An adic dynamical system related to the Delannoy numbers

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Outline

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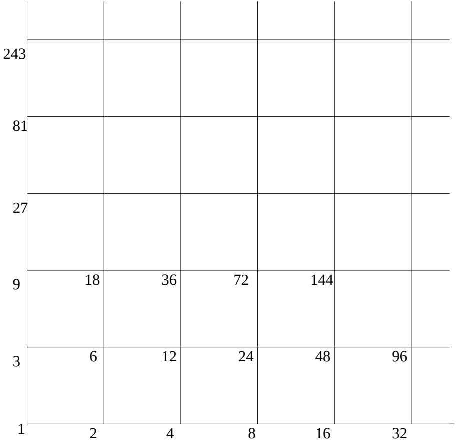
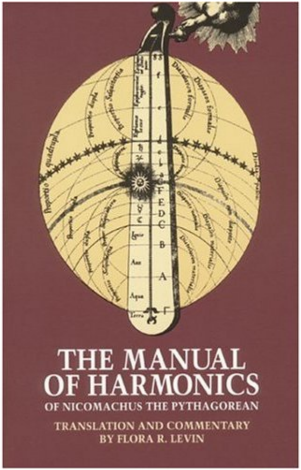
The first part of the Nicomachus diagram



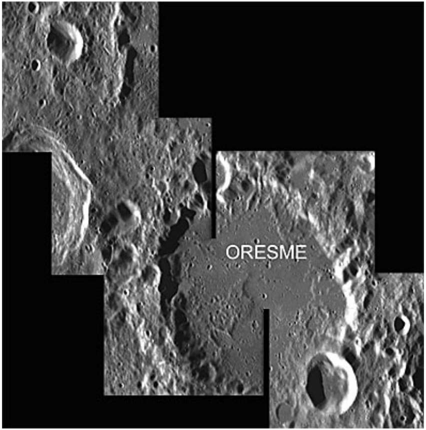
Nicomachus of Gerasa, c. 100

Nicomachus on Music

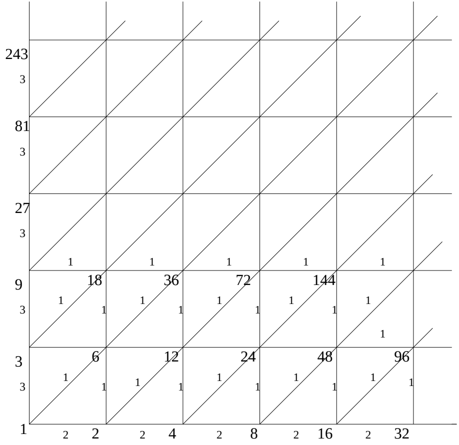


Nicole Oresme, c. 1350

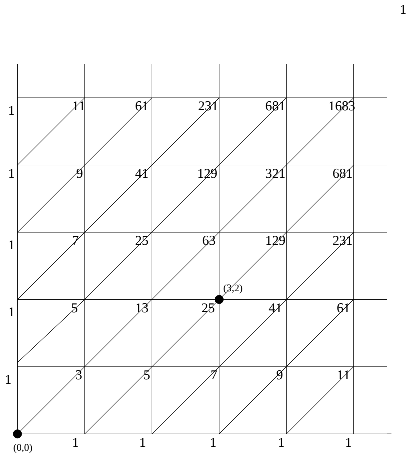
Oresme



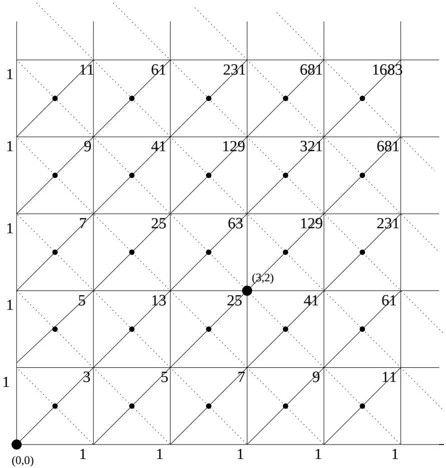
The Nicomachus diagram with added diagonals



The Delannoy graph



The Delannoy graph made into a Bratteli diagram



Recurrence formula and generating function for Delannoy numbers

- $$\begin{aligned} D(n,0) = D(0,n) &= 1 \text{ for all } n \geq 0; \\ D(n,k) &= 0 \text{ if either } n \text{ or } k < 0; \\ D(n,k) = D(n,k-1) + D(n-1,k-1) + D(n-1,k) &\text{ for all } n,k. \end{aligned}$$

