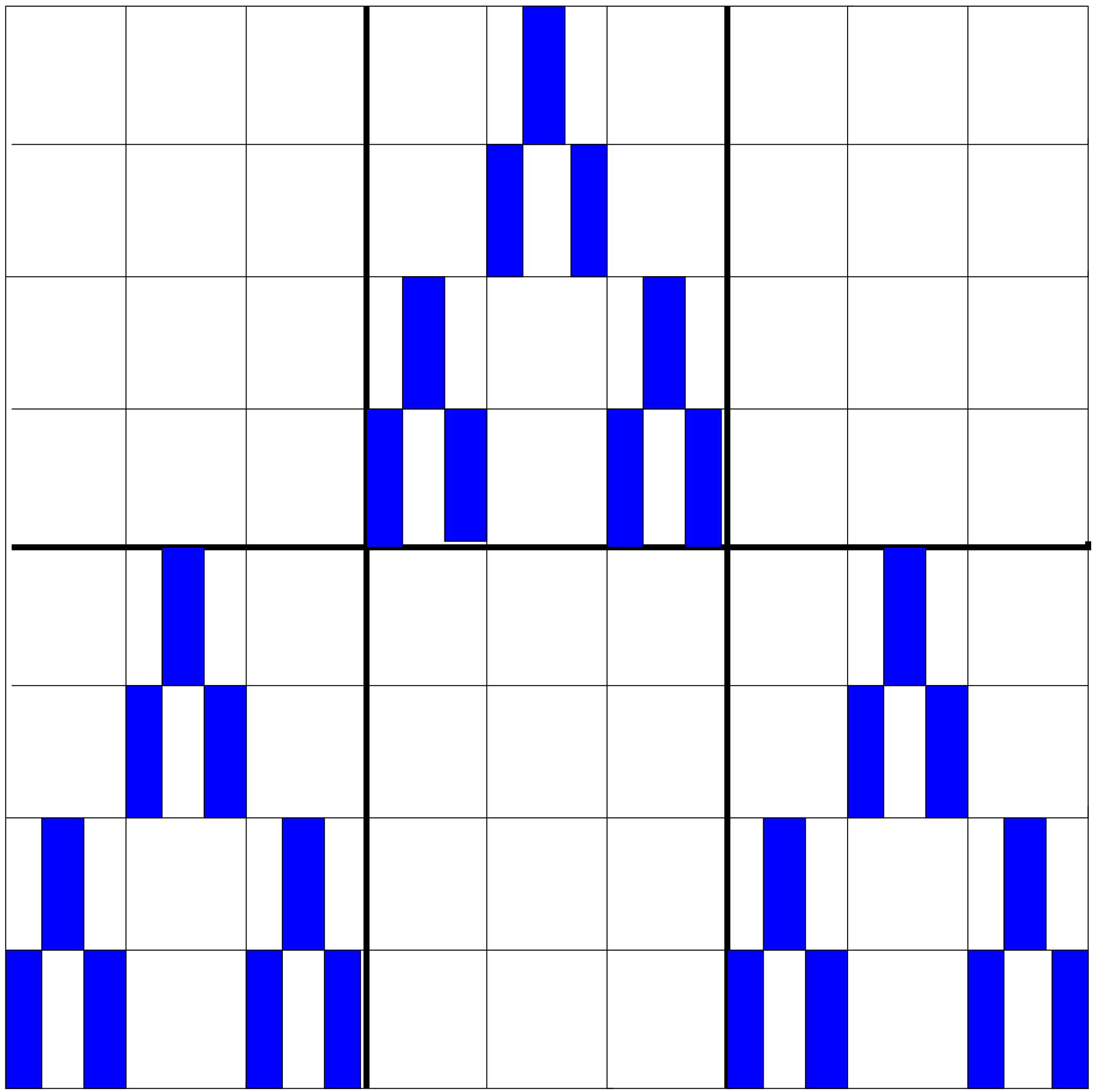
Sierpinski Carpet



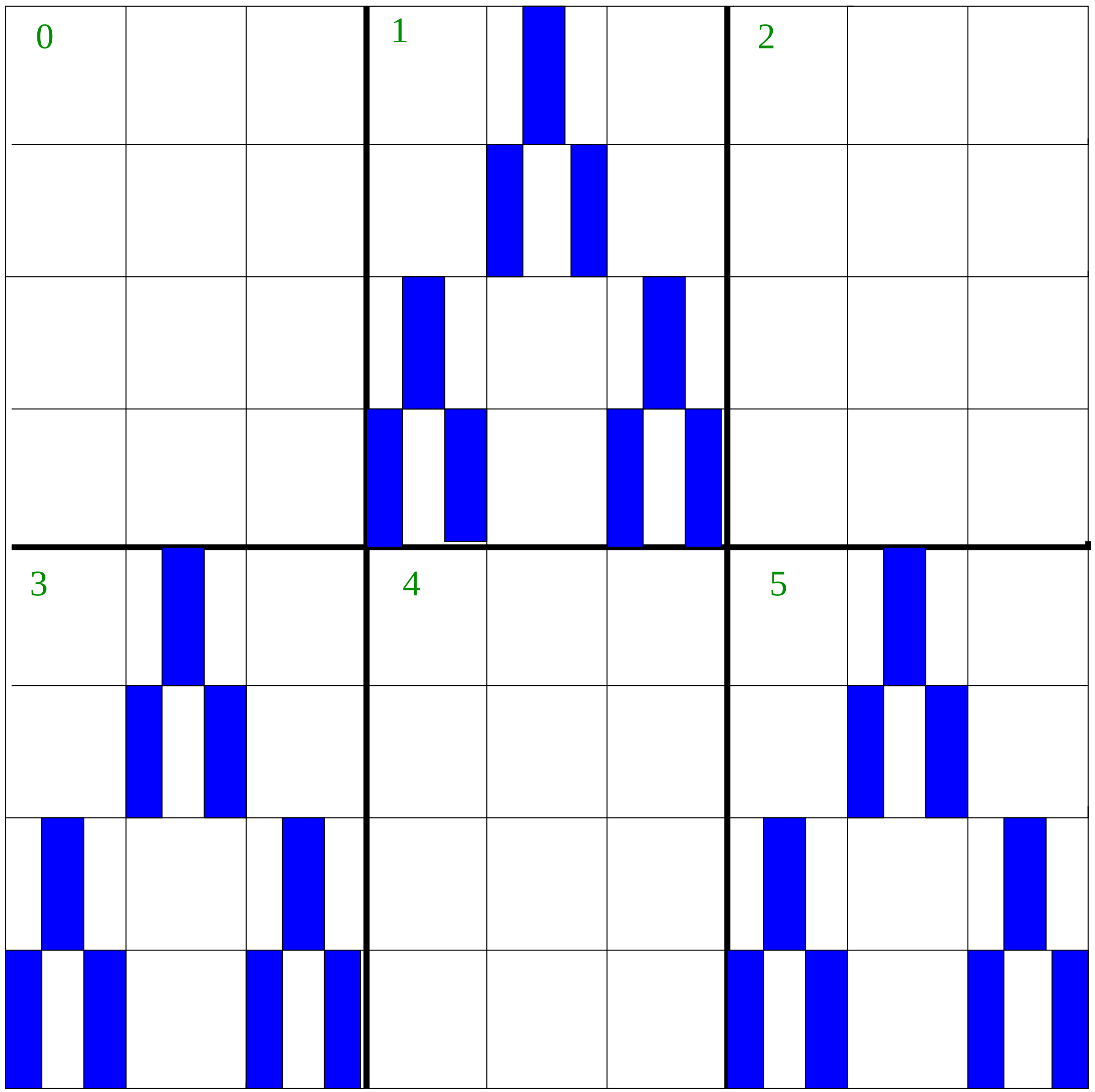
Sierpinski (or Dean Smith) Carpet



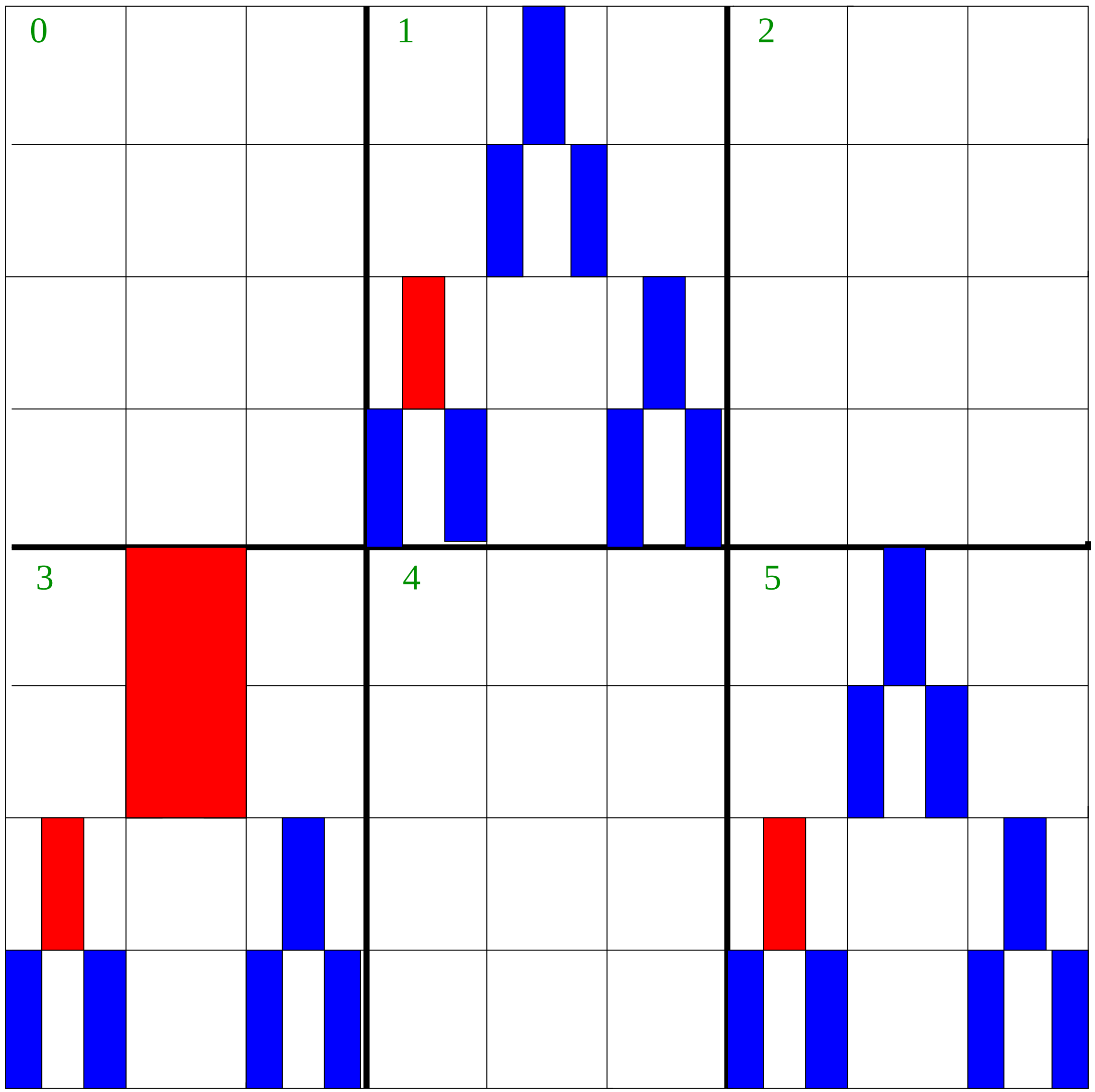
Nonconformal Carpet



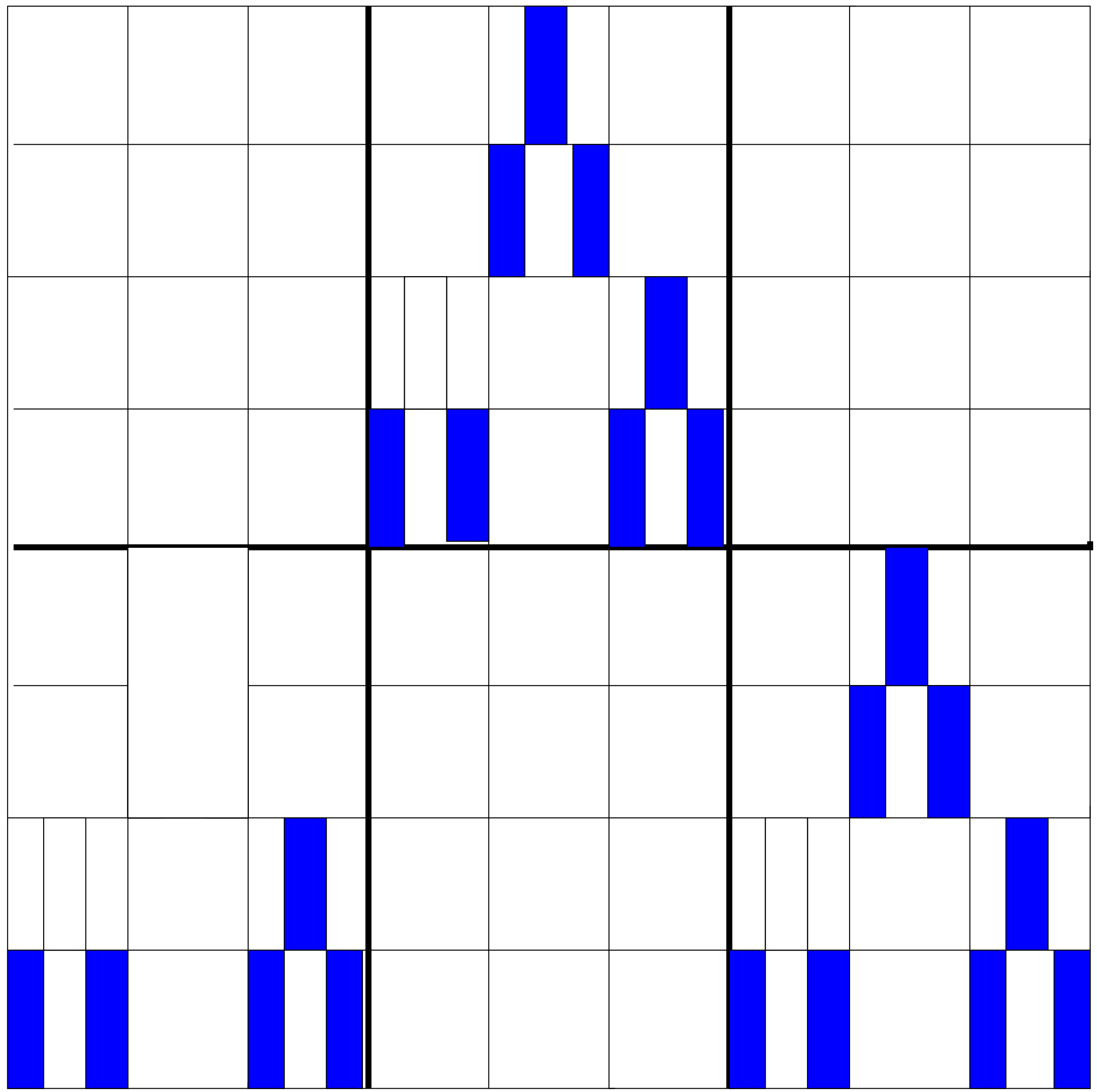
