CONSTRUCTIVE SYMBOLIC PRESENTATIONS OF
RANK ONE MEASURE-PRESERVING SYSTEMS

BY

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Abstract. Given a rank one measure-preserving system defined by cutting and stacking with spacers, we produce a rank one binary sequence such that its orbit closure under the shift transformation, with its unique nonatomic invariant probability, is isomorphic to the given system. In particular, the classical dyadic odometer is presented in terms of a recursive sequence of blocks on the two-symbol alphabet $\{0, 1\}$. The construction is accomplished using a definition of rank one in the setting of adic, or Bratteli–Vershik, systems.

1. Introduction. *Rank one* measure-preserving systems were introduced in [1], and got their present name in [17]; they have been extensively studied, including recent developments such as [13, 12, 14, 2, 3]. The survey [8] tried to collate the various definitions of rank one, but it has two shortcomings: first, it ignores the theory of *adic systems* (see for example [18], and, for background, [6]), though rank one systems have a particularly simple and useful adic presentation (which, to our knowledge, has never appeared in print). Then, the nicest (and most used in recent papers) definition of rank one, called the constructive symbolic definition in the survey, was not known at the time to be equivalent to the others, because the *odometers*, which are rank one systems by all the other definitions, do not satisfy it (see the discussion in Section 3 below).

The present note aims to fill both of these gaps, by giving one more definition of rank one systems, the *adic definition*, and then by using it to show that any rank one system is measure-theoretically isomorphic to a rank one system with an extra property we call *essential expansiveness*, which in turn implies that it satisfies the constructive symbolic definition.

Rank one systems, as described in Definition 2.1 below, were modified, in a different context, by the second author in [9, Theorem 6.3] and [10, The-

[1]

orem 4.8], by replacing, at suitably rare steps n, a relatively small number of n-towers by stacks of spacers of the same height, and this technique may well have been known before. In 2013 the use of this kind of construction to give a symbolic constructive model in the particular case of odometers was discussed by the second author with Aaron Hill, but nothing was written. In [2, Remark 2.10] it is stated that each rank one system is isomorphic to one with $a_{n,q_n-1} > 0$ for infinitely many n. Indeed, combining the proof of [2, Theorem 2.8] with [4] and [16, Appendix], one may produce an isomorphism with a constructive symbolic rank one system. Here, we show explicitly how to build for each rank one $\mathbb{Z}$-action an isomorphic system that is essentially expansive, i.e., satisfies the constructive symbolic definition of rank one. Our new system will satisfy the hypothesis of Proposition 5.2, a stronger condition than the one in [2, Remark 2.10].

2. Geometric and symbolic definitions. Henceforth we take as definition of a rank one system the constructive geometric definition of [8]. Note that the property of being rank one is invariant under measure-theoretic isomorphism; indeed, it is equivalent to building our system on the unit interval by cutting and stacking in a certain way, and allowing such isomorphisms. By a *measure-preserving system* we mean a nonatomic Lebesgue probability space together with an invertible measure-preserving transformation.

DEFINITION 2.1. A measure-preserving system $(X, \mathcal{A}, T, \mu)$ is *rank one* if there exist sequences of positive integers q_n, $n \in \mathbb{N} \cup \{0\}$, with $q_n > 1$ for infinitely many n, and $a_{n,i}$, $n \in \mathbb{N} \cup \{0\}$, $0 \leq i \leq q_n - 1$, such that, if h_n are defined by

$$(2.1) \qquad h_0 = 1, \quad h_{n+1} = q_n h_n + \sum_{i=0}^{q_n-1} a_{n,i},$$

then

$$(2.2) \qquad \sum_{n=0}^{\infty} \frac{h_{n+1} - q_n h_n}{h_{n+1}} = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{q_n-1} a_{n,i}}{h_{n+1}} < \infty,$$

$$\text{or equivalently} \quad \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{q_n-1} a_{n,i}}{q_n h_n} < \infty,$$

and subsets of X, denoted by F_n, $n \in \mathbb{N} \cup \{0\}$, $F_{n,i}$, $n \in \mathbb{N}$, $0 \leq i \leq q_n - 1$, and $S_{n,i,j}$, $n \in \mathbb{N}$, $0 \leq i \leq q_n - 1$, $0 \leq j \leq a_{n,i} - 1$ (if $a_{n,i} = 0$ there are no $S_{n,i,j}$), such that for all n:

