Generalization of Neural Complexity to Dynamical Systems

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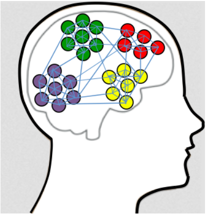
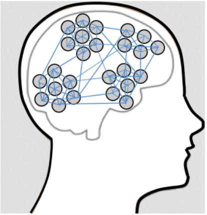
Two viewpoints

Contrasting viewpoints of brain organization in higher vertebrates:

1. Emphasis on specificity and modularity (functional segregation)
2. Emphasis on global functions and mass actions (integration in perception and behavior)





Neural Complexity (G. Edelman, O. Sporns, G. Tononi, 1994)

- Neither view alone adequately accounts for interactions that occur during brain activity.
- So they propose a general measure that encompasses these fundamental aspects of brain organization.
- High values are associated with non-trivial organization of the network. This is the case when segregation coexists with integration.
- Low values are associated with systems that are either completely independent (segregated, disordered) or completely dependent (integrated, ordered).

Mutual Information

Entropy of a random variable X taking values in a discrete set E:

$$H(X) = -\sum_{x \in E} Pr\{X = x\} \log Pr\{X = x\}.$$

Mutual information between random variables X and Y over the same probability space:

$$\begin{aligned} MI(X, Y) &= H(X) + H(Y) - H(X, Y). \\ &= H(X) - H(X|Y) = H(Y) - H(Y|X) \end{aligned}$$

- $MI(X, Y)$ is a measure of how much Y tells about X (equivalently, how much X tells about Y)
- $MI(X, Y) = 0 \Leftrightarrow X$ and Y are independent

Some notation:

- $n^* = \{0, 1 \ldots, n-1\}$
- $X = \{X_i : i \in n^*\}$ a family of random variables representing an isolated neural system with n elementary components (neuronal groups)
- For $S \subset n^*$, $X_S = \{X_i : i \in S\}$
- $S^c = n^* \setminus S$.

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Neural Complexity, C_N

Average of mutual information over subfamilies of a family of random variables

$$C_N(X) = \frac{1}{n+1} \sum_{S \subset n^*} \frac{1}{\binom{n}{|S|}} MI(X_S, X_{S^c}).$$

Intricacy (J. Buzzi, L. Zambotti, 2009)

- Give a general probabilisitic representation of neural complexity.
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System of coefficients

A *system of coefficients*, c_S^n, is a family of numbers satisfying for all $n \in \mathbb{N}$ and $S \subset n^*$

1. $c_S^n \geqslant 0$;
2. $\sum_{S \subset n^*} c_S^n = 1$;
3. $c_{S^c}^n = c_S^n$.

Mutual information functional

- For a fixed $n \in \mathbb{N}$ let $X = \{X_i : i \in n^*\}$ be a collection of random variables all taking values in the same finite set.
- Given a system of coefficients, c_S^n, the corresponding *mutual information functional*, $\mathcal{I}^c(X)$ is defined by

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Intricacy

An *intricacy* is a mutual information functional satisfying:

1. Exchangeability: invariance by permutations of n;
2. Weak additivity: $\mathcal{I}^c(X, Y) = \mathcal{I}^c(X) + \mathcal{I}^c(Y)$ for any two independent systems $X = \{X_i : i \in n^*\}$ and $Y = \{Y_j : j \in m^*\}$.

Theorem (Buzzi, Zambotti)

Let c_S^n be a system of coefficients and $\mathcal{I}^c$ the associated mutual information functional. $\mathcal{I}^c$ is an intricacy if and only if there exists a symmetric probability measure λ_c on $[0,1]$ such that

$$c_S^n = \int_{[0,1]} x^{|S|}(1-x)^{n-|S|}\lambda_c(dx)$$

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Example

1. $c_S^n = \dfrac{1}{(n+1)}\dfrac{1}{\binom{n}{|S|}}$ (Edelman-Sporns-Tononi);
2. For $0 < p < 1$,

$$c_S^n = \frac{1}{2}(p^{|S|}(1-p)^{n-|S|} + (1-p)^{|S|}p^{n-|S|}) \text{ (p-symmetric);}$$

3. For $p = 1/2$, $c_S^n = 2^{-n}$ (uniform).

Topological dynamical system, (X, T)

- X a compact Hausdorff (often metric) space;
- $T : X \to X$ a homeomorphism.