Nonconformal Carpet Coded



Disallow some transitions 31



More worn carpet



Information Loss

Models information loss, “deterministic noise”:

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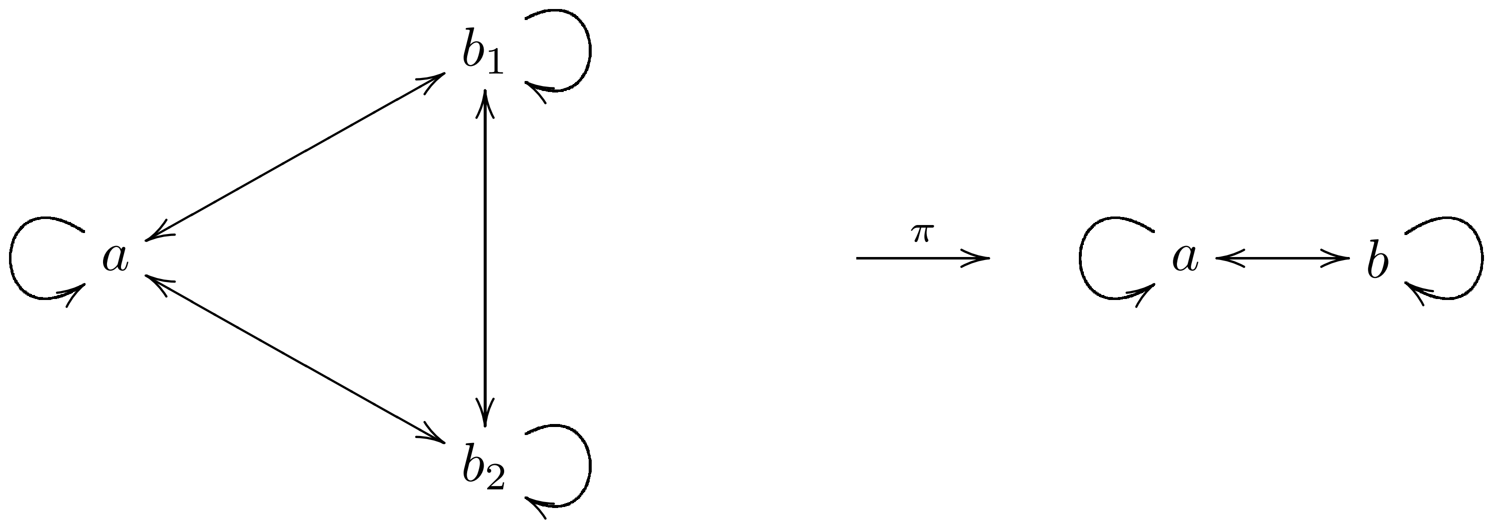

