Some results and systems related to the super-K property

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Outline

Introduction

Ordinary tail fields

Fine tail fields

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Super-K plus generators

Systems that present tail fields

Some questions about the systems

Dresden

A Little Earlier

Help

Thouvenot, Schmidt, Weiss

Stationary processes

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The time-0 partition of Ω is a generator for the m.p. system (Ω, μ, σ).

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When α is a generator, $\mathcal{T}^+(\alpha)$ is the *Pinsker algebra* of $(X, \mathcal{B}, \mu, T)$.

The K property

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Therefore, for any partition α, $\mathcal{T}^-(\alpha)$ is trivial if and only if $\mathcal{T}^+(\alpha)$ is trivial (because for any $\beta \leq \alpha$, $h_\mu(T, \beta) = h_\mu(T^{-1}, \beta)$).

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$$\psi_m^n(x) = \psi(T^m x) \cdots \psi(T^n x), \text{ in abelian case } \sum_{k=m}^{n} \psi(T^k x)$$

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E.g., if $\psi : \Omega \to \mathbb{Z}^d$ is defined by $\psi(\omega) = e_i \in \mathbb{Z}^d$ if $\omega_0 = a_i$, then $\psi_0^{n-1}(\omega)$ gives in each entry i the number of times that a_i appears in the first n entries in ω.

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When ψ is the symbol-counting cocycle, these equivalence relations are the orbit relation of the group of *finite coordinate permutations*.

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There are also such results for the 2-sided case by Blackwell-Freedman for Markov processes, Georgii for Gibbs states, Berbee-den Hollander for integer-valued processes, and others.

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And unlike the *K* property, super-*K* depends on the choice of generating partition.

We can have $\mathcal{F}_\psi^+(\alpha)$ trivial and find a refinement $\beta \geq \alpha$ with $\mathcal{F}_\psi^+(\beta)$ nontrivial (in fact equal to $\mathcal{B}$).

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Corollary: Any process (could be countable-state) with 2-sided trivial tail field $\mathcal{T}^\pm$ is super-$K^\pm$: $\mathcal{F}_\psi^\pm(\alpha)$ is trivial.

Super-K^+ generators

JPT-KP, 2004: If an ergodic system $(X, \mathcal{B}, \mu, T)$, with generator α, is isomorphic to the direct product of a positive-entropy Bernoulli system (B, σ) and some other system (Y, S), then there is a generator β for $(X, \mathcal{B}, \mu, T)$ such that $\mathcal{F}^+(\beta) = \mathcal{T}^+(\beta) = \mathcal{T}^+$.

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The idea of the proof is to construct a partition β with $\mathcal{F}^+(\beta) \subset \mathcal{T}^+(\beta)$, so that no new information is provided by counting β-symbols.

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On each marked interval, where W appears in B, we do not change the B coding, but we change the Y coding so that each β_0 symbol appears the same number of times.

Coding free intervals

On each free interval, we recode the $\gamma \times \rho$ name by cutting into subintervals and using permutations of a string of all $\beta_0 \times \rho$ symbols (one of each symbol), plus we add *one extra symbol*, which depends only on the length of the free interval.

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If the count ends inside a marked or free interval, with high probability we have a bounded translate of a count across a complete union of intervals, so it is not too different.

Super-K generators for K systems

JPT, 2008: If $(X, \mathcal{B}, \mu, T)$ is ergodic, finite entropy, and weak Pinsker (for every $\epsilon > 0$, $X \approx B \times Y$ with B Bernoulli and $h(Y) < \epsilon$), then there is a finite generator α with $\mathcal{F}^{\pm}_{\psi}(\alpha) = \mathcal{P}(T)$.

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Corollary: If $(X, \mathcal{B}, \mu, T)$ is K, it has a super-$K^{\pm}$ generator.

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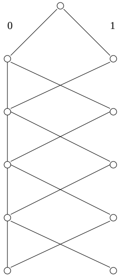
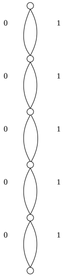
Similarly for a SFT Σ_M: $\mathcal{T}^+$ is the field of invariant sets for the *stationary adic*.