- $(F_{n,i}, 0 \leq i \leq q_n - 1)$ is a partition of F_n,
- the $T^k F_n$, $0 \leq k \leq h_n - 1$, are disjoint,

- $T^{h_n} F_{n,i} = S_{n,i,1}$ if $a_{n,i} \neq 0$,
- $T^{h_n} F_{n,i} = F_{n,i+1}$ if $a_{n,i} = 0$ and $i < q_n - 1$,
- $T S_{n,i,j} = S_{n,i,j+1}$ if $j < a_{n,i} - 1$,
- $T S_{n,i,a_{n,i}-1} = F_{n,i+1}$ if $i < q_n - 1$,
- $F_{n+1} = F_{n,0}$,

and the partitions $P_n = \{F_n, T F_n, \dots, T^{h_n-1} F_n, X \setminus \bigcup_{k=0}^{h_n-1} T^k F_n\}$ are increasing to $\mathcal{A}$.

The parameters q_n and $a_{n,i}$ determine the transformation T up to a set of measure 0, but of course for a given T several sets of parameters are possible (this is the point of the whole paper). The sets $T^k F_n$, $0 \leq k \leq h_n - 1$, are called the *levels* of the *n-towers* (or stacks). The sets $S_{n,i,j}$, $0 \leq i \leq q_n - 1$, $0 \leq j \leq a_{n,i} - 1$ are the *n-spacer* levels.

Note that if we had $q_n = 1$ ultimately, then because of the condition on h_n we should have also $a_{n,0} = 0$ for all n large enough, and then T would not be defined almost everywhere.

This construction determines naturally an associated constructive symbolic definition.

DEFINITION 2.2. A system is *constructive symbolic rank one* if it consists of the shift on the orbit closure of the sequence u defined as follows, with its unique nonatomic invariant measure. For each $n = 0, 1, \dots$ an initial block of u is B_n. The blocks B_n are defined recursively by concatenation:

$$(2.3) \qquad B_0 = 0, \qquad B_{n+1} = B_n 1^{a_{n,0}} B_n \dots B_n 1^{a_{n,q_n-1}} \qquad \text{for } n \geq 0.$$

The parameters $a_{n,i}$ are as in Definition 2.1 and are assumed to satisfy (2.1) and (2.2).

The symbols 1, corresponding to spacer levels, are called *spacers*.

3. Odometers. The classical *dyadic odometer*, also called the von Neumann–Kakutani adding machine, can be defined by the constructive geometric definition as a rank one system with $q_n = 2$ for all n, and $a_{n,i} = 0$ for all n and i; but then the blocks B_n on the alphabet $\{0, 1\}$ defined recursively by $B_0 = 0$, $B_{n+1} = B_n B_n$ lead to a trivial one-point symbolic dynamical system, while the phase space of the dyadic odometer is the unit interval. Thus this set of parameters q_n and $a_{n,i}$ do not provide a symbolic constructive definition of the odometer; in [8, p. 43] and [11, p. 124] it is stated that the dyadic odometer is not known to be rank one by the constructive symbolic definition.

Note that there exists a symbolic model for the dyadic odometer: namely, the shift on the orbit closure of the fixed point u of the substitution σ defined by $0 \to 01$, $1 \to 00$ (a Toeplitz sequence, called the period-doubling sequence

and constructed by García and Hedlund [15]) is measure-theoretically iso-
morphic to the dyadic odometer. As an aside, we show we cannot use it to
find a constructive symbolic definition for this rank one system:

PROPOSITION 3.1. *No sequence x in the orbit closure of u can be built
by blocks B_n as in Definition* 2.2.

Proof. Any sequence x in the orbit closure of u is the concatenation of the
blocks 0100 and 0101. Therefore spaces between appearances of 0100 in x are
multiples of 4 (there is no appearance of 0100 across a juncture of 0100 and
0101). Suppose $x = B1^{s_0}B1^{s_1}B\ldots$, where B is some block in a supposed
constructive symbolic rank one construction of x that is long enough to
contain 0100, and B starts and ends with the symbol 0. The s_i are bounded
because the sequence has minimal orbit closure. The sequence x does not
contain the block 11, so each s_i is 0 or 1. If both 0 and 1 appear among the
s_i, then we can find in x a stretch where the spaces between appearances
of B, and hence between appearances of 0100, are not all multiples of 4.
Therefore either $s_i = 0$ for all i or $s_i = 1$ for all i, and in both cases x is
periodic. ∎

The general odometers are rank one systems with $a_{n,i} = 0$ for all n and i.