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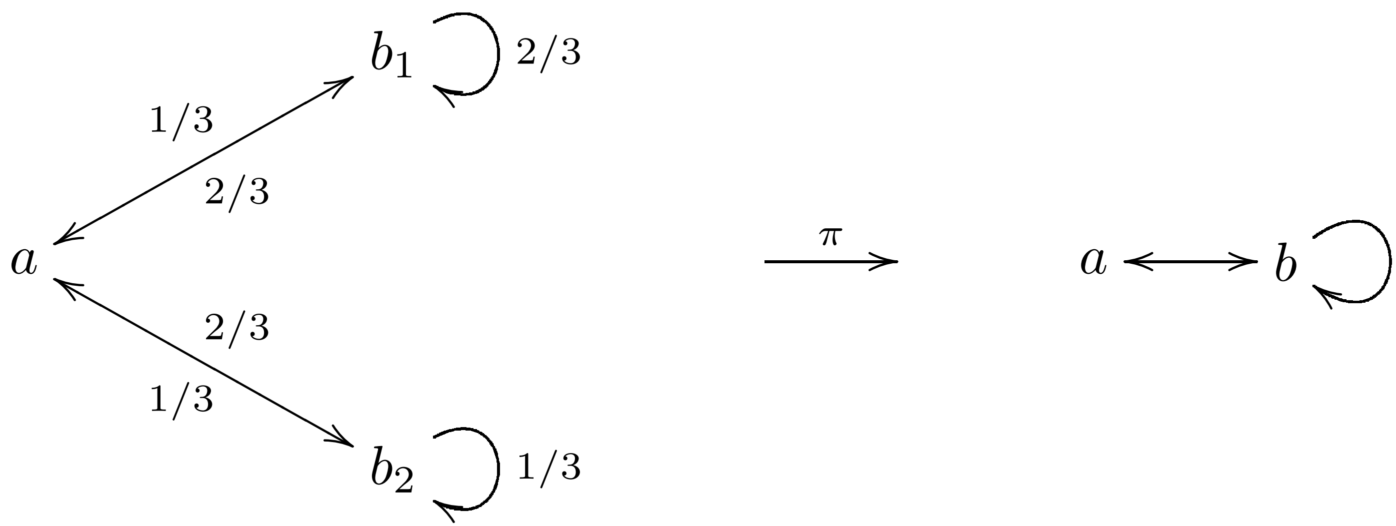

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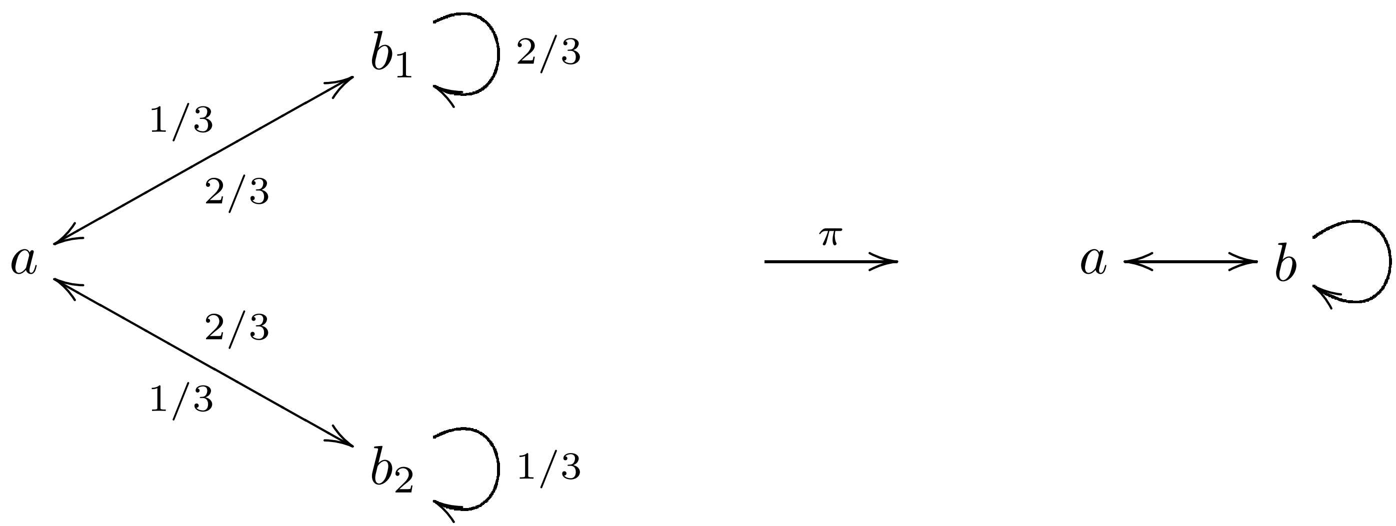

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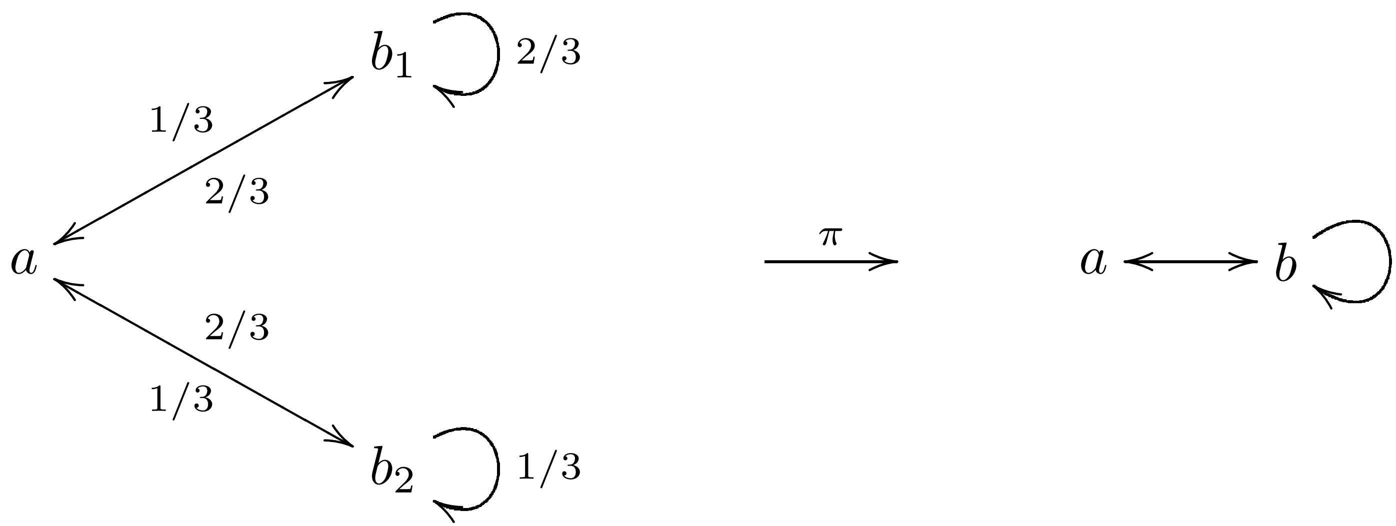

Image measure is not Markov.

Its entropy is hard to compute.

Markovian

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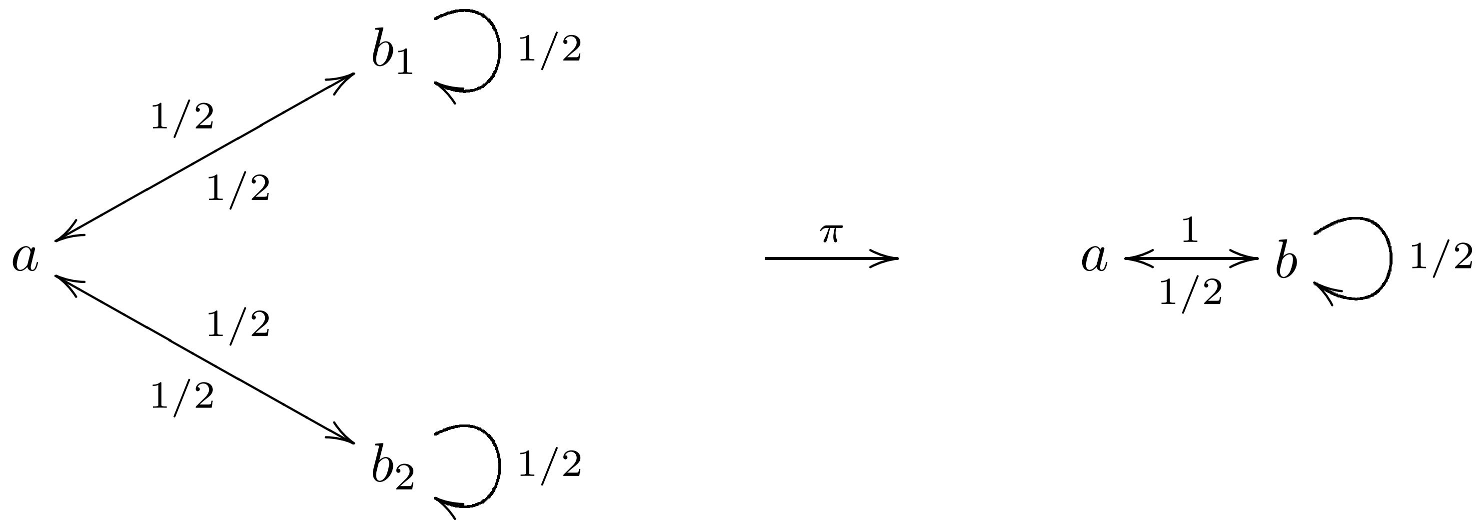