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Definition (Adler, Konheim, McAndrew, 1965)

The *topological entropy* of (X, T) is defined by

$$h_{\mathrm{top}}(X, T) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U}).$$

Topological entropy is a measure of the amount of randomness or disorder in a system.

Let (X, T) be a topological dynamical system and $\mathcal{U}$ an open cover of X. Given $n \in \mathbb{N}$ and a subset $S \subset n^*$ define

$$\mathcal{U}_S = \bigvee_{i \in S} T^{-i}\mathcal{U}.$$

Definition (P-W)

Let c_S^n be a system of coefficients. Define the *topological intricacy of* (X, T) *with respect to the open cover* $\mathcal{U}$ to be

$$\mathrm{Int}(X, \mathcal{U}, T) := \lim_{n \to \infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \log \left(\frac{N(\mathcal{U}_S) N(\mathcal{U}_{S^c})}{N(\mathcal{U}_{n^*})} \right).$$

$$\mathrm{Int}(X, \mathcal{U}, T) = 2\,\mathrm{Asc}(X, \mathcal{U}, T) - h_{\mathrm{top}}(X, \mathcal{U}, T).$$

$$\operatorname{Int}(X, \mathcal{U}, T) = 2 \operatorname{Asc}(X, \mathcal{U}, T) - h_{\mathrm{top}}(X, \mathcal{U}, T).$$

Definition (P-W)

The *topological average sample complexity of* T *with respect to the open cover* $\mathcal{U}$ is defined to be

$$\operatorname{Asc}(X, \mathcal{U}, T) := \lim_{n \to \infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \log N(\mathcal{U}_S).$$

Theorem

The limits in the definitions of $\mathrm{Int}(X, \mathcal{U}, T)$ *and* $\mathrm{Asc}(X, \mathcal{U}, T)$ *exist.*

The proof is based on subadditivity of the sequence

$$b_n := \sum_{S \subset n^*} c_S^n \log N(\mathcal{U}_S)$$

and Fekete's Subadditive Lemma: for every subadditive sequence a_n, the limit $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_n a_n/n$.

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Proposition

For each open cover $\mathcal{U}$*,*
$\mathrm{Asc}(X, \mathcal{U}, T) \leqslant h_{\mathrm{top}}(X, \mathcal{U}, T) \leqslant h_{\mathrm{top}}(X, T)$*, and hence*
$\mathrm{Int}(X, \mathcal{U}, T) \leqslant h_{\mathrm{top}}(X, \mathcal{U}, T) \leqslant h_{\mathrm{top}}(X, T)$.

In particular, a dynamical system with zero (or relatively low) topological entropy (integrated, ordered) has zero (or relatively low) topological intricacy.

Definition

For $r \in \mathbb{N}$ consider the finite set $\mathcal{A} = \{0, 1, \ldots r-1\}$. We call $\mathcal{A}$ an *alphabet* and give it the discrete topology. The (two-sided) *full shift space*, $\Sigma(\mathcal{A})$, is defined as

$$\Sigma(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{-\infty}^{\infty} : x_i \in \mathcal{A} \text{ for each } i\},$$

and is given the product topology. The shift transformation $\sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{A})$ is defined by

$$(\sigma x)_i = x_{i+1} \quad \text{for } -\infty < i < \infty,$$

Definition

A *subshift* is a pair (X, σ) where $X \subset \Sigma(\mathcal{A})$ is a nonempty, closed, shift-invariant ($\sigma X = X$) set.

Definition

A *block* or *word* is an element of $\mathcal{A}^k$ for $k = 0, 1, 2 \ldots$, i.e. a finite string on the alphabet $\mathcal{A}$.

Denote the set of words of length n in a subshift X by $\mathcal{L}_n(X)$.

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For a subset $S \subset n^*$, $S = \{s_0, s_1, \ldots, s_{|S|-1}\}$, denote the set of words we can see at the places in S for all words in $\mathcal{L}_n(X)$ by $\mathcal{L}_S(X)$,

$$\mathcal{L}_S(X) = \{w_{s_0} w_{s_1} \ldots w_{s_{|S|-1}} : w = w_0 w_1 \ldots w_{n-1} \in \mathcal{L}_n(X)\}.$$

Notice $\mathcal{L}_{n^*}(X) = \mathcal{L}_n(X)$.