Invariant measures

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The measure of maximal entropy on Σ_M assigns pretty much the same measure of all cylinder sets of a fixed length.

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x_k labels the edge from $s_{k-1}(x)$ to $s_k(x)$.

Adic transformations

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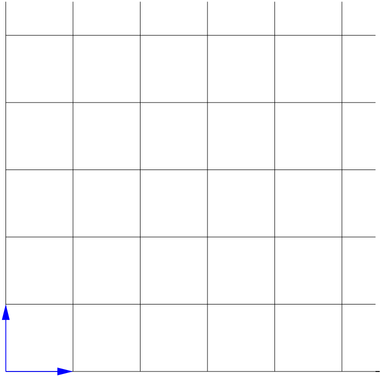
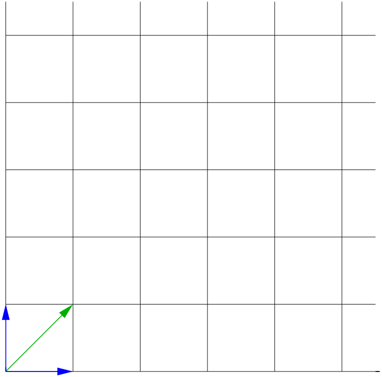
Thus these systems visually present the future fine tail fields—we can see the corresponding equivalence relations.

The Pascal walk

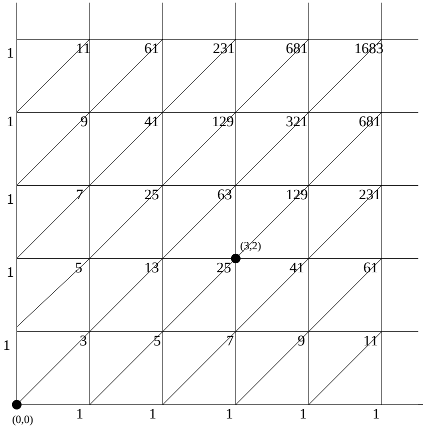


The Delannoy walk

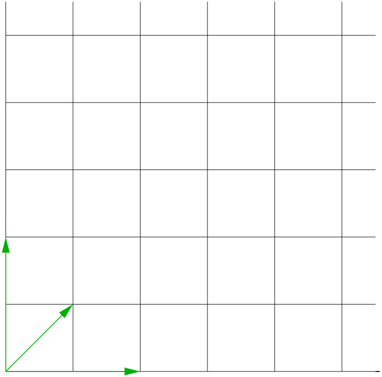


The Delannoy graph

1



Xavier Méla's X_3 walk



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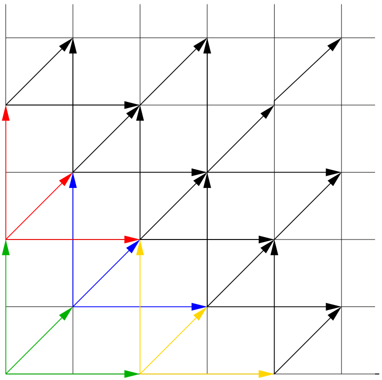
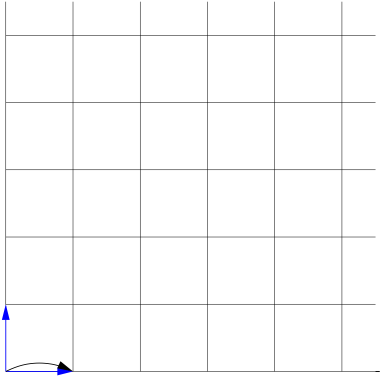
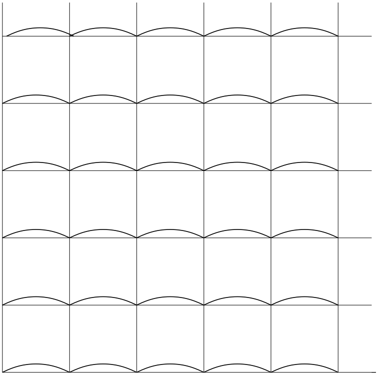