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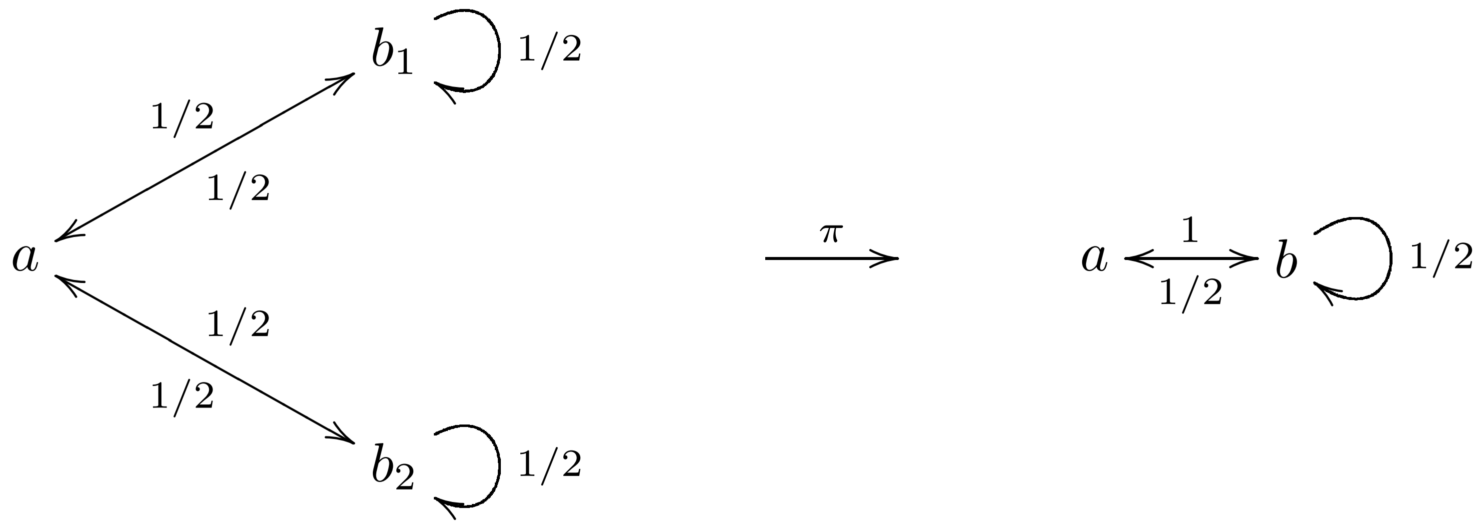
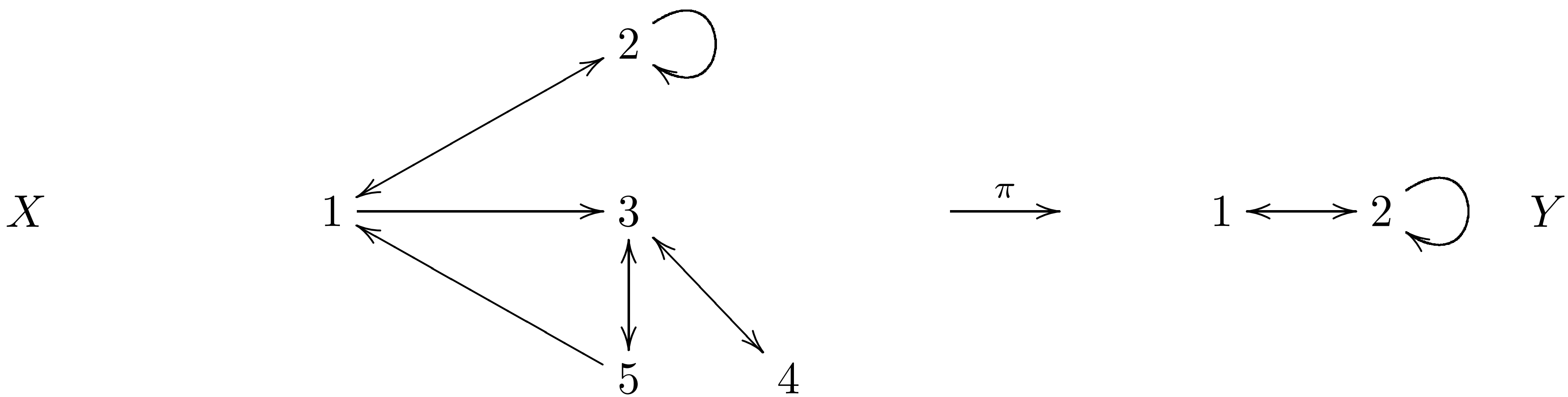


So the code is *Markovian*:

some Markov measure maps to a Markov measure.

Shin non-Markovian

2. Shin non-Markovian example: some Markov measure does not lift to a Markov measure:



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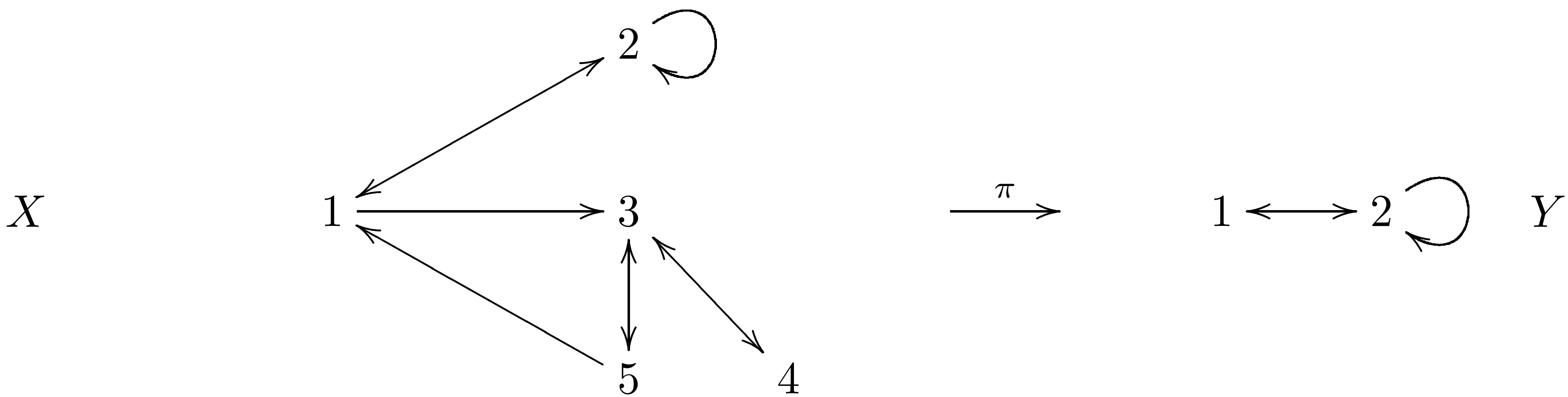

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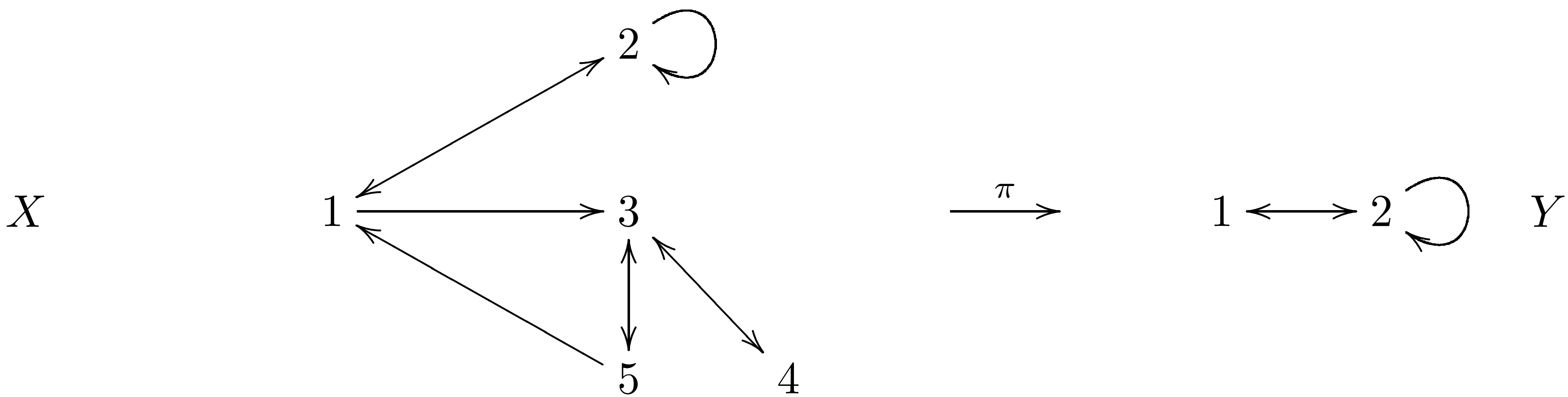

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MPW: Blackwell-type example of a metrically sofic ν on Y that is not the

finite-to-one image of any Markov measure of any order anywhere.