Definition

- A *shift of finite type* (SFT) is defined by specifying a finite collection, $\mathcal{F}$, of forbidden words on a given alphabet, $\mathcal{A} = \{0, 1, \ldots, r\}$.
- Define $X_{\mathcal{F}} \subset \Sigma_r$ to be the set of all sequences none of whose words are in $\mathcal{F}$.

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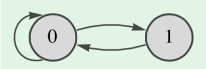
Example

Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11\}$. $(X_{\mathcal{F}}, \sigma)$ is called the *golden mean shift*.

Adjacency Matrix	Graph
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	

Intricacy of a subshift, X

$$\mathrm{Int}(X, \mathcal{U}_0, \sigma) = \lim_{n\to\infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \log \left(\frac{|\mathcal{L}_S(X)||\mathcal{L}_{S^c}(X)|}{|\mathcal{L}_{n^*}(X)|} \right)$$

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Example (Computing $|\mathcal{L}_S(X)|$ for the golden mean sft)

Let $n = 3$, $n^* = \{0, 1, 2\}$.

$S = \{0, 1\}$

$$\begin{array}{ccc} 0 & 0 & \\ 0 & 1 & \\ 1 & 0 & \end{array}$$

$|\mathcal{L}_S(X)| = 3$

$S = \{0, 2\}$

$$\begin{array}{ccc} 0 & & 0 \\ 0 & & 1 \\ 1 & & 0 \\ 1 & & 1 \end{array}$$

$|\mathcal{L}_S(X)| = 4$

Example (Computing $|\mathcal{L}_S(X)|$ for the golden mean sft)

S	S^c	$\lvert\mathcal{L}_S(X)\rvert$	$\lvert\mathcal{L}_{S^c}(X)\rvert$
$\emptyset$	$\{0,1,2\}$	1	5
$\{0\}$	$\{1,2\}$	2	3
$\{1\}$	$\{0,2\}$	2	4
$\{2\}$	$\{0,1\}$	2	3
$\{0,1\}$	$\{2\}$	3	2
$\{0,2\}$	$\{1\}$	4	2
$\{1,2\}$	$\{0\}$	3	2
$\{0,1,2\}$	$\emptyset$	5	1

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$\{0,1\}$	$\{2\}$	3	2
$\{0,2\}$	$\{1\}$	4	2
$\{1,2\}$	$\{0\}$	3	2
$\{0,1,2\}$	$\emptyset$	5	1

$$\frac{1}{3\cdot 2^3}\sum_{S\subset 3^*}\log\left(\frac{|\mathcal{L}_S(X)||\mathcal{L}_{S^c}(X)|}{|\mathcal{L}_{n^*}(X)|}\right)=\frac{1}{24}\log\left(\frac{6^4\cdot 8^2}{5^6}\right)\approx 0.070$$

Theorem

Let X be a shift of finite type with adjacency matrix M such that $M^2 > 0$. Let $c_S^n = 2^{-n}$ for all S. Then

$$\mathrm{Asc}(X, \mathcal{U}_0, \sigma) = \frac{1}{4} \sum_{k=1}^{\infty} \frac{\log |\mathcal{L}_{k^*}(X)|}{2^k}.$$

Asc is sensitive to word counts of all lengths, so is a finer measurement than h_{top}, which just gives the asymptotic exponential growth rate.

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Proof idea: Most subsets $S \subset n^*$ are also subsets of $(n-1)^*$.

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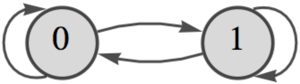
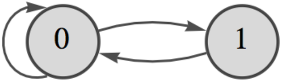
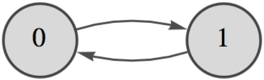
Corollary

For the full r-shift with $c_S^n = 2^{-n}$ for all S,

$$\mathrm{Asc}(\Sigma_r, \mathfrak{U}_0, \sigma) = \frac{\log r}{2} \quad \textit{and} \quad \mathrm{Int}(\Sigma_r, \mathfrak{U}_0, \sigma) = 0.$$

	Adjacency Graph	Entropy	Asc	Int
Disordered		0.693	0.347	0
		0.481	0.286	0.090
Ordered		0	0	0

Theorem

Let (X, T) be a topological dynamical system and fix the system of coefficients to be $c_S^n = 2^{-n}$. Then

$$\sup_{\mathcal{U}} \operatorname{Asc}(X, \mathcal{U}, T) = h_{\mathrm{top}}(X, T).$$

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- Most $S \subset n^*$ have size about $n/2$, so are not too sparse.
- In ordinary topological entropy of a subshift, using the time-0 partition (or open cover) α, when we replace α by $\alpha_{k^*} = \alpha_0^{k-1}$ in counting the number of cells or calculating the entropy of the refined partition, instead of α_{n^*}, we are looking at $\alpha_{(n+k)^*}$, and when k is fixed, as n grows the result is the same.