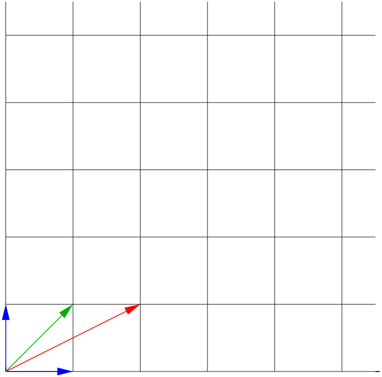
Frick's $2x + 1$ walk



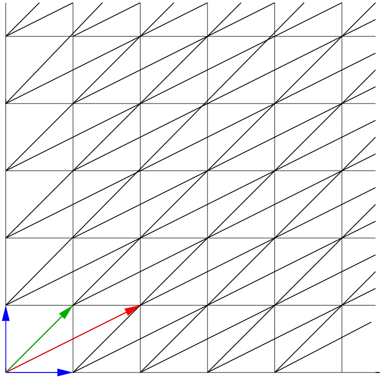
Frick's $2x + 1$ system



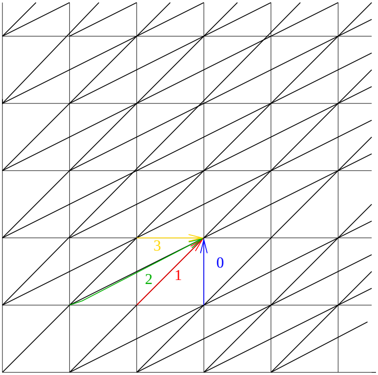
A walk with 4 vectors



An isotropic adic system based on a walk with 4 vectors



Ordering incoming edges to define the transformation



Ergodic measures

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Expansiveness and complexity

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For the Delannoy, $p(n) \sim n^3/24$ (Frick).

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It seems that for the Pascal, for every order $p(n)$ is asymptotically no more than $n^5/3$.

More questions

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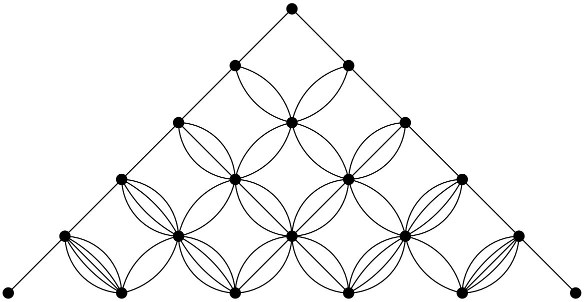
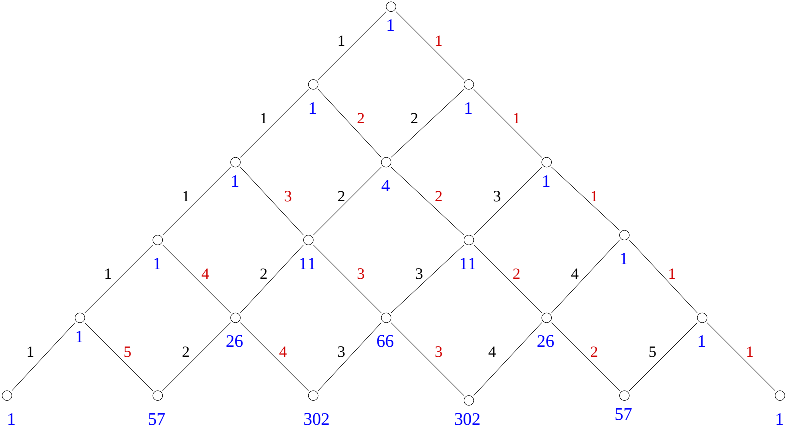
When the simple walks that give rise to isotropic adic systems are allowed to evolve according to reinforcement schemes, even more interesting systems arise,

for example the *Eulerian system* studied by Frick-Keane-KP-Salama, Frick-KP, KP-Varchenko, Gnedin-Olshanski.

The Eulerian adic



The Eulerian adic with path counts



C^* algebra connections

Study of such systems leads to interesting combinatorial questions and connections with C^* algebras and group representations (Kerov).

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Indeed, the Pascal graph is an example of an AF C^* algebra (the “CCR" algebra) in Bratteli’s 1972 paper.