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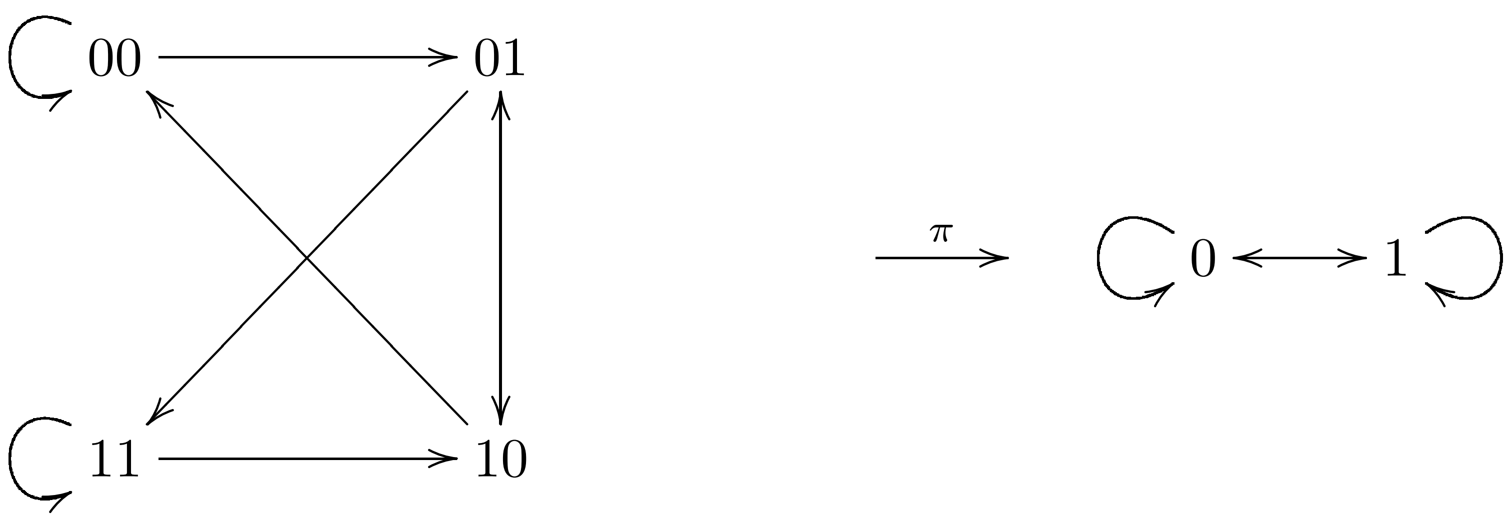
2-block recoding:



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- Bernoulli $1/2, 1/2$ measure on Σ_2 is mapped to itself.
- Every Markov ν on Y has a unique relatively maximal lift (in fact unique preimage), which is Markov
- For every ergodic ν on Y, *all* of $\pi^{-1}\{\nu\}$ consists of relatively maximal measures over ν, all having the same entropy as ν.

Walters—3

- If $p \neq 1/2$, the two measures on X that correspond to $\mathcal{B}(p, 1-p)$ and $\mathcal{B}(1-p, p)$ both map to ν_p on Y, which is fully supported.

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- So this potential function $V_p \circ \pi$ has many equilibrium states.

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Thus $h(\mu^1) > h(\mu)$, while $h(\pi\mu^1) < h(\pi\mu)$.

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$$P(\pi, V)(y) = \limsup_{n \to \infty} \frac{1}{n} \log \Bigg[\sum_{x \in D_n(y)} \exp \Big(\sum_{i=0}^{n-1} V(\sigma^i x) \Big) \Bigg].$$

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$$P(\pi, 0)(y) = \limsup_{n \to \infty} \frac{1}{n} \log \left| D_n(y) \right|$$

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In particular, for a fixed $\nu \in M(Y)$,

$$\sup\{h_\mu(X|Y) : \pi\mu = \nu\} = \sup\{h(\mu) - h(\nu) : \pi\mu = \nu\} = \int_Y P(\pi, 0)\, d\nu.$$

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Then for each $V \in C(Y)$,

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Thus, we obtain the value of $P(\pi, V)(y)$ a.e. with respect to every invariant measure on Y if we delete from the definition of $D_n(y)$ the requirement that $x \in \pi^{-1}(y)$.

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F takes into account, for all potential functions V on Y at once, the extra freedom, information, or free energy available in X as compared to Y because of the ability to move around in fibers over points of Y.

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- $\pi(\max_X) = \max_Y$ if and only if there is a *constant* compensation function.
- If $G \in \mathcal{F}(Y)$ (Walters class), then $G \circ \pi$ is a (*saturated*) compensation function if and only if there is $c > 0$ such that

$$\frac{1}{c} \leq e^{S_n G(y)} \left|\pi^{-1}[y_0 \dots y_{n-1}]\right| \leq c \text{ for all } y, n.$$

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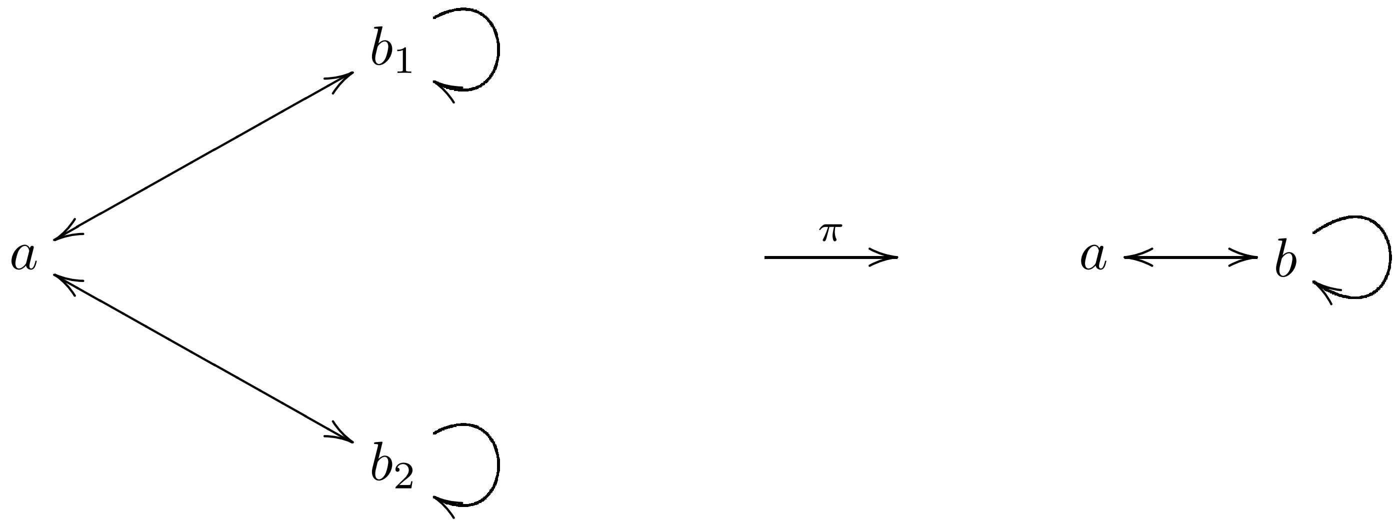

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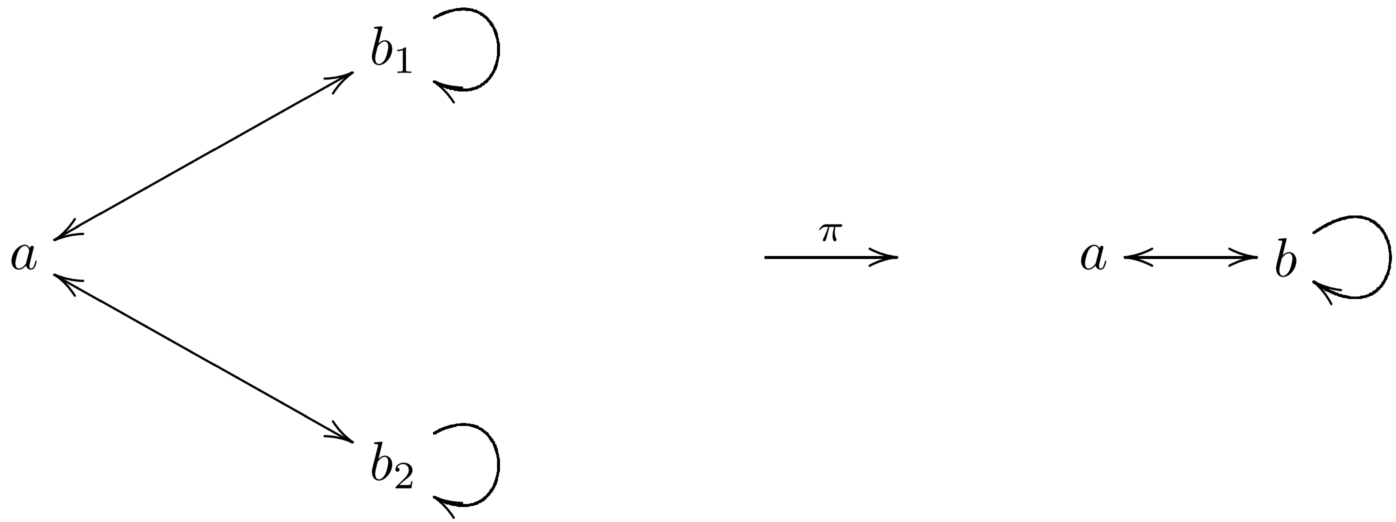

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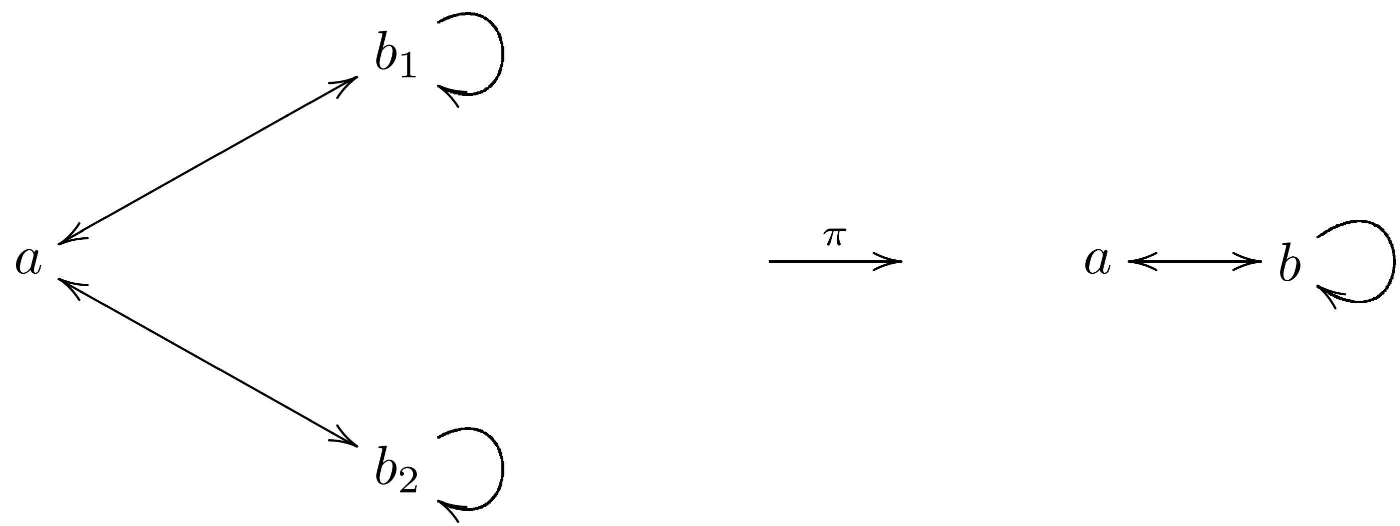