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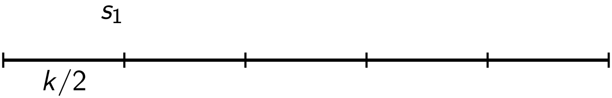
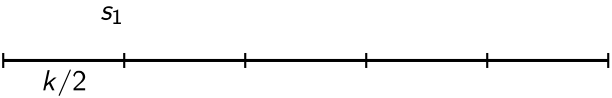
- When we code by k-blocks, $S \subset n^*$ is replaced by $S + k^*$, and the effect on α_{S+k^*} as compared to α_S is similar, since it acts similarly on each of the long subintervals comprising S.



- Fix a k for coding by k-blocks (or looking at $N((\mathcal{U}_k)_S)$ or $H((\alpha_k)_S)$).



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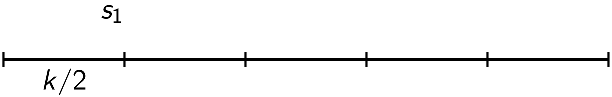

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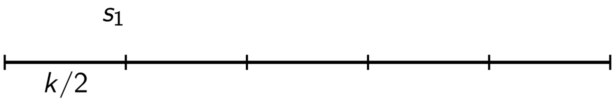

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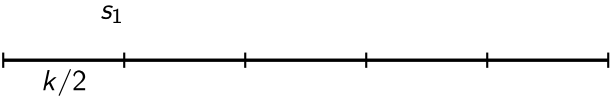

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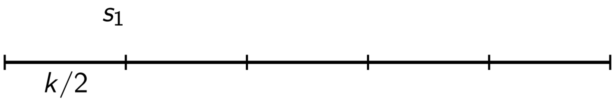
- So if S hits many of the intervals of length $k/2$, then $S + k^*$ starts to look like a union of long intervals, say each with $|E_j| > k$.
- By shaving a little off each of these relatively long intervals, we can assume that also the gaps have length at least k.



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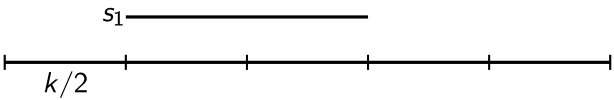

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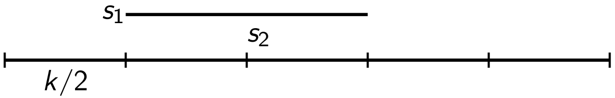

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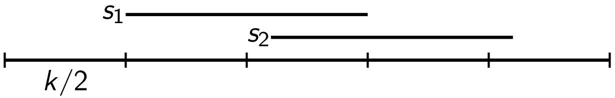
- So if S hits many of the intervals of length $k/2$, then $S + k^*$ starts to look like a union of long intervals, say each with $|E_j| > k$.
- By shaving a little off each of these relatively long intervals, we can assume that also the gaps have length at least k.



- Fix a k for coding by k-blocks (or looking at $N((\mathcal{U}_k)_S)$ or $H((\alpha_k)_S)$).
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- Given $\epsilon > 0$, we may assume k is large enough that for every interval $I \subset \mathbb{N}$ with $|I| \geqslant k/2$,

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Measure-theoretic dynamical systems

Measure-theoretic dynamical system $(X, \mathcal{B}, \mu, T)$

- X is a measure space
- $\mathcal{B}$ is a σ-algebra of measurable subsets of X
- μ is a probability measure on X, i.e., $\mu(X) = 1$
- $T : X \to X$ is a measure-preserving transformation on X, i.e., T is a one-to-one onto map such that $\mu(T^{-1}E) = \mu(E)$ for all $E \in \mathcal{B}$

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Entropy of a partition

The *entropy of a finite measurable partition* $\alpha = \{A_1, \ldots, A_n\}$ *of* X is defined by

$$H_\mu(\alpha) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).$$

Definition

The *entropy of X and T with respect to μ and a partition α* is

$$h_\mu(X,\alpha,T) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha \vee T^{-1}\alpha \vee \cdots \vee T^{-n+1}\alpha).$$