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- $$\sum_{n,k\geq 0} D(n,k)x^n y^k = \frac{1}{1-(x+y+xy)}$$

Various formulas for Delannoy numbers

Assuming $n \geq k$,

- $$D(n,k) = \sum_{d=0}^{k} \binom{k}{d} \binom{n+k-d}{k} = \sum_{d=0}^{k} 2^d \binom{n}{d} \binom{k}{d}$$

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- $$= \sum_{d=0}^{k} \binom{n+k-d}{k-d}\binom{n}{d} = \sum_{d=0}^{k} \binom{n+d}{d}\binom{n}{k-d}.$$

Asymptotics of Delannoy numbers on the diagonal

$$D(n,n) \sim (3+2\sqrt{2})^n(.57\sqrt{n} - .067n^{-3/2} + .006n^{-5/2} + \dots).$$

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- Then $x < y$ if $e_N(x) < e_N(y)$.
- Tx = smallest $y > x$ (if there is one).

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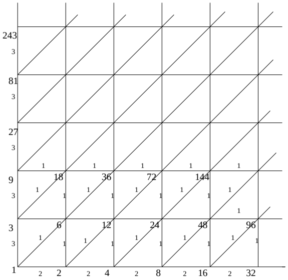

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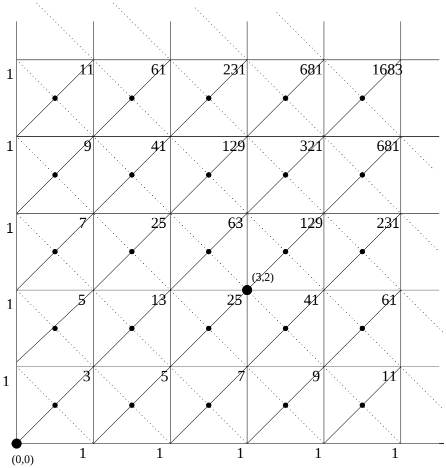
Theorem

The non-atomic ergodic (invariant probability) measures for the Delannoy adic dynamical system are a one-parameter family $\{\mu_\alpha : \alpha \in [0, 1]\}$ *given by choosing nonnegative* α, β, γ *with* $\alpha + \beta + \gamma = 1$ *and* $\beta\gamma = \alpha$ *and then putting weight* β *on each horizontal edge, weight* γ *on each vertical edge, and weight* α *on each diagonal edge. (The measure of any cylinder set is then determined by multiplying the weights on the edges that define it.)*

The Delannoy adic



Ingredients of the proofs

- Pemantle-Wilson asymptotics for the Delannoy numbers:

$$D(n,k) \sim \left(\frac{\sqrt{n^2+k^2}-k}{n} \right)^{-n} \left(\frac{\sqrt{n^2+k^2}-n}{k} \right)^{-k} \times$$
$$\sqrt{\frac{1}{2\pi}} \sqrt{\frac{nk}{(n+k-\sqrt{n^2+k^2})^2 \sqrt{n^2+k^2}}},$$

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- X. Méla's isotropy argument

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With respect to each of its ergodic (invariant probability) measures, the Delannoy adic dynamical system is totally ergodic (i.e., has among its eigenvalues no roots of unity besides 1*).*

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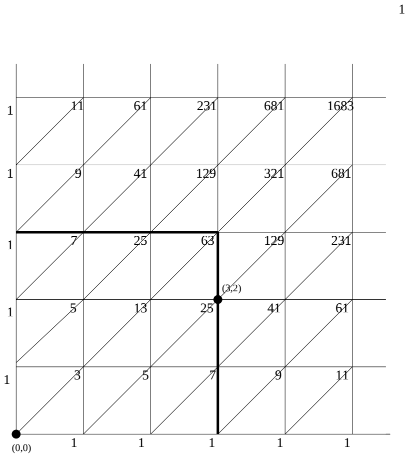
For p *prime,* $r \geq 0$*, and* $n = 0, 1, 2, \ldots,$

$$D(n, p^r - 1) \equiv_p (-1)^{(n \mod p^r)}.$$

The Delannoy graph with a “blocking set”



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- We do not know about limit laws for return times, weak mixing, multiplicity of the spectrum, or joinings.
- But there is some progress on the complexity and on generalizing these considerations to a class of systems.