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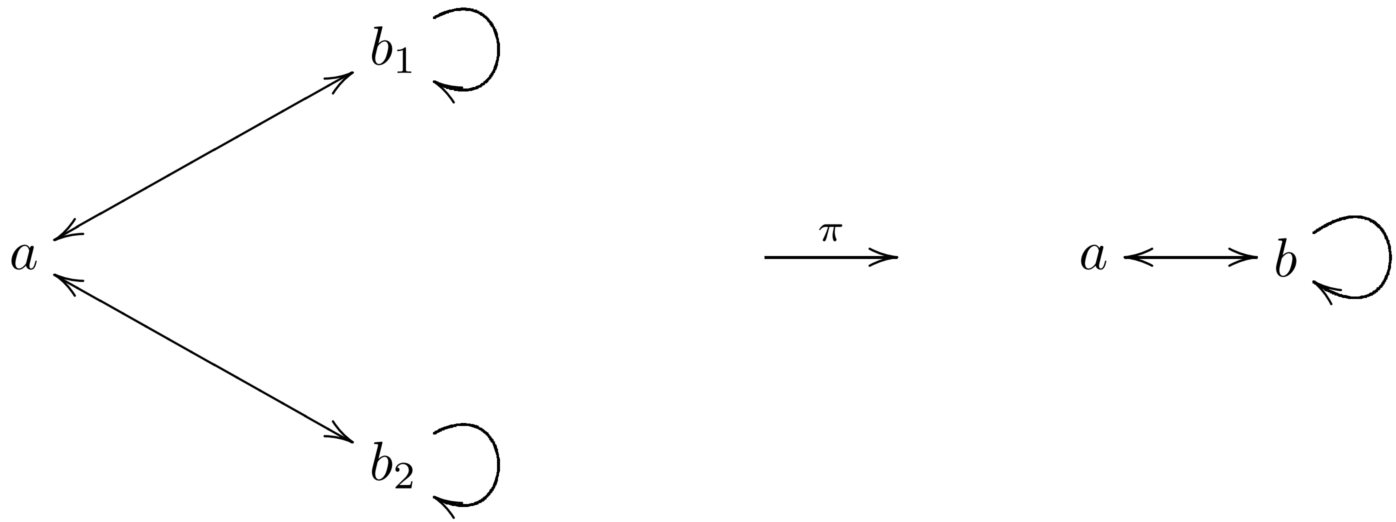

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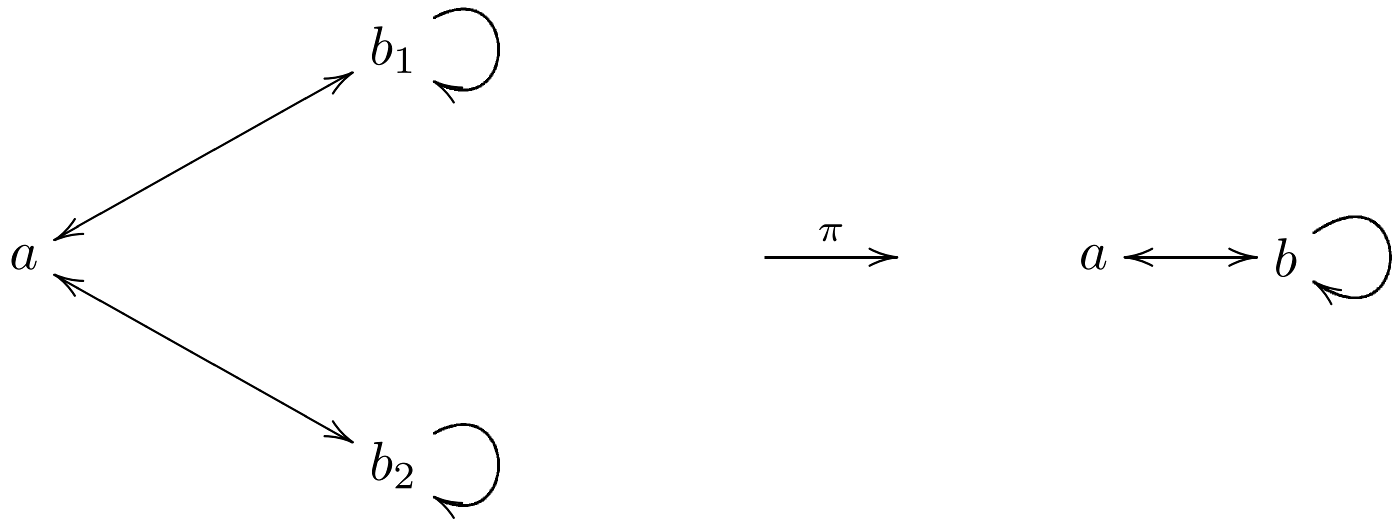

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When in y we see $ab^{k_1}ab^{k_2}a\ldots ab^{k_r}a$, multiply in: 1 at each b, 2 at each a .

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Theorem (Shin)**.** *Suppose that* $\nu \in \mathcal{E}(Y)$ *and* $\pi\mu = \nu$*. Then* μ *is relatively maximal over* ν *if and only if there is* $V \in \mathcal{C}(Y)$ *such that* μ *is an equilibrium state of* $V \circ \pi$*.*

Lifting Markov Measures

- If there is a *locally constant* saturated compensation function $G \circ \pi$, then every Markov measure on Y has a unique relatively maximal lift, which is Markov, because then the relatively maximal measures over an equilibrium state of $V \in \mathcal{C}(Y)$ are the equilibrium states of $V \circ \pi + G \circ \pi$ (Walters).

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- Further, μ_X is the unique equilibrium state of the potential function 0 on X; and the relatively maximal measures over μ_Y are the equilibrium states of $G \circ \pi$.

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Theorem (P-Quas-Shin)**.** *For each ergodic* ν *on* Y*, any two distinct ergodic measures on* X *of maximal entropy in the fiber* $\pi^{-1}\{\nu\}$ *are* relatively orthogonal*.*

Relatively Independent Joining

For $\mu_1, \ldots, \mu_n \in \mathcal{M}(X)$ with $\pi\mu_i = \nu$ for all i, their *relatively independent joining* $\hat{\mu}$ over ν is defined by:

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There is zero probability of coincidence.

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Let b be a symbol in the alphabet of Y such that b has $N_\nu(\pi)$ preimages

$a_1, \ldots, a_{N_\nu(\pi)}$ under the block map π.

Pigeonholing

Since $n > N_\nu(\pi)$, for every $\hat{x} \in \phi^{-1}[b]$ there are $i \neq j$ with $(p_i \hat{x})_0 = (p_j \hat{x})_0$.

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(If you have more measures than preimage symbols, two of those measures have to coincide on one of the symbols: with respect to each measure, that symbol a.s. appears infinitely many times in the same place.)

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Idea of the proof of the second theorem.

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We concatenate words from the two processes, using the the fact that the two measures are supported on sequences that agree infinitely often. Since X is a 1-step SFT, we can switch over whenever a coincidence occurs.

Weaving In More Entropy

Let $w \in \mathcal{B}(1/2, 1/2)$, symbols 1 and 2.

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$\pi_3 : X \times X \times \mathcal{B}(1/2, 1/2) \to X$,

$$\pi_3(u, v, w) = \ldots (u_s u_{s+1} \ldots u_{t-1})(v_t v_{t+1} \ldots v_{r-1})(u_r u_{r+1} \ldots) \ldots$$

Why Does It Go Up?

The switching increases entropy.

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The argument uses

- strict concavity of $-t \log t$
- lots of calculations with conditional expectations .

V. Recognizing the hidden Markov measures

1. Identify images of Markov measures (metrically sofic, hidden Markov). Heller, Robertson, Furstenberg, Binkowska-Kaminski.

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$\phi(\epsilon) = 1, \phi(y_1 \dots y_n) = \nu[y_1 \dots y_n]$ extends to linear functional on $\mathcal{A}$.

Metrically Sofic vs. Finitary

$\mathcal{N}$=largest left ideal in kernel$(\phi) = \{a \in \mathcal{A} : \phi(wa) = 0$ for all $w \in A^*\}$

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Furstenberg: Characterization of metrically sofic in terms of finite-dimensionality of a related algebra by a different left ideal.

2. Formal languages characterization

Kleene, Schützenberger, Hansel-Perrin, etc.