The *entropy of the transformation T* is defined to be

$$h_\mu(X,T) = \sup_\alpha h_\mu(X,\alpha,T).$$

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Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system, $\alpha = \{A_1, \ldots, A_n\}$ a finite measurable partition of X, and c_S^n a system of coefficients. The *measure-theoretic intricacy of* T *with respect to the partition* α is

$$\mathrm{Int}_\mu(X, \alpha, T) = \lim_{n\to\infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \left[H_\mu(\alpha_S) + H_\mu(\alpha_{S^c}) - H_\mu(\alpha_{n^*})\right].$$

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The *measure-theoretic average sample complexity of T with respect to the partition* α is

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Theorem

The limits in the definitions of measure-theoretic intricacy and measure-theoretic average sample complexity exist.

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Theorem (Ornstein-Weiss, 2007)

If J is a finitely observable functional defined for ergodic finite-valued processes that is an isomorphism invariant, then J is a continuous function of the measure-theoretic entropy.

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- Fix a k and find the topological average sample complexity $Asc(X, \mathcal{U}_k, \sigma) = \lim_{n\to\infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \log N((\mathcal{U}_k)_S)$,

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- or do not take the limit on n, and study it as a function of n,
- analogously to the symbolic or topological complexity functions.
- Similarly for the measure-theoretic version: fix a partition α and study the limit, or the function of n.

$$\mathrm{Asc}_\mu(X, T, \alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{S\subset n^*} c_S^n H_\mu(\alpha_S).$$

So we begin study of Asc for a fixed open cover as a function of n.

$$\mathrm{Asc}(X, \sigma, \mathcal{U}_k, n) = \frac{1}{n} \sum_{S \subset n^*} c_S^n \log N(S).$$

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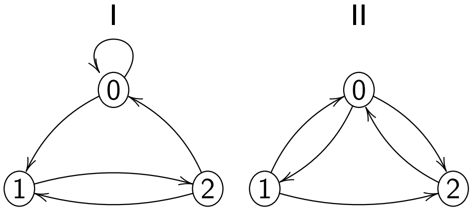
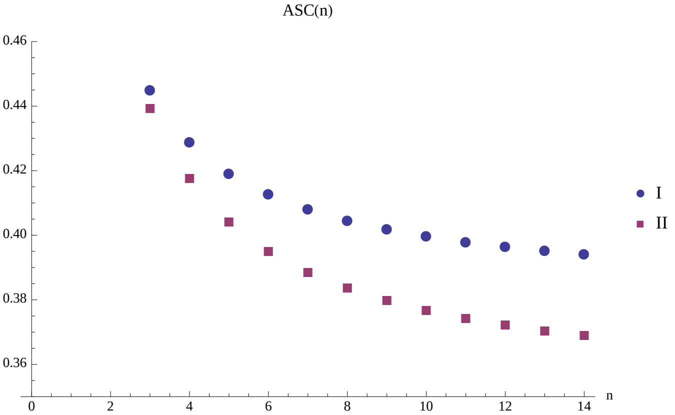
Example



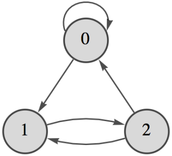
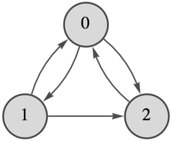
$(\mathrm{M}_I)^3 > 0$ $\qquad$ $(\mathrm{M}_{II})^4 > 0$

Figure: Graphs of two subshifts with the same complexity function but different average sample complexity functions.

$$\mathrm{Asc}(n) = \frac{1}{n}\frac{1}{2^n}\sum_{S\subset n^*} \log N(S)$$



Interesting example

Adjacency Graph	h_{top}	Asc(10)	Int(10)
	0.481	0.399	0.254
	0.481	0.377	0.208

These SFTs have the same entropy and complexity functions (words of length n) but different Asc and Int functions.

Results in measure-theoretic setting

For a fixed partition α, we give a relationship between $\mathrm{Asc}_{\mu}(X, \alpha, T)$ and a series summed over i involving the conditional entropies $H_{\mu}(\alpha \mid \alpha_i^{\infty})$.

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Idea

- View a subset $S \subset n^*$ as corresponding to a random binary string of length n generated by Bernoulli measure $\mathcal{B}(1/2, 1/2)$ on the full 2-shift.
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- For example $\{0, 2, 3\} \subset 5^* \leftrightarrow 10110$.
- The average entropy, $H_\mu(\alpha_S)$, over all $S \subset n^*$, is then an integral and can be interpreted in terms of the entropy of a first-return map to the cylinder $A = [1]$ in a cross product of our system X and the full 2-shift, Σ_2.