4. Adic definition. We now give a definition of rank one which was
not known at the time of writing [8]; it is a translation of the constructive
geometric definition into the language of adic transformations defined on
paths in Bratteli diagrams.

DEFINITION 4.1. A system is *rank one* if it is measure-theoretically iso-
morphic to the following system, where the q_n and $a_{n,i}$ are as in Definition
2.1 and satisfy the sum condition (2.2).

Let $V = \{v_{-1}, v_{0,0}, v_{0,1}, \ldots, v_{n,0}, v_{n,1}, \ldots\}$ represent an infinite set of
nodes (or vertices). The integer n is called the *level* of the vertex $v_{n,i}$, $i = 0, 1$.
For each $n = -1, 0, 1, \ldots$ there are directed edges from level n to level $n+1$.
Let $e_{-1,0}$ be an edge connecting v_{-1} to $v_{0,0}$, and $e_{-1,1}$ be an edge connecting
v_{-1} to $v_{0,1}$. For $n \in \mathbb{N} \cup \{0\}$, let $e_{n,i}$ for $0 \le i < q_n$ be edges connecting $v_{n,0}$
to $v_{n+1,0}$. Let $e_{n,i,j}$ for $0 \le i < q_n$ and $0 \le j < a_{n,i}$ be edges connecting $v_{n,1}$
to $v_{n+1,0}$. Let d_n be a single edge connecting $v_{n,1}$ to $v_{n+1,1}$. Order the edges
entering $v_{n+1,0}$ as

$$\cdots \prec e_{n,i} \prec e_{n,i,0} \prec e_{n,i,1} \prec \cdots \prec e_{n,i,a_{n,i}-1} \prec e_{n,i+1} \prec \cdots .$$

Let E be the set of all edges, and let $G = \{V, E\}$ be the infinite directed
acyclic graph with the ordering defined above. Let X_0 be the set of infinite
paths beginning at the root node v_{-1}. The ordering of edges arriving at the
nodes defines a lexicographic ordering of infinite paths that start at v_{-1}: we
say that $x < y$ if there is an N such that $x_n = y_n$ for all $n > N$ and $x_N < y_N$.

If $X_{\max}$ denotes the set of maximal paths, then the adic transformation T on $X = X_0 \setminus X_{\max}$ is defined to be the map that assigns to each path its successor in this ordering.

A *cylinder set* is determined by a finite set E_0 of edges forming a path from the root vertex v_{-1} to a vertex v: it is the set of all the infinite paths in X that pass through all the edges in E_0. Let $\mathcal{A}$ be the σ-algebra generated by all cylinder sets. There is a unique probability measure that for each $v \in V$ assigns the same mass to all cylinder sets determined by paths from v_{-1} to v. The adic transformation T preserves this measure. Denote by P'_k the partition of X into the cylinder sets determined by paths from v_{-1} to vertices at level k.

We show how to translate the constructive geometric definition into the adic definition: for almost all x in the geometric system, either there is a unique $M = M(x) \geq 0$ such that x is in $S_{M,i,j}$ for some $0 \leq i = i(x) \leq q_M - 1$, $0 \leq j = j(x) \leq a_{M,i} - 1$, or else x is in F_0, in which case we take $M(x) = -1$. Then, for all $n > M$, x is in some $T^k F_{n,i}$ for some $0 \leq i = i(n,x) \leq q_n - 1$, which means that x is in the ith (from the bottom) copy of the n-tower inside the $n+1$-tower.