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$$(s_1 s_2)(w) = \sum_{u,v \in A^*, uv=w} s_1(u)\, s_2(v).$$

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A submodule $M \subset \mathcal{F}(A)$ is *stable* if $w^{-1}M \subset M$ for all $w \in A^*$.

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4. μ is the image under a 1-block map of a 1-step Markov measure.

VI. Measures of Maximal Hausdorff Dimension

Find measures of maximal Hausdorff dimension for expanding (not necessarily conformal) maps on manifolds restricted to compact invariant sets.

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So in some cases they are unique, Bernoulli, etc.

Carpets

Ledrappier-Young:

$$\mathrm{HD}(\mu) = \frac{h_\mu(f)}{\lambda_\mu^1(f)} + \left[\frac{1}{\lambda_\mu^2(f)} - \frac{1}{\lambda_\mu^1(f)}\right] h_{\pi\mu}(f_*)$$

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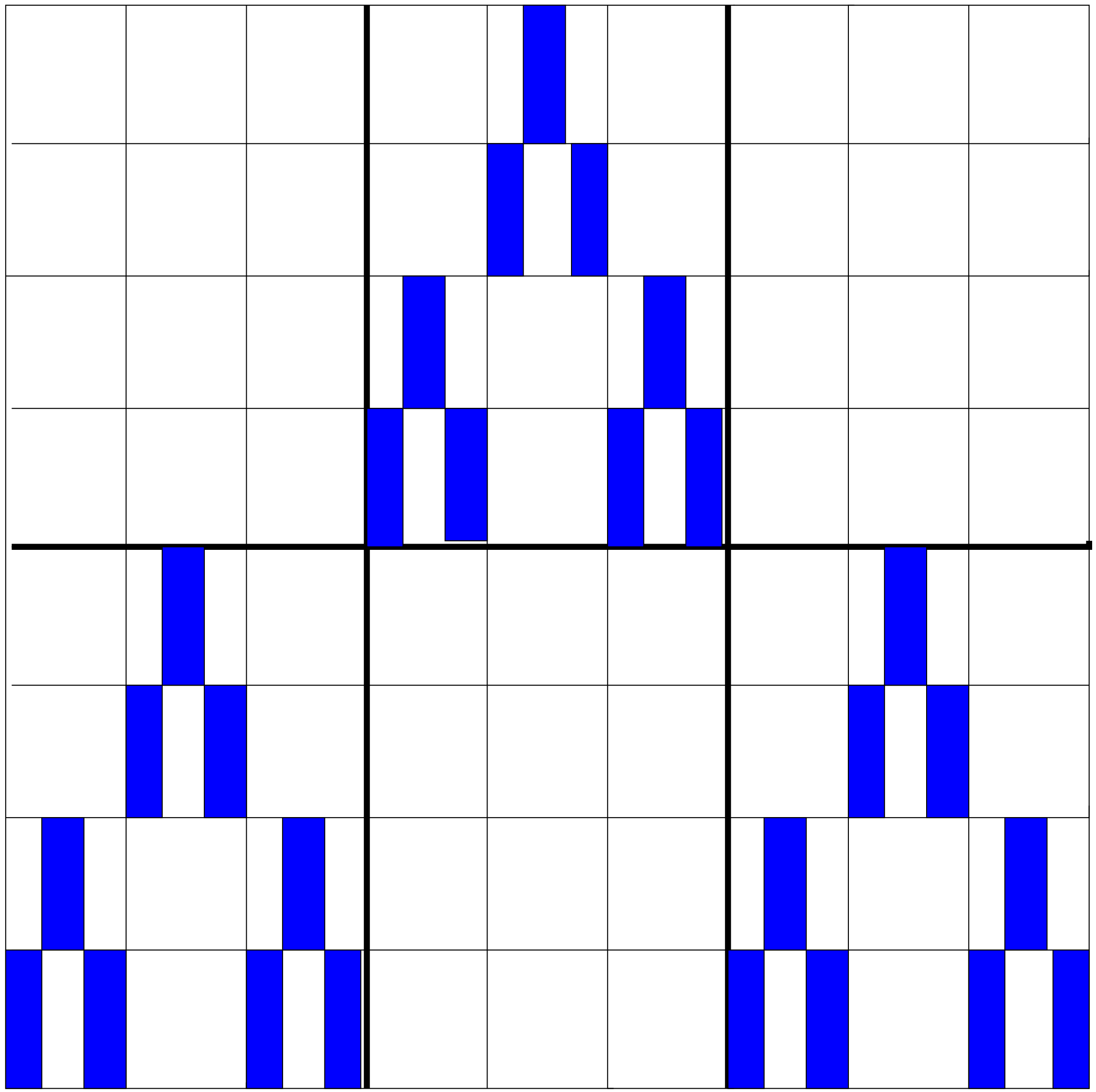
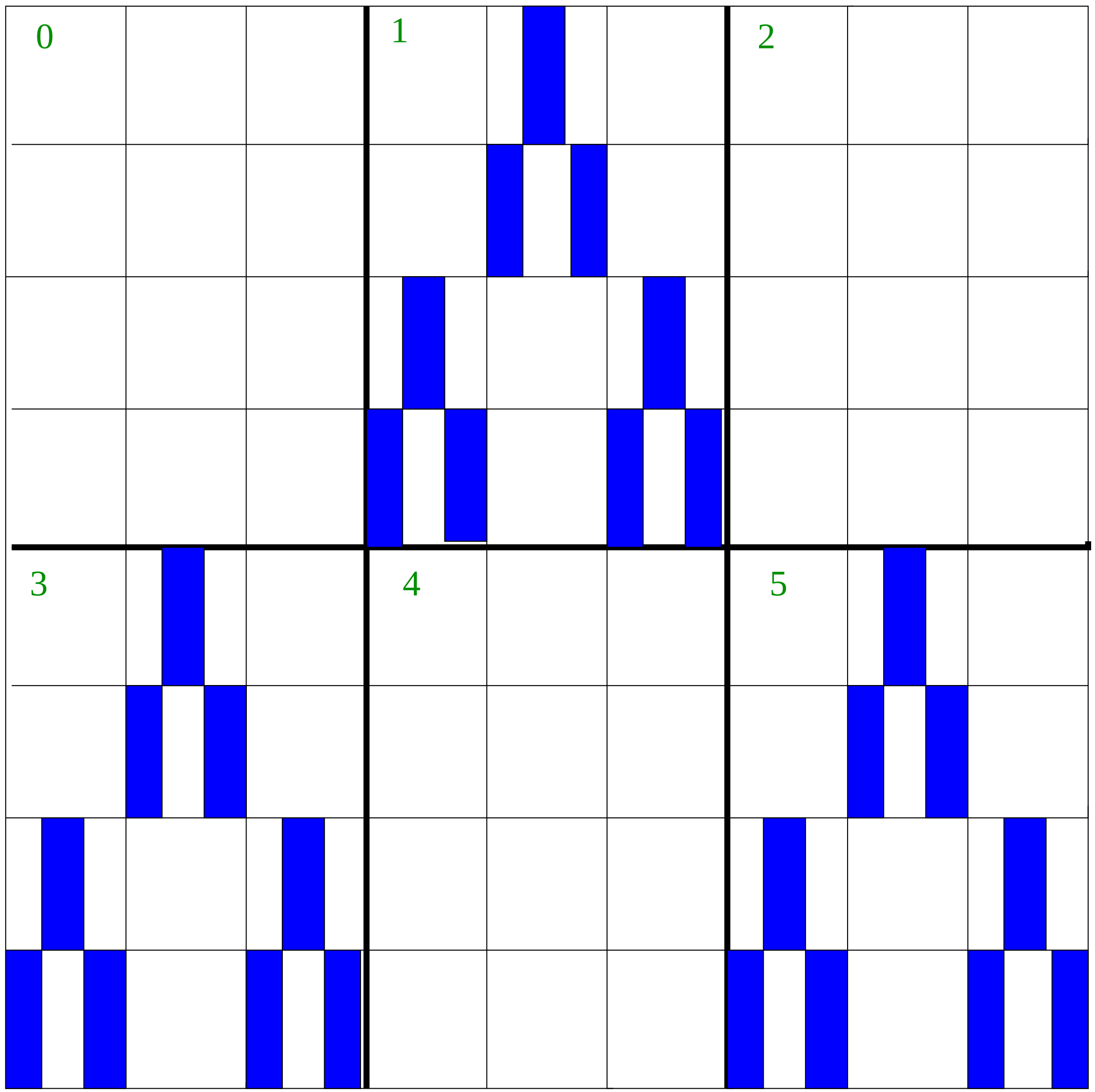
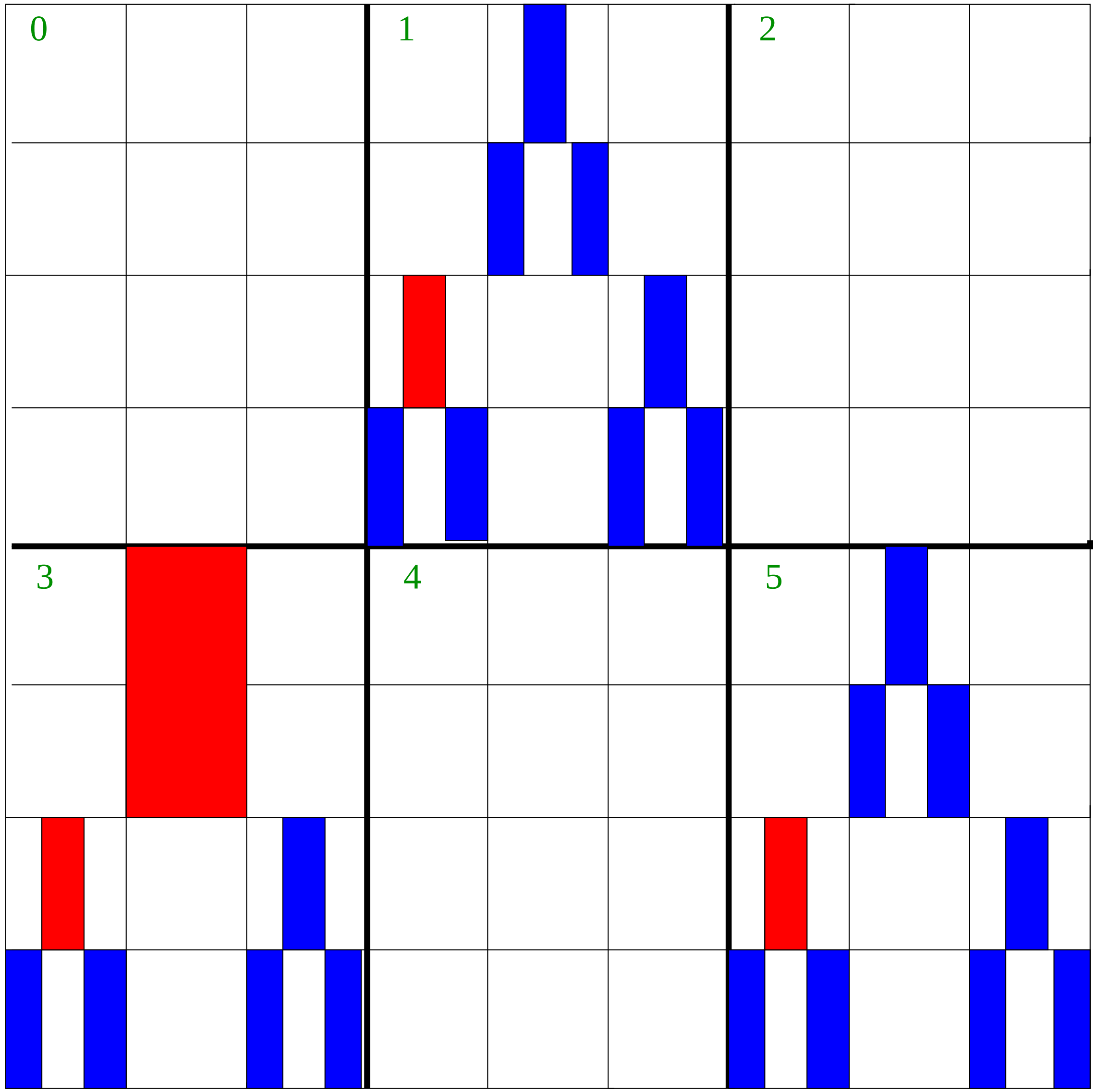
Nonconformal Carpet