Theorem

Let $(X, \mathcal{B}, \mu, T)$ *be a measure-preserving system and* α *a finite measurable partition of* X*. Let* $A = [1] = \{\xi \in \Sigma_2^+ : \xi_0 = 1\}$ *and* $\beta = \alpha \times A$ *the related finite partition of* $X \times A$*. Denote by* $T_{X\times A}$ *the first-return map on* $X \times A$ *and let* $P_A = P/P[1]$ *denote the measure* P *restricted to* A *and normalized. Let* $c_S^n = 2^{-n}$ *for all* $S \subset n^*$*. Then*

$$\mathrm{Asc}_\mu(X, \alpha, T) = \frac{1}{2} h_{\mu\times P_A}(X \times A, \beta, T_{X\times A}).$$

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$$\mathrm{Asc}_\mu(X, \alpha, T) \geqslant \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2^i} H_\mu\left(\alpha \mid \alpha_i^\infty\right).$$

Equality holds in certain cases (in particular, for Markov shifts)

In the topological case the first-return map $T_{X \times A}$ is not continuous nor expansive nor even defined on all of $X \times A$ in general, so known results about measures of maximal entropy and equilibrium states do not apply.

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$$\mathrm{Int}(X, \mathcal{U}, T) = 2\,\mathrm{Asc}(X, \mathcal{U}, T) - h_{\mathrm{top}}(X, \mathcal{U}, T).$$

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But the above theorem does give up some information immediately:

Proposition

When $T : X \to X$ is an expansive homeomorphism on a compact metric space (e.g., a subshift), $\mathrm{Asc}_\mu(X, T, \alpha)$ is an affine upper semicontinuous (in the weak topology) function of μ, so the set of maximal measures for $\mathrm{Asc}_\mu(X, T, \alpha)$ is nonempty, compact, and convex and contains ergodic measures (see Walters, p. 198 ff.).*

Markov Shift

- Consider the measure on the shift space (Σ_n, σ) given by s stochastic matrix $P = (P_{ij})$ and fixed probability vector $p = \begin{pmatrix} p_0 & p_1 & \dots & p_{n-1} \end{pmatrix}$, i.e. $\sum p_i = 1$ and $pP = p$.
- The measure $\mu_{P,p}$ is defined as usual on cylinder sets by $\mu_{p,P}[i_0 i_1 \dots i_k] = p_{i_0} P_{i_0 i_1} \cdots P_{i_{k-1} i_k}$.

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Example (1-step Markov measure on the golden mean shift)

Denote by $P_{00} \in [0,1]$ the probability of going from 0 to 0 in a sequence of $X_{\{11\}} \subset \Sigma_2$. Then

$$P = \begin{pmatrix} P_{00} & 1 - P_{00} \\ 1 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} \frac{1}{2-P_{00}} & \frac{1-P_{00}}{2-P_{00}} \end{pmatrix}$$

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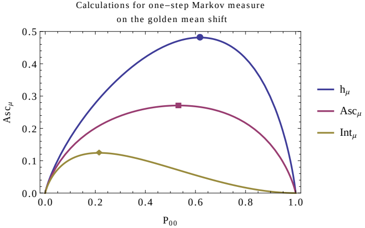
Using the series formula and known equations for conditional entropy, we approximate Asc_μ and Int_μ for Markov measures on SFTs.

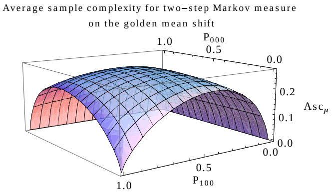
1-step Markov measures on the golden mean shift

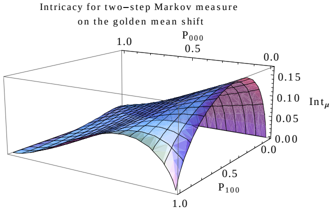


P_{00}	h_μ	Asc_μ	Int_μ
0.618	**0.481**	0.266	0.051
0.533	0.471	**0.271**	0.071
0.216	0.292	0.208	**0.124**

- Maximum value of $h_\mu = h_{\mathrm{top}} = \log \phi$ when $P_{00} = 1/\phi$
- Unique maxima among 1-step Markov measures for Asc_μ and Int_μ
- Maxima for Asc_μ, Int_μ, and h_μ achieved by different measures