Then in the adic system x becomes the path (still denoted by x)

$$(e_{-1,1}, d_0, \ldots, d_{M-1}, e_{M,i(x),j(x)}, e_{M+1,i(M+1,x)}, e_{M+2,i(M+2,x)}, \ldots) \text{ if } M > 0,$$
$$(e_{-1,1}, e_{0,i(x),j(x)}, e_{1,i(1,x)}, e_{2,i(2,x)}, \ldots) \qquad\qquad\qquad\qquad \text{ if } M = 0,$$
$$(e_{-1,0}, e_{0,i(0,x)}, e_{1,i(1,x)}, e_{2,i(2,x)}, \ldots) \qquad\qquad\qquad\qquad \text{ if } M = -1.$$

On the other hand, in the adic system, for each $n \in \mathbb{N} \cup \{0\}$, define the cylinder set

$$(4.1) \qquad\qquad C_n = \{x : x(m) = e_{m,0}, \, m < n\}.$$

Then the h_n of Definition 2.1 is the number of paths from the root node v_{-1} to the node $v_{n,0}$, often called the *dimension* of the vertex $v_{n,0}$, and the basis F_n of the tower in Definition 2.1 is the cylinder C_n. For $n = 0$, the partition P'_0 is $\{C_0, D_1\}$, where $D_1 = C_0^c$ is the set of *spacer paths* that go through the vertex $v_{0,1}$, while C_0 is the set of *nonspacer paths* that do not.

What determines a point x in a rank one system is the sequence J_n of the numbers of the level where x lies in the n-tower. The adic definition allows us to compute them explicitly from the path x.

LEMMA 4.2. *Let x be defined by (4.1) with $M(x) = M$. Then for all $n > M$, x is in $T^{J_n(x)} C_n$ where*

$$J_0(x) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{ if } M = -1,$$
$$J_{M+1}(x) = (i(x)+1)h_M + a_{M,0} + \ldots + a_{M,i(x)-1} + j(x) \quad \text{ if } M \geq 0,$$
$$J_{n+1}(x) = i(n,x)h_n + a_{n,0} + \ldots + a_{n,i(n,x)-1} + J_n(x) \qquad \text{ if } n \geq M+1.$$

Proof. This is a direct consequence of the construction of the towers in Definition 2.1 and its adic translation above. ∎

5. Essential expansiveness. We look now at cases where the constructive symbolic definition of rank one is equivalent to the other definitions. For this, we use a notion of expansiveness which comes from the general theory of adic and topological dynamical systems (see for example [5]).

DEFINITION 5.1. For $k \in \mathbb{N} \cup \{0\}$, a rank one system as defined in Definition 2.1, resp. Definition 4.1, is *essentially k-expansive* if the partition P_k, resp. P'_k, of X generates the full σ-algebra under the transformation T.

When a rank one system as defined in Definition 4.1 is essentially 0-expansive, the symbolic system that results from coding the orbits of its paths according to their initial edges and taking the closure coincides with the symbolic system specified in Definition 2.2. With the same parameters in each case, it is measure-theoretically isomorphic to both the adic system (X, T) of Definition 4.1 and a measure-preserving system satisfying Definition 2.1.

A necessary and sufficient condition for essential 0-expansiveness of rank one systems, due to Kalikow [16, Appendix], is that the sequence u in Definition 2.2 is not periodic. This in turn is realized for example when the parameters in Definition 2.1 or 4.1 satisfy

$$(5.1) \quad \sup_{0 \le m \le n, \, 0 \le i \le q_{n+1}-1} a_{m,q_m-1} + a_{m+1,q_{m+1}-1} + \cdots + a_{n,q_n-1} + a_{n+1,i} = \infty,$$

as this implies u has unbounded strings of 1, while it has symbols 0 infinitely often because q_n is at least 2 for infinitely many n.

Here, in order to make the present paper more self-contained, we introduce a stronger sufficient condition, allowing an elementary proof of essential expansiveness.

PROPOSITION 5.2. *A rank one system as defined in Definition* 2.1 *or* 4.1 *is essentially 0-expansive if $a_{n,q_n-1} > a_{n,i}$ for all n and $i < q_n - 1$.*

Proof. Let $\mathcal{D}$ be the smallest S-invariant σ-algebra that contains C_0 and D_1; we shall show that it contains the cylinder set C_n of Section 4 for all n, thus all cylinder sets, and therefore equals the full σ-algebra of Y.

Indeed, C_0 is in $\mathcal{D}$. Suppose that $n \ge 0$ and C_n is in $\mathcal{D}$.

Let $W_1 = T^{h_n} C_n \cap D_1$. For i such that $0 \le i < a_{n,q_n-1}$, define $W_{i+1} = TW_i \cap D_1$. Then, by our hypothesis, $W_{a_{n,q_n-1}}$ is exactly $T^{h_{n+1}-1} C_{n+1}$. As all the W_i are in $\mathcal{D}$, so is C_{n+1}, which completes our induction. ∎

Note that our proposition gives a sufficient condition for essential expansiveness, but it is not necessary: the so-called Chacon map given in Definition 2.2 by the recursion rule $B_{n+1} = B_n B_n 1 B_n$ does not satisfy it,

but, as u is not periodic, the system given by Definition 2.1 with the same parameters is essentially expansive.

Note that if $q_n = 2$ and there is a single spacer, essential expansivity implies that $a_{n,q_n-1} = a_{n,1} > a_{n,0}$ at least for infinitely many n; otherwise, the recursion rule is $B_{n+1} = B_n 1 B_n$ ultimately, and u is periodic while the corresponding system in Definition 2.1 is an odometer. Indeed, it is proved in [7, Lemma 10] that if the sequence u as in Definition 2.2 is periodic, then the corresponding rank one system from Definition 2.1 is isomorphic to an odometer.

6. Essentially expansive construction. We turn now to our main objective, the construction of an essentially expansive model (and thus of a constructive symbolic model) for every rank one transformation. We use the adic definition, which is well adapted to the building of isomorphisms.

THEOREM 6.1. *Every rank one system is measure-theoretically isomorphic to an essentially 0-expansive rank one system.*

The remainder of this section is devoted to the proof of the theorem.

6.1. Definition of the new system. We start from the rank one system (X, T) of Definition 2.1 or 4.1. We choose a strictly increasing sequence m_n with $m_0 = 0$ and such that if $H_n = h_{m_n}$ then

$$(6.1) \qquad \sum_{n=0}^{\infty} \frac{H_n}{H_{n+1}} < \infty.$$

We shall now look at the same system (X, T), but we build it by considering only the steps m_n. Thus the parameters q_n and $a_{n,i}$ are replaced by new parameters Q_n and $A_{n,i}$, which we compute now.

Namely, we have

$$(6.2) \qquad Q_n = \prod_{i=m_n}^{m_{n+1}-1} q_i.$$

We compute $A_{n,i}$ for $0 \le i \le Q_n - 1$. To do that, we write in a unique way

$$(6.3) \qquad i = g_0 + \sum_{j=1}^{m_{n+1}-m_n-1} g_j \prod_{k=0}^{j-1} q_{m_n+k}$$

with $0 \le g_j \le q_{m_n+j} - 1$. Then let l be such that $g_k = q_{m_n+k} - 1$ for all $k < l$ and $g_l \ne q_{m_n+l} - 1$, or $l = m_{n+1} - m_n - 1$ if this last inequality is

never satisfied; then we have

$$(6.4) \qquad A_{n,i} = \sum_{k=0}^{l} a_{m_n+k,g_k}.$$

Thus, the H_n are indeed the heights of the stacks and the $A_{n,i}$ the numbers of spacers for the system (X,T) after collapsing the stages between $m_n + 1$ and $m_{n+1} - 1$. As (X,T) preserves a finite measure, the equivalent of condition (2.2) still applies, thus

$$(6.5) \qquad \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{Q_n-1} A_{n,i}}{H_{n+1}} < \infty.$$

We consider the rank one system (X,T), as defined by the parameters Q_n and $A_{n,i}$, from Definition 4.1; its edges are denoted as in that definition, by d_n, $e_{n,i}$ and $e_{n,i,j}$.

Let $\overline{A}_n = \max\{A_{n,i} : 0 \le i \le Q_n - 1\}$. Choose $0 \le i_n \le Q_n - 1$ to be the last i such that $A_{n,i} + H_n + A_{n,i+1} + H_n + \cdots + A_{n,Q_n-1} > \overline{A}_n$, namely

$$(6.6) \qquad \overline{H}_n = (Q_n - i_n - 1)H_n + \sum_{i \ge i_n} A_{n,i} > \overline{A}_n,$$

but

$$(6.7) \qquad (Q_n - i_n - 2)H_n + \sum_{i > i_n} A_{n,i} \le \overline{A}_n.$$

We define now a new rank one system (Y,S) by modifying the system (X,T) (now built with the towers at stages m_n) in the following way: in the building of the tower at stage m_{n+1} by subsets of the tower at stage m_n and spacers, we replace the upper $Q_n - i_n - 1$ of these subtowers by spacers. The index i_n has been chosen such that this change is the smallest one allowing the new system to satisfy Proposition 5.2, and we shall show below that the change is small enough, first to allow the new system to preserve a finite measure, and then to allow it to be isomorphic to the initial system.

We define now this system precisely, using the adic formalism.

The set of vertices of our Bratteli diagram is $\{u_{-1}, u_{i,0}, u_{i,1} : i \in \mathbb{N} \cup 0\}$.

Let $r_{-1,0}$ be a directed edge from u_{-1} to $u_{0,0}$, and $r_{-1,1}$ from u_{-1} to $u_{0,1}$. Let d_n be a single edge connecting $u_{n,1}$ to $u_{n+1,1}$. For $i \le i_n$, let $r_{n,i}$ be edges from $u_{n,0}$ to $u_{n+1,0}$; for $i > i_n$, there is no edge $r_{n,i}$; thus there are $i_n + 1$ edges $r_{n,i}$.

Let $s_{n,i,j}$ be edges connecting $u_{n,1}$ to $u_{n+1,0}$ for $0 \le j < A_{n,i}$ and $i < i_n$. Let $s_{n,i_n,j}$ be edges connecting $u_{n,1}$ to $u_{n+1,0}$ for $0 \le j < \overline{H}_n$.

Order the edges entering $u_{n+1,0}$ in the following manner (with obvious modifications if i_n is the last element of U_n):

- $r_{n,i} \prec s_{n,i,0} \prec s_{n,i,1} \prec \cdots \prec s_{n,i,A_{n,i}-1} \prec r_{n,i+1}$ for $i < i_n$;
- $r_{n,i_n} \prec s_{n,i_n,0} \prec s_{n,i_n,1} \prec \cdots \prec s_{n,i_n,\overline{H}_n-1}$.

This ordering on the Bratteli diagram defines an invertible transformation S on the set of all nonmaximal paths Y. This new system is rank one, and we get its parameters by replacing Q_n by $Q'_n = i_n + 1$, $A_{n,i}$ by $A'_{n,i} = A_{n,i}$ if $i < i_n$, $A'_{n,i_n} = \overline{H}_n$, the $A'_{n,i}$ being defined for $0 \le i < Q'_n$. The heights of the stacks remain the same, $H'_n = H_n$. As in Section 4, we define quantities $M'(x)$ and $J'_n(x)$.

We have

$$(6.8) \qquad \sum_{i=0}^{Q'_n-1} A'_{n,i} = \sum_{i=0}^{Q_n-1} A_{n,i} + (Q_n - i_n - 1)H_n$$

$$\le \sum_{i=0}^{Q_n-1} A_{n,i} + \overline{A}_n + H_n \le 2\sum_{i=0}^{Q_n-1} A_{n,i} + H_n.$$

Since $\sum_{n=0}^{\infty} H_n/H_{n+1} < \infty$ by construction, condition (6.5) implies that

$$(6.9) \qquad \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{Q'_n-1} A'_{n,i}}{H'_{n+1}} < \infty.$$

Thus there is a unique invariant nonatomic probability measure for the rank one transformation S which assigns the same measure to all cylinder sets ending at a given vertex in the left column.

Observe that S is essentially 0-expansive by Proposition 5.2, as its last stacks of spacers are of height $\overline{H}_n$, while the others are of height at most $\overline{A}_n$ for all n. Now we show that S is isomorphic to T.

6.2. Definition of the isomorphism. Let E_n denote the set of $x \in X$ which are either in the n-spacers or in the top $\overline{H}_n$ levels of the $n+1$-tower; namely E_n is the set of x such that either

- $x(n) = e_{n,i,j}$ for some i, j, or
- $x(n) = e_{n,i}$ for some $i > i_n$.

Let

$$(6.10) \qquad E = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} E_n,$$

$$(6.11) \qquad \overline{E} = \bigcup_{k=-\infty}^{\infty} T^k E.$$

Now E_n is made with exactly $\sum_{i=0}^{Q_n-1} A_{n,i} + (Q_n - i_n - 1)H_n$ iterates of C_{n+1},

and we have $(Q_n - i_n - 2)H_n \leq \overline{A}_n$, thus

$$(6.12) \qquad \mu(E_n) \leq \frac{\sum_{i=0}^{Q_n-1} A_{n,i} + \overline{A}_n + H_n}{H_{n+1}}.$$

Because of (6.5) and $\sum_{n=0}^{\infty} H_n/H_{n+1} < \infty$ we have $\sum_{n=0}^{\infty} \mu(E_n) < \infty$ and

$$(6.13) \qquad \mu(E) = \mu(\overline{E}) = 0.$$

If $x \notin \bigcup_{n=0}^{\infty} E_n$ (i.e., x is in no E_n), define $N(x) = -1$; otherwise, for $x \notin E$, define

$$(6.14) \qquad N(x) = \max\{n : x \in E_n\}.$$

Note that, by definition of the E_n, $N(x) \geq M(x)$ (of Section 4). Hence $J_{N(x)+1}(x)$ exists and can be computed from x by Lemma 4.2.

Now we define our isomorphism $\phi : X \to Y$. For $x \in X \setminus E$, define $y = \phi(x)$ as follows: if $N(x) = -1$, define $y(-1) = r_{-1,0}$ and

$$(6.15) \qquad y(n) = r_{n,i} \quad \text{whenever} \quad x(n) = e_{n,i}.$$

Otherwise, if $0 \leq N(x) = N < \infty$, define $y(-1) = r_{-1,1}$ and

$$y_n = \begin{cases} d_n & \text{for } 0 < n < N, \\ s_{N,i,j} & \text{if } x(N) = e_{N,i,j},\, i \leq i_N & \text{for } n = N, \\ s_{N,i_N,J_{N+1}(x)-H_{N+1}+\overline{H}_N} & \text{if } x(N) = e_{N,i},\, i > i_N & \text{for } n = N, \\ s_{N,i_N,J_{N+1}(x)-H_{N+1}+\overline{H}_N} & \text{if } x(N) = e_{N,i,j},\, i > i_N & \text{for } n = N, \\ r_{n,i} & \text{if } x(n) = e_{n,i} & \text{for } n > N. \end{cases}$$

6.3. Proof of the isomorphism. Let $N = N(x)$. From the definition of ϕ, $\phi(x)$ is never in the n-spacers if $N = -1$; otherwise the last n for which $\phi(x)$ is in the n-spacers is N. Thus

$$(6.16) \qquad M'(\phi(x)) = N(x).$$

We compute $J'_{N+1}(\phi(x))$: if $x(N) = e_{N,i,j}$, then, by the definition of ϕ, $J'_{N+1}(\phi(x)) = J_N(x)$; in the other cases, $\phi(x)(N) = s_{N,i_N,J_{N+1}(x)-H_{N+1}+\overline{H}_N}$, which implies that $\phi(x)$ is at distance

$$\overline{H}_N - 1 - (J_{N+1}(x) - H_{N+1} + \overline{H}_N) = H_{N+1} - 1 - J_{N+1}(x)$$

from the top of the $N + 1$-tower, while, by definition of J'_{N+1}, $\phi(x)$ is at distance $H'_{N+1} - 1 - J'_{N+1}(\phi(x))$ from that top. So because $H'_{N+1} = H_{N+1}$ we get

$$(6.17) \qquad J'_{N+1}(\phi(x)) = J_{N+1}(x).$$

For $n > N + 1$, neither x nor $\phi(x)$ can be in the top $\overline{H}_{n-1}$ levels in their respective n-towers. Then, by the definition of $\phi(x)(n)$ for $n \geq N + 1$ and

Lemma 4.2, we get

$$J'_n(\phi(x)) = J_n(x) \quad \text{for all } n > N(x) = M'(\phi(x)).$$
(6.18)

Now, $\phi(x_1) = \phi(x_2)$ implies $J'_n(\phi(x_1)) = J'_n(\phi(x_2))$ for all n larger than $M'(\phi(x_1)) \vee M'(\phi(x_2))$. Thus $J_n(x_1) = J_n(x_2)$ for all n large enough, and by the rank one property this implies $x_1 = x_2$. Thus ϕ is injective on $X \setminus E$.

Let E'_n denote the set of $y \in Y$ which are in the n-spacers, i.e. $E' = \bigcap_{k=0}^\infty \bigcup_{n=k}^\infty E'_n$. For y in $Y \setminus E'$, $M' = M'(y)$ is finite. Thus $y(M') = s_{M',i,j}$, and we can compute $J'_{M'+1}(y)$ by Lemma 4.2. Then $y = \phi(x)$ for the (unique by the injectivity proved above) x such that $J_{M'+1}(x) = J'_{M'+1}(y)$, and $x(n) = e_{n;i}$ whenever $y(n) = r_{n,i}$ for all $n > M'$. This x satisfies $N(x) = M'$ by construction. Thus

$$\phi(X \setminus E) = Y \setminus E'.$$
(6.19)

Finally, suppose x is in $\overline{E}$, so that both x and Tx are in E. Then, for $n > (N(x) \vee N(Tx)) + 1$, both x and $\phi(x)$ are in the n-tower and not in the topmost level (indeed not in the topmost $\overline{H}_{n-1}$ levels). Thus $J_n(Tx) = J_n(x) + 1$, $J'_n(S\phi(x)) = J'_n(\phi(x)) + 1$, and

$$J'_n(S\phi(x)) = J_n(Tx) = J'_n(\phi(Tx)),$$
(6.20)

for all n large enough. Thus ϕ is equivariant on $X \setminus \overline{E}$, which completes the proof.

6.4. Variant. A variation on the same method would be to replace only one tower by spacers: for each n we choose one i_n and throughout the above section we replace "all $i > i_n$" by "$i = i_n$". Then the isomorphism works in the same way, but essential expansiveness is proved using the machinery of Kalikow's result and condition (5.1), as the new system has strings of at least H_n spacers.

7. Back to the odometer. We can apply this construction to the dyadic odometer. We have $q_n = 2$, $a_{n,i} = 0$ for all n and i, $h_n = 2^n$, and choose $m_n = n(n+1)/2$, so that $H_{n+1} = 2^{n+1}H_n$. Then the odometer is built with $Q_n = 2^{n+1}$ and $A_{n,i} = 0$ for all n and i. This implies $\overline{A}_n = 0$, and $i_n = Q_n - 2$. We build the new system by replacing only the top subtower with spacers. Then the new system admits a constructive symbolic definition, with recursion formula

$$B_{n+1} = B_n^{2^{n+1}-1} 1^{2^{n(n+1)/2}},$$

with corresponding constructive geometric and adic definitions. The following picture gives the first stages of the adic construction, corresponding to $B_1 = B_0 1$ and $B_2 = B_1 B_1 B_1 11$.
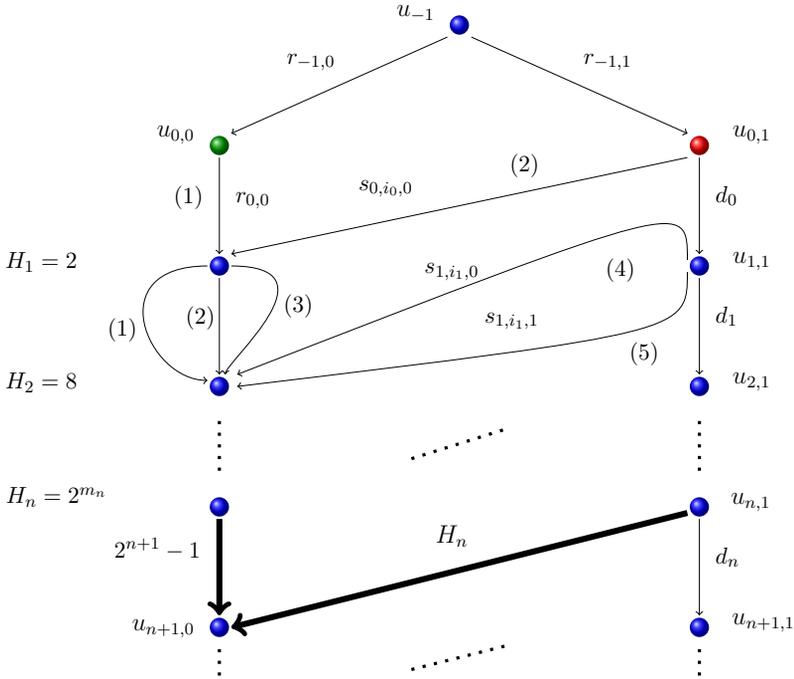
Fig. 1. Constructive odometer

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