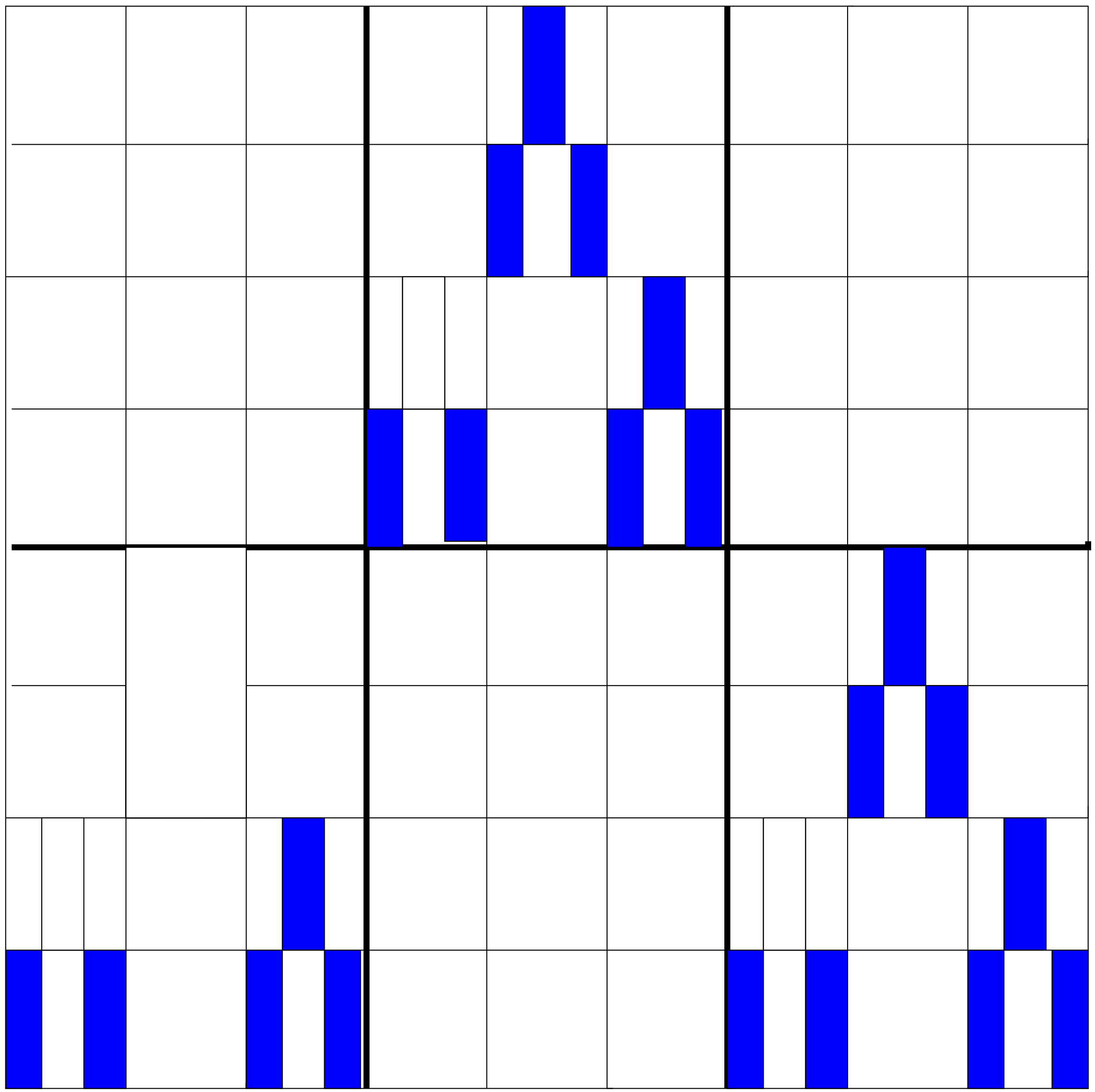
Nonconformal Carpet Coded



Disallow some transitions 31

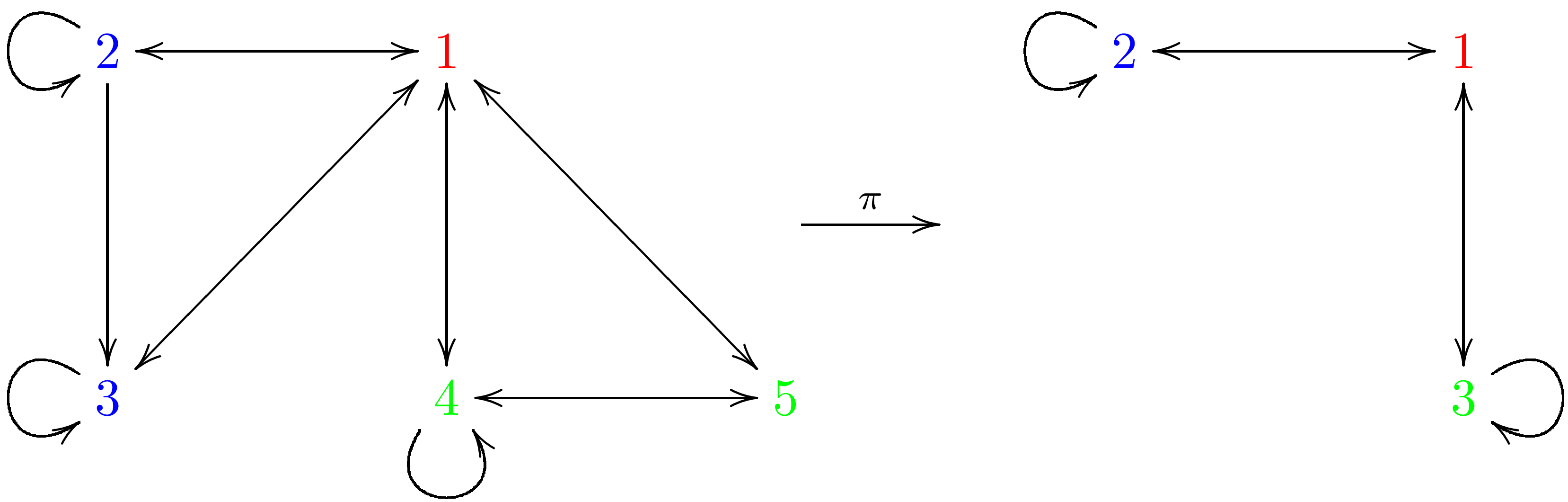


More worn carpet



A candidate for nonuniqueness

$\pi(1) = 1, \pi(2) = \pi(3) = 2, \pi(4) = \pi(5) = 3.$



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Construct them as weak* limits of well-distributed measures on periodic orbits?