2-step Markov measures on the golden mean shift

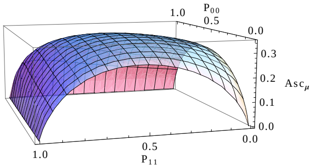


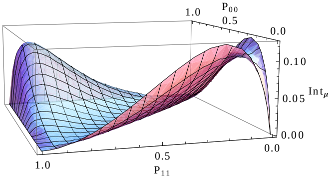


P_{000}	P_{100}	h_μ	Asc_μ	Int_μ
0.618	0.618	**0.481**	0.266	0.051
0.483	0.569	0.466	**0.272**	0.078
0	0.275	0.344	0.221	**0.167**

- Asc_μ appears to be strictly convex, so it would have a unique maximum among 2-step Markov measures
- Int_μ appears to have a unique maximum among 2-step Markov measures on a proper subshift ($P_{000} = 0$)
- The maxima for Asc_μ, Int_μ, and h_μ are achieved by different measures

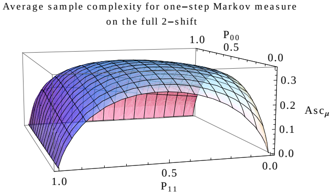
1-step Markov measures on the full 2-shift

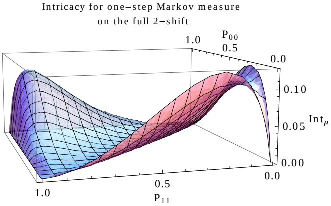




P_{00}	P_{11}	h_μ	Asc_μ	Int_μ
0.5	0.5	**0.693**	**0.347**	0
0.216	0	0.292	0.208	**0.124**
0	0.216	0.292	0.208	**0.124**
0.905	0.905	0.315	0.209	0.104

1-step Markov measures on the full 2-shift





- Asc_μ appears to be strictly convex, so it would have a unique maximum among 1-step Markov measures
- Int_μ appears to have two maxima among 1-step Markov measures on proper subshifts ($P_{00} = 0$ and $P_{11} = 0$).
- There seems to be a 1-step Markov measure that is fully supported and is a local maximum for Int_μ among all 1-step Markov measures.
- The maxima for Asc_μ, Int_μ, and h_μ are achieved by different measures.

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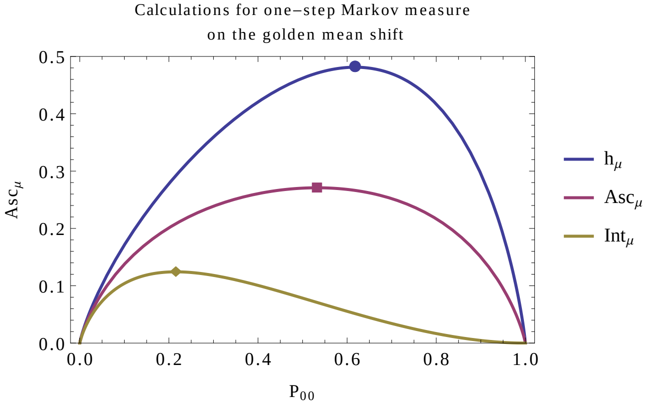
We summarize some of the questions generated above.
Conj. 1: On the golden mean SFT, for each r there is a unique r-step Markov measure μ_r that maximizes $\mathrm{Asc}_\mu(X, \sigma, \alpha)$ among all r-step Markov measures.



Conj. 2: $\mu_2 \neq \mu_1$

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P_{00}	h_μ	Asc_μ	Int_μ
0.618	**0.481**	0.266	0.051
0.533	0.471	**0.271**	0.071
0.216	0.292	0.208	**0.124**

Table: Calculations for one-step Markov measures on the golden mean shift. Bolded numbers are maxima for given category.

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0.618	0.618	**0.481**	0.266	0.051
0.483	0.569	0.466	**0.272**	0.078
0	0.275	0.344	0.221	**0.167**

Table: Calculations for two-step Markov measures on the golden mean shift.

Conj. 3: On the golden mean SFT there is a unique measure that maximizes $\mathrm{Asc}_{\mu}(X, T, \alpha)$. It is not Markov of any order (and of course is not the same as μ_{max}).

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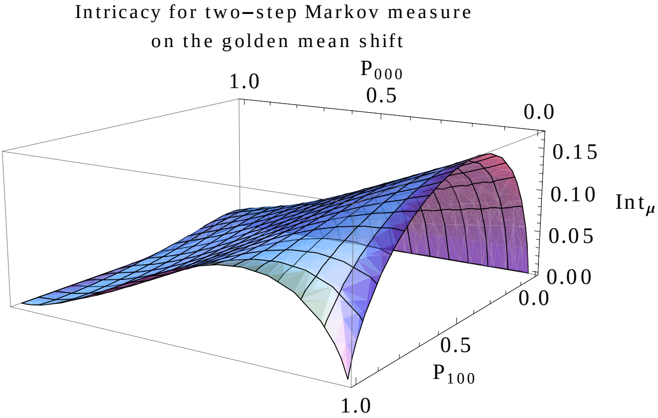

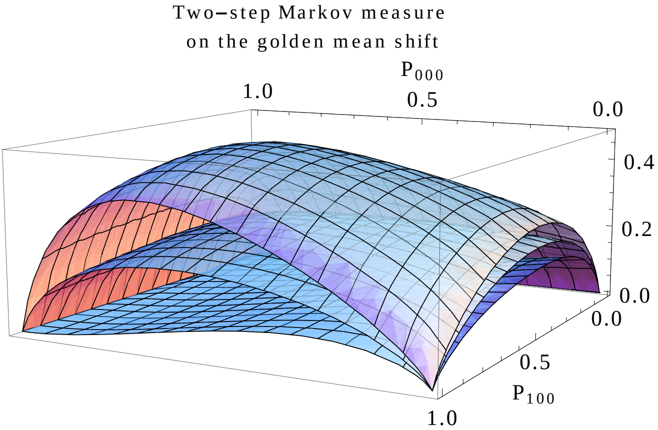

Figure: Combination of the plots of h_μ, Asc_μ, and Int_μ for two-step Markov measures on the golden mean shift.

Conj. 5: On the 2-shift there are *two* 1-step Markov measures that maximize $\mathrm{Int}_{\mu}(X, T, \alpha)$ among all 1-step Markov measures. They are supported on the golden mean SFT and its image under the dualizing map $0 \leftrightarrow 1$.

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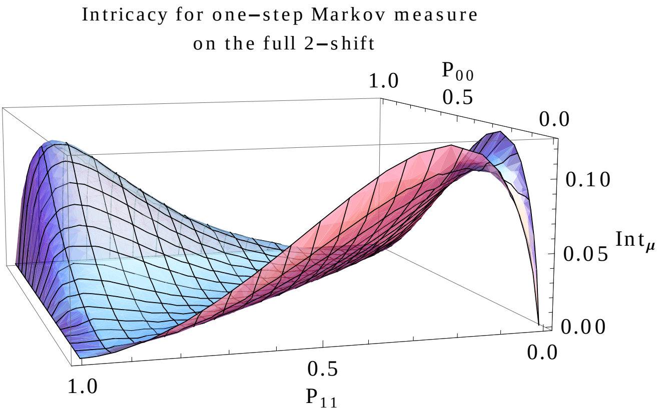

Conj. 6: On the 2-shift there is a 1-step Markov measure that is *fully supported* and is a local maximum point for $\mathrm{Int}_{\mu}(X, T, \alpha)$ among all 1-step Markov measures.

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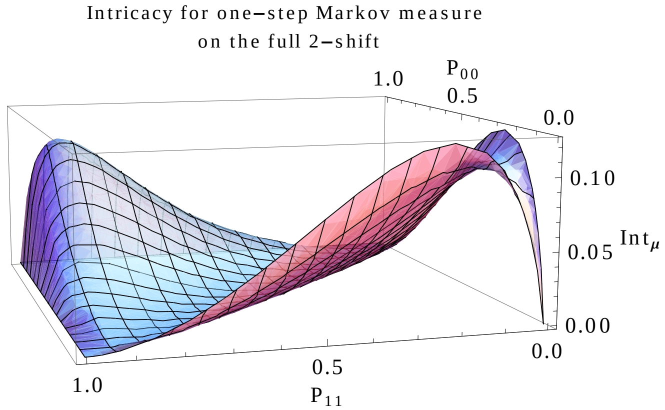

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- First one can consider a function of just a single coordinate that gives the value of each symbol.
- Maximum intricacy may be useful for finding areas of high information activity, such as working regions in a brain (Edelman-Sporns-Tononi) or coding regions in genetic material (Koslicki-Thompson).

The end

The end (of